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GLOBAL REGULARITY IN HOMOGENEOUS MORREY-HERZ SPACES OF SOLUTIONS TO NONDIVERGENCE ELLIPTIC EQUATIONS WITH VMO COEFFICIENTS*

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Abstract In this paper, by establishing the boundedness results of singular integral operators and linear commutators, we obtain the global regularity, in homogeneous Morrey-Herz spaces, of strong solutions to nondivergence elliptic equations with VMO coefficients.

Key Words Elliptic equation; Morrey-Herz space; VMO; singular integral; commutator.

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1. Introduction

Let Ω be an open subset of $\mathbf{R}^n$. Let $B_k = B(0, 2^k) = \{x \in \mathbf{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbf{Z}$. Let $\mathcal{X}_k = \mathcal{X}_{A_k}$ be the characteristic function of the set A_k for $k \in \mathbf{Z}$.

Definition 1.1 *Let $\alpha \in \mathbf{R}, 0 < p \leq \infty, 0 < q < \infty$, and $\lambda \geq 0$. The homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$ are defined by*

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega) = \{f \in L^q_{loc}(\Omega \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} = \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\mathcal{X}_k\|^p_{L^q(\Omega)} \right)^{1/p}$$

with the usual modifications made when $p = \infty$.

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The more general spaces are first introduced recently by Lu and Xu in [1]. Compare the homogeneous Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$ with the homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\Omega)$ (see [2]) and the Morrey spaces $M_q^{\lambda}(\Omega)$ (see [3]), where $\dot{K}_q^{\alpha,p}(\Omega)$ is defined by

$$\dot{K}_q^{\alpha,p}(\Omega) = \{f \in L_{loc}^q(\Omega \setminus 0) : \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\mathcal{X}_k\|_{L^q(\Omega)}^p < \infty\},$$

and $M_q^{\lambda}(\Omega)$ is defined by

$$M_q^{\lambda}(\Omega) = \{f \in L_{loc}^q(\Omega) : \sup_{\rho>0, x\in\Omega} \frac{1}{\rho^\lambda} \int_{B(x,\rho)\cap\Omega} |f(y)|^q \, dy < \infty\}.$$

It is easy to see that $M\dot{K}_{p,q}^{\alpha,0}(\Omega) = \dot{K}_q^{\alpha,p}(\Omega)$ and $M_q^{\lambda}(\Omega) \subset M\dot{K}_{q,q}^{0,\lambda}(\Omega)$.

Assume $f \in M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$, the main purpose of this paper is to investigate the regularity, in the homogeneous Morrey-Herz spaces, of the strong solutions to the following Dirichlet problem on the second-order elliptic equations in nondivergence form:

$$\begin{cases} Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} = f \quad \text{a.e. in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$, Ω is a bounded domain $C^{1,1}$ of $\mathbf{R}^n$, the coefficients $\{a_{ij}\}_{i,j=1}^n$ are symmetric and uniformly elliptic in Ω , i.e., for some $\Lambda > 0$ and every $\xi \in \mathbf{R}^n$,

$$a_{ij}(x) = a_{ji}(x), \quad \text{and} \quad \Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{1.2}$$

with a.e. $x \in \Omega$. Moreover, we assume that $a_{ij}(x) \in VMO(\Omega)$, the spaces of the functions of vanishing mean oscillation introduced by Sarason in [4].

By establishing the boundedness results of singular integral operators and linear commutators (see Section 2), owing to the integral representation formulas given in [5, 6], we obtain the following global regularity, in homogeneous Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$, of the strong solution u to (1.1):

$$\|u\|_{W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)} \le C\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)}, \tag{1.3}$$

where $0 < p < \infty$, $1 < q < \infty$, $\lambda > 0$, $-n/q+\lambda \le \alpha \le n(1-1/q)+\lambda$, and $W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$ is the homogeneous Sobolev-Merry-Herz spaces defined in Section 3. We can see that, when $\alpha = 0$ and $p = q$, the above results coordinate with those in the setting of the Merry spaces $M_q^{\lambda}(\Omega)$, which are proved by Fan, Lu, and Yang in [7].

2. Estimates about Singular Integrals and Linear Commutators

In this section we shall deal with the boundedness results about some more general classes of singular integral operators and linear commutators on homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)$ which will be needed for proving our main boundary estimate result (1.3).

Definition 2.1 *Let $k : \mathbf{R}^n \setminus \{0\} \longrightarrow \mathbf{R}$. We say that $k(x)$ is a constant Calderón-Zygmund kernel (constant C-Z kernel) if*

(i) $k \in C^\infty(\mathbf{R}^n \setminus \{0\})$;

(ii) *k is homogeneous of degree $-n$;*

(iii) $\int_{S^{n-1}} k(x)\, d\sigma = 0$, *where, and in what follows,* $S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$.

Definition 2.2 *Let Ω be an open set of $\mathbf{R}^n$ and $k : \Omega \times \{\mathbf{R}^n \setminus \{0\}\} \longrightarrow \mathbf{R}$. We say that $k(x, y)$ is a variable Calderón-Zygmund kernel (variable C-Z kernel) on Ω if*

(i) *$k(x, \cdot)$ is a constant C-Z kernel for a.e. $x \in \Omega$;*

(ii) $\max_{|j|\le 2n} \|(\partial^j/\partial z^j)k(x, z)\|_{L^\infty(\Omega\times S^{n-1})} \equiv M < \infty$.

Let k be a a constant or a variable C-Z kernel on Ω. We define the corresponding C-Z operator by

$$Tf(x) = P.V. \int_{\mathbf{R}^n} k(x-y)f(y)\, dy \quad \text{or} \quad Tf(x) = P.V. \int_{\Omega} k(x, x-y)f(y)\, dy.$$

First we have

Lemma 2.1 *Let $0 < p < \infty,\ 1 < q < \infty$, $\lambda > 0$, and $-n/q+\lambda \le \alpha \le n(1-1/q)+\lambda$. If k is a constant or a variable C-Z kernel on $\mathbf{R}^n$ and T is the corresponding C-Z operator, then there exists a constant $C = C(n, p, q, \alpha, \lambda, k)$ or $C = C(n, p, q, \alpha, \lambda, k, M)$ such that for all $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)$,*

$$\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}. \tag{2.1}$$

Proof It is easy to see that (2.1) can be obtained from Theorem 2.2 in [1] with $r = \infty$.

From this lemma, by a proof similar to that of Theorem 2.11 in [5], we obtain the following corollary:

Corollary 2.1 *Let $0 < p < \infty,\ 1 < q < \infty,\ \lambda > 0,\ -n/q + \lambda \le \alpha \le n(1 - 1/q) + \lambda$ and Ω be an open set of $\mathbf{R}^n$. If k is a variable C-Z kernel on Ω and T is the corresponding C-Z operator, then there exists a constant $C = C(n, p, q, \alpha, \lambda, k, M)$ such that for all $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$,*

$$\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)}. \tag{2.2}$$

Before considering the boundedness on homogeneous Morrey-Herz spaces of the linear commutator $[a, T]$ defined by $[a, T]f(x) = T(af)(x) - a(x)T(f)(x)$, we recall the definitions of the spaces BMO and VMO. For the properties of these spaces, we refer to [4, 8].

Definition 2.3 *Let Ω be an open set of $\mathbf{R}^n$. We say that any $a \in L^1_{loc}(\Omega)$ is in the space $BMO(\Omega)$ if*

$$\sup_{\rho>0,x\in\Omega} \frac{1}{|B(x,\rho)\cap\Omega|}\int_{B(x,\rho)\cap\Omega} |a(y)-a_{B(x,\rho)\cap\Omega}|\,dy<\infty, \tag{2.3}$$

where, and in what follows, a_Q denotes the average over the set Q of the function a, i.e., $a_Q = 1/|Q|\int_Q a(y)\,dy$. The $BMO(\Omega)$ norm of $a(x)$ will be denoted by $\|a\|_*$ and given by

$$\|a\|_* = \sup_{\rho>0,x\in\Omega} \frac{1}{|B(x,\rho)\cap\Omega|}\int_{B(x,\rho)\cap\Omega} |a(y)-a_{B(x,\rho)\cap\Omega}|\,dy.$$

Definition 2.4 *Let Ω be an open set of $\mathbf{R}^n$. For any $f \in BMO(\Omega)$ and $r>0$, we set*

$$\eta(r) \equiv \sup_{\rho\le r,x\in\Omega} \frac{1}{|B(x,\rho)\cap\Omega|}\int_{B(x,\rho)\cap\Omega} |a(y)-a_{B(x,\rho)\cap\Omega}|\,dy. \tag{2.4}$$

We say that any $a \in BMO(\Omega)$ is in the space $VMO(\Omega)$ if $\eta(r)\longrightarrow 0$ as $r \longrightarrow 0^+$ and refer to $\eta(r)$ as the VMO modulus of f.

Theorem 2.1 *Let $0<p<\infty$, $1<q<\infty$, $\lambda>0$, $-n/q+\lambda\le\alpha\le n(1-1/q)+\lambda$ and $a\in BMO(\mathbf{R}^n)$. If a linear operator T satisfies*

$$|T(f)(x)| \le C\int_{\mathbf{R}^n}\frac{|f(y)|}{|x-y|^n}\,dy,\ \ x\notin supp\ f \tag{2.5}$$

for any $f\in L^1(\mathbf{R}^n)$ with compact support and $[a,T]$ is bounded on $L^q(\mathbf{R}^n)$, then $[a,T]$ is also bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)$.

Proof Let $f\in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)$ and $a\in BMO(\mathbf{R}^n)$. Write

$$f(x)=\sum_{j=-\infty}^{\infty} f(x)\mathcal{X}_j(x)\equiv\sum_{j=-\infty}^{\infty} f_j(x)$$

and

$$\begin{aligned}
\|[a,T]f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)} &= \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|([a,T]f)\mathcal{X}_k\|^p_{L^q(\mathbf{R}^n)}\right)^{1/p}\\
&\le C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\|([a,T]f_j)\mathcal{X}_k\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\quad + C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\sum_{j=k-1}^{k+1}\|([a,T]f_j)\mathcal{X}_k\|^p_{L^q(\mathbf{R}^n)}\right)^{1/p}\\
&\quad + C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\|([a,T]f_j)\mathcal{X}_k\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}
\end{aligned}$$

$$\equiv I_1 + I_2 + I_3.$$

For I_2, by the $L^q(\mathbf{R}^n)$-boundedness of $[a,T]$, we have

$$\begin{aligned}
I_2 &\le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \sum_{j=k-1}^{k+1} \|a\|_*^p \|f_j \mathcal{X}_k\|^p_{L^q(\mathbf{R}^n)} \right)^{1/p} \\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\mathcal{X}_k\|^p_{L^q(\mathbf{R}^n)} \right)^{1/p} \\
&= C\|a\|_* \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}.
\end{aligned}$$

Thus, the desired estimate is obtained. For I_1, note that when $x \in A_k$, $y \in A_j$, and $j \le k-2$, $|x-y| \sim |x|$. So from the condition (2.5) we have

$$\begin{aligned}
|[a,T]f_j(x)| \le& C \int_{\mathbf{R}^n} \frac{|a(x)-a(y)|}{|x-y|^n} |f_j(y)|\, dy \\
\le& C2^{-nk}|a(x)-a_{A_k}| \int_{\mathbf{R}^n} |f_j(y)|\, dy + \\
&+ C2^{-nk}|a_{A_j}-a_{A_k}| \int_{\mathbf{R}^n} |f_j(y)|\, dy + C2^{-nk} \int_{\mathbf{R}^n} |a(y)-a_{A_j}||f_j(y)|\, dy.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
I_1 \le& C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda} \Bigg(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \Bigg(\sum_{j=-\infty}^{k-2} 2^{-nk} \|(a(x)-a_{A_k})\mathcal{X}_k(x)\|_{L^q(\mathbf{R}^n)} \\
&\qquad \int_{\mathbf{R}^n} |f_j(y)|\mathcal{X}_j(y)\, dy \Bigg)^p \Bigg)^{1/p} \\
&+ C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda} \Bigg(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \Bigg(\sum_{j=-\infty}^{k-2} 2^{-nk} \|(a_{A_j}-a_{A_k})\mathcal{X}_k(x)\|_{L^q(\mathbf{R}^n)} \\
&\qquad \int_{\mathbf{R}^n} |f_j(y)|\mathcal{X}_j(y)\, dy \Bigg)^p \Bigg)^{1/p} \\
&+ C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} 2^{-nk} 2^{nk/q} \int_{\mathbf{R}^n} |f_j(y)||a(y)-a_{A_j}|\, dy \right)^p \right)^{1/p} \\
\equiv& I_{11} + I_{12} + I_{13}.
\end{aligned}$$

Noting that $\alpha \le n(1-1/q)+\lambda$, by Hölder's inequality and the John-Nirenberg' lemma on $BMO(\mathbf{R}^n)$ functions, we have

$$I_{11} \le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} 2^{-nk} 2^{nk/q} \|a\|_* 2^{jn(1-1/q)} \|f_j\|_{L^q(\mathbf{R}^n)} \right) \right)^{1/p}$$

$$
\begin{aligned}
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{n(j-k)(1-1/q)}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\right.\right.\\
&\qquad\left.\left.(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|^p_{L^q(\mathbf{R}^n)})^{1/p}\right)^p\right)^{1/p}\\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{n(j-k)(1-1/q)}2^{-j\alpha}2^{j\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n(1-1/q)+\lambda-\alpha)}\right)^p\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\\
&\le C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}.
\end{aligned}
$$

It is easy to see that $|a_{A_j}-a_{A_k}|\le C(k-j)\|a\|_*$. Therefore, similar to I_{11}, we have

$$
\begin{aligned}
I_{12} &\le C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{-nk}2^{nk/q}(k-j)\|a\|_* 2^{jn(1-1/q)}\|f_j\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} (k-j)2^{n(j-k)(1-1/q)-j\alpha+j\lambda}2^{-j\lambda}\right.\right.\\
&\qquad\left.\left.(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|^p_{L^q(\mathbf{R}^n)})^{1/p}\right)^p\right)^{1/p}\\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} (k-j)2^{n(j-k)(1-1/q)}2^{-j\alpha}2^{j\lambda}\right.\right.\\
&\qquad\left.\left.\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\sum_{j=-\infty}^{k-2} (k-j)2^{(j-k)(n(1-1/q)+\lambda-\alpha)}\right)^p\right)^{1/p}\\
&\qquad\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\\
&\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\\
&\le C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}.
\end{aligned}
$$

For I_{13}, also by Hölder's inequality and the John-Nirenberg' lemma on $BMO(\mathbf{R}^n)$ functions, we have

$$\begin{aligned}I_{13} \le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-2} 2^{-nk}2^{nk/q}\|(a(y)-a_{A_j})\mathcal{X}_j(y)\|_{L^{q'}(\mathbf{R}^n)} \\ \|f_j\|_{L^q(\mathbf{R}^n)}\Big)^p\Big)^{1/p} \\ \le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{-nk+nk/q}2^{jn(1-1/q)}\|a\|_*\|f_j\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p},\end{aligned}$$

Thus, with a similar argument to I_{11}, we obtain that

$$I_{13} \le C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}.$$

Now let us turn to estimate for I_3. Note that when $x\in A_k$, $y\in A_j$, and $j\ge k+2$, $|x-y|\sim|y|$. Therefore, by the condition (2.5) we have

$$\begin{aligned}|[a,T]f_j(x)| \le & C\int_{\mathbf{R}^n}\frac{|a(x)-a(y)|}{|x-y|^n}|f_j(y)|\,dy \\ \le & C2^{-nj}|a(x)-a_{A_k}|\int_{\mathbf{R}^n}|f_j(y)|\,dy+ \\ & + C2^{-nj}|a_{A_j}-a_{A_k}|\int_{\mathbf{R}^n}|f_j(y)|\,dy + C2^{-nj}\int_{\mathbf{R}^n}|a(y)-a_{A_j}||f_j(y)|\,dy.\end{aligned}$$

Thus,we have

$$\begin{aligned}I_3 \le & C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}\|(a(x)-a_{A_k})\mathcal{X}_k(x)\|_{L^q(\mathbf{R}^n)}\right.\right. \\ & \left.\left.\int_{\mathbf{R}^n}|f_j(y)|\mathcal{X}_j(y)\,dy\right)^p\right)^{1/p} \\ & + C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}\|(a_{A_j}-a_{A_k})\mathcal{X}_k(x)\|_{L^q(\mathbf{R}^n)}\right.\right. \\ & \left.\left.\int_{\mathbf{R}^n}|f_j(y)|\mathcal{X}_j(y)\,dy\right)^p\right)^{1/p} \\ & + C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}2^{nk/q}\int_{\mathbf{R}^n}|f_j(y)||a(y)-a_{A_j}|\,dy\right)^p\right)^{1/p} \\ & \equiv I_{31}+I_{32}+I_{33}.\end{aligned}$$

Noting that $\alpha \geq -n/q+\lambda$, by Hölder's inequality and the John-Nirenberg' lemma on $BMO(\mathbf{R}^n)$ functions, we have

$$\begin{aligned}
I_{31} &\leq C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}\|(a(x)-a_{A_k})\mathcal{X}_k(x)\|_{L^q(\mathbf{R}^n)}2^{jn(1-1/q)}\right.\right.\\
&\qquad\left.\left.\|f_j\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\leq C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}2^{jn(1-1/q)}2^{kn/q}\|a\|_*\|f_j\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\leq C\|a\|_*\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{n(k-j)/q}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\right.\right.\\
&\qquad\left.\left.\Big(\sum_{l=-\infty}^{j}2^{l\alpha p}\|f_l\|^p_{L^q(\mathbf{R}^n)}\Big)^{1/p}\right)^p\right)^{1/p}\\
&\leq C\|a\|_*\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{n(k-j)/q}2^{-j\alpha}2^{j\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\leq C\|a\|_*\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n/q-\lambda+\alpha)}\right)^p\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\\
&\leq C\|a\|_*\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\\
&\leq C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}.
\end{aligned}$$

For I_{32}, by the fact of $|a_{A_j}-a_{A_k}|\leq C(j-k)\|a\|_*$, therefore, similar to I_{31} we have

$$\begin{aligned}
I_{32} &\leq C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}2^{jn(1-1/q)}2^{kn/q}(j-k)\|a\|_*\|f_j\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}\\
&\leq C\|a\|_*\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} (j-k)2^{n(k-j)/q}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\right.\right.\\
&\qquad\left.\left.\Big(\sum_{l=-\infty}^{j}2^{l\alpha p}\|f_l\|^p_{L^q(\mathbf{R}^n)}\Big)^{1/p}\right)^p\right)^{1/p}\\
&\leq C\|a\|_*\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} (j-k)2^{n(k-j)/q}2^{-j\alpha}2^{j\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}\right)^p\right)^{1/p}
\end{aligned}$$

$$\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\sum_{j=k+2}^{\infty}(j-k)2^{(k-j)(n/q-\lambda+\alpha)}\right)^p\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}$$

$$\le C\|a\|_* \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}$$

$$\le C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}.$$

Finally, we turn to estimate for I_{33}. We have

$$I_{33} \le C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}2^{nk/q}\|(a(y)-a_{A_j})\mathcal{X}_j(y)\|_{L^{q'}(\mathbf{R}^n)}\right.\right.$$
$$\left.\left.\|f_j\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}$$
$$\le C\sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}2^{kn/q}2^{jn(1-1/q)}\|a\|_*\|f_j\|_{L^q(\mathbf{R}^n)}\right)^p\right)^{1/p}$$

Hence, with a similar argument to I_{31}, we obtain that

$$I_{33} \le C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}.$$

This finishes the proof of Theorem 2.1 .

The condition (2.5) in Theorem 2.1 can be satisfied by many operators such as Bochner-Riesz operators at the critical index, Ricci-Stein's oscillatory singular integrals, Fefferman's multipler, and the C-Z operators. From this theorem and Theorem 2.7 and 2.10 in [5], we easily deduce the following corollary.

Corollary 2.2 *Let $0<p<\infty$, $1<q<\infty$, $\lambda>0$, $-n/q+\lambda\le\alpha\le n(1-1/q)+\lambda$, and $a\in BMO(\mathbf{R}^n)$. If k is a constant or a variable C-Z kernel on $\mathbf{R}^n$ and T is the corresponding C-Z operator, then there exists a constant $C=C(n,p,q,\alpha,\lambda,k)$ or $C=C(n,p,q,\alpha,\lambda,k,M)$ such that for all $f\in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)$,*

$$\|[a,T]f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)} \le C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n)}. \tag{2.6}$$

From this corollary, by a procedure similar to that of Theorem 2.11 in [5], we obtain the following corollary.

Corollary 2.3 *Let $0<p<\infty$, $1<q<\infty$, $\lambda>0$, and $-n/q+\lambda\le\alpha\le n(1-1/q)+\lambda$. Suppose Ω be an open set of $\mathbf{R}^n$ and $a\in BMO(\Omega)$. If k is a variable C-Z kernel on Ω and T is the corresponding C-Z operator, then there exists a constant $C=C(n,p,q,\alpha,\lambda,k,M)$ such that for all $f\in M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$,*

$$\|[a,T]f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} \le C\|a\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)}. \tag{2.7}$$

With a similar proof to that of Theorem 2.13 in [4], we can also have the following local version of Corollary 2.3.

Corollary 2.4 *Let $0 < p < \infty$, $1 < q < \infty$, $\lambda > 0$, and $-n/q + \lambda \le \alpha \le n(1-1/q) + \lambda$. Suppose Ω be an open set of $\mathbf{R}^n$ and $a \in VMO(\Omega)$. If k is a variable C-Z kernel on Ω and T is the corresponding C-Z operator, then for any $\varepsilon > 0$, there exists a positive number $\rho_0 = \rho_0(\varepsilon, \eta)$ such that for any ball $B(0,r)$ with the radius $r \in (0, \rho_0)$, $B(0,r) \cap \Omega \equiv \Omega_r \neq \phi$ and all $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega_r)$,*

$$\|[a,T]f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega_r)} \le C\varepsilon \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega_r)}, \tag{2.8}$$

where $C = C(n,p,q,\alpha,\lambda,k,a)$ is independent of ε, f, and r.

In order to establish the regularity estimates (1.3), we still need to study the boundedness, on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)$, of some other integral operators and linear commutators, where $\mathbf{R}^n_+ = \{x = (x', x_n) : x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\ x_n > 0\}$.

Theorem 2.2 *Let $0 < p < \infty$, $1 < q < \infty$, $\lambda > 0$, $-n/q + \lambda \le \alpha \le n(1-1/q) + \lambda$ and $\tilde{x} = (x', -x_n)$ with $x = (x', x_n)$. If a sublinear operator T satisfies*

$$|T(f)(x)| \le C \int_{\mathbf{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy,\ x \in \mathbf{R}^n_+ \tag{2.9}$$

for any $f \in L^1(\mathbf{R}^n_+)$ with compact support and T is bounded on $L^q(\mathbf{R}^n_+)$, then T is also bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)$.

Proof Let $A^+_k = (B_k \setminus B_{k-1}) \cap \mathbf{R}^n_+$ and $\mathcal{X}^+_k = \mathcal{X}_{A^+_k}$. Let $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)$, we write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\mathcal{X}^+_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x),\ x \in \mathbf{R}^n_+$$

and

$$\begin{aligned}
\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)} &= \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|Tf\mathcal{X}^+_k\|^p_{L^q(\mathbf{R}^n_+)} \right)^{1/p} \\
&\le C \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} \|Tf_j\mathcal{X}^+_k\|_{L^q(\mathbf{R}^n_+)} \right)^p \right)^{1/p} \\
&\le C \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \sum_{j=k-1}^{k+1} \|Tf_j\mathcal{X}^+_k\|^p_{L^q(\mathbf{R}^n_+)} \right)^{1/p} \\
&\le C \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{j=k+2}^{\infty} \|Tf_j\mathcal{X}^+_k\|_{L^q(\mathbf{R}^n_+)} \right)^p \right)^{1/p} \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

For I_2, by the $L^q(\mathbf{R}^n_+)$-boundedness of T, we have

$$I_2 \le C \sup_{k_0 \in \mathbf{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \sum_{j=k-1}^{k+1} \|f_j\mathcal{X}^+_k\|^p_{L^q(\mathbf{R}^n_+)} \right)^{1/p}$$

$$\le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\mathcal{X}_k^+\|^p_{L^q(\mathbf{R}^n_+)}\right)^{1/p}$$
$$= C\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}.$$

Thus, the desired estimate is obtained. For I_1, note that when $x=(x',x_n)\in A_k^+$, $y\in A_j^+$, and $j\le k-2$, $|\tilde x-y|\sim|x|$ with $\tilde x=(x',-x_n)$. Therefore, noticing that $\alpha\le n(1-1/q)+\lambda$, by the condition (2.9) and the Hölder's inequality , we have

$$I_1 \le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{-nk}2^{nk/q}\int_{\mathbf{R}^n_+}|f_j(y)|\,dy\right)^p\right)^{1/p}$$
$$\le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{-nk}2^{nk/q}2^{jn(1-1/q)}\|f_j\|_{L^q(\mathbf{R}^n_+)}\right)^p\right)^{1/p}$$
$$\le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{n(j-k)(1-1/q)}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\right.\right.$$
$$\left.\left.(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|^p_{L^q(\mathbf{R}^n_+)})^{1/p}\right)^p\right)^{1/p}$$
$$\le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} 2^{n(j-k)(1-1/q)}2^{-j\alpha}2^{j\lambda}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}\right)^p\right)^{1/p}$$
$$\le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n(1-1/q)+\lambda-\alpha)}\right)^p\right)^{1/p}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}$$
$$\le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right)^{1/p}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}$$
$$\le C\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}.$$

Finally, let's turn to estimate for I_3. As $x=(x',x_n)\in A_k^+$, $y\in A_j^+$, and $j\ge k+2$, $|\tilde x-y|\sim|y|$ with $\tilde x=(x',-x_n)$. Thus, noticing that $\alpha\ge -n/q+\lambda$, by the condition (2.9) and the Hölder's inequality , similar to I_1 we have

$$I_3 \le C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}2^{nk/q}\int_{\mathbf{R}^n_+}|f_j(y)|\,dy\right)^p\right)^{1/p}$$

$$\leq C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{-nj}2^{nk/q}2^{jn(1-1/q)}\|f_j\|_{L^q(\mathbf{R}^n_+)}\right)^p\right)^{1/p}$$

$$\leq C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{n(k-j)/q}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\right.\right.$$

$$\left.\left.(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|^p_{L^q(\mathbf{R}^n_+)})^{1/p}\right)^p\right)^{1/p}$$

$$\leq C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty} 2^{n(k-j)/q}2^{-j\alpha}2^{j\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}\right)^p\right)^{1/p}$$

$$\leq C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\left(\sum_{j=k+2}^{\infty} 2^{(k-j)(n/q-\lambda+\alpha)}\right)^p\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}$$

$$\leq C \sup_{k_0\in\mathbf{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}$$

$$\leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)}.$$

This finishes the the proof of Theorem 2.2 .

By Theorem 2.2, similar to Lemma 3.1 in [6], we have

Corollary 2.5 *Let* $0<p<\infty,\ 1<q<\infty,\ \lambda>0,\ -n/q+\lambda\leq\alpha\leq n(1-1/q)+\lambda$ *and* Ω *be an open set of* $\mathbf{R}^n_+$*. If* k *is a variable C-Z kernel on* Ω *and for* $x\in\Omega$ *we define*

$$\tilde{T}f(x)=\int_\Omega k(x,T(x)-y)\,dy,$$

where

$$T(x,y)\equiv x-\frac{2x_n}{a_{nn}(y)}a(y),\ T(x)\equiv T(x,x),\ \text{and } a(x)=\{a_{in}(x)\}_{i=1}^n \text{ as in (1.1)},$$

then there exists a constant $C=C(n,p,q,\alpha,\lambda,k,M)$ *such that for all* $f\in M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$,

$$\|\widetilde{T}f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)}\leq C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)}. \tag{2.10}$$

For the linear commutator on $\mathbf{R}^n_+$, we have

Theorem 2.3 *Let* $0<p<\infty,\ 1<q<\infty,\ \lambda>0,\ -n/q+\lambda\leq\alpha\leq n(1-1/q)+\lambda$ *and* $a\in BMO(\mathbf{R}^n_+)$*. If a linear operator* T *satisfies* (2.9) *in Theorem* 2.2 *and* $[a,T]$ *is bounded on* $L^q(\mathbf{R}^n_+)$*, then* $[a,T]$ *is also bounded on* $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbf{R}^n_+)$.

The proof of Theorem 2.3 is similar to that of Theorem 1.1 . Thus, we omit the details here. We can also obtain the following local version of Theorem 2.3; see Theorem 2.13 in [5].

Corollary 2.6 *Let $0 < p < \infty$, $1 < q < \infty$, $\lambda > 0$, and $-n/q + \lambda \le \alpha \le n(1-1/q) + \lambda$. Let $a \in VMO(\mathbf{R}^n_+)$ and η be its VMO modulus. If k is a variable C-Z kernel on $\mathbf{R}^n_+$ and $\widetilde{T}$ is the same as in Corollary 2.5, then for any $\varepsilon > 0$, there exists a positive number $\rho_0 = \rho_0(\varepsilon, \eta)$ such that for any ball $B^+(0,r) \equiv B(0,r) \cap \mathbf{R}^n_+$ with the radius $r \in (0, \rho_0)$ and all $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(B^+(0,r))$,*

$$\|[a, \widetilde{T}]f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(B^+(0,r))} \le C\varepsilon \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(B^+(0,r))}, \tag{2.11}$$

where $C = C(n, p, q, \alpha, \lambda, k, a)$ is independent of ε, f, and r .

3. Regularity of Solutions to Elliptic Equations with VMO Coefficients

In this section we shall establish the global regularity of the strong solutions to the Dirichlet problem (1.1) in homogeneous Morrey-Herz spaces by applying the estimates about singular integral operators and linear commutators obtained in the above section.

Definition 3.1 *Let $\alpha \in \mathbf{R}, 0 < p \le \infty, 0 < q < \infty$, and $\lambda \ge 0$. Let Ω be an open subset of $\mathbf{R}^n$. $u \in L^1_{loc}(\Omega)$ is said to be the homogeneous Sobolev-Morrey-Herz spaces $W^2 M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$ if and only if u and its distributional derivatives, u_{x_i}, $u_{x_i x_j}$ $(i, j = 1, 2, \cdots, n)$ are in $M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$. Moreover, $\|\cdot\|_{W^2 M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)}$ is given by*

$$\|u\|_{W^2 M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} \equiv \|u\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)} + \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)}. \tag{3.1}$$

Now, let Ω be an open bounded subset of $\mathbf{R}^n$ with $n \ge 3$ and $\partial\Omega \in C^{1,1}$. Let $a_{ij}(x) \in VMO(\Omega)$ and

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

with a_{ij}'s satisfying (1.2). Without loss of generality, we may assume that a_{ij}'s belong to $VMO(\mathbf{R}^n)$ (see [8]). Let $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$, where $0 < p < \infty$, $1 < q < \infty$, $\lambda > 0$, and $-n/q + \lambda \le \alpha \le n(1-1/q) + \lambda$. We are interested in the following Dirichlet problem:

$$\begin{cases} Lu = f & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Note that Ω is bounded, then $f \in M\dot{K}^{\alpha,\lambda}_{p,q}(\Omega)$ implies $f \in L^q(\Omega)$. Therefore, by the results in [6], we see that the Dirichlet problem (3.2) has a unique solution $u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ satisfying the following estimate:

$$\|u\|_{W^{2,q}(\Omega)} \le C\|f\|_{L^q(\Omega)}, \tag{3.3}$$

where constant C is independent of f. Fan, Lu, and Yang established the above global regularity in the Morrey spaces $M_q^{\lambda}(\Omega)$ (see [Theorem 3.2, 7]). Owing to the estimates about singular integral operators and linear commutators obtained in Section 2, we can improve (3.3) in the homogeneous Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbf{R}^n)$. More precisely, we have

Theorem 3.1 *Let L satisfy the above assumption, $0 < p < \infty$, $1 < q_1 \le q < \infty$, $\lambda > 0$, $-n/q+\lambda \le \alpha \le n(1-1/q)+\lambda$, and $\eta = (\sum_{i,j=1}^n \eta_{i,j}^2)^{1/2}$, where $\eta_{i,j}$ is the VMO modulus of $a_{i,j}$. Let $f \in M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega), u \in W^2 M\dot{K}_{p,q_1}^{\alpha,\lambda}(\Omega) \cap W_0^{1,q_1}(\Omega)$, and $Lu = f$ a.e. in Ω. Then $u \in W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$ and there exists a constant $C = C(n,p,q,\alpha,\lambda,\Lambda,\partial\Omega,\eta)$ such that*

$$\|u\|_{W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)} \le C\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)}.$$

By a standard procedure, the proof of Theorem 3.1 consists in establishing the interior and boundary estimates of solution to (3.2). Indeed, by a similar proof to Theorem 4.2 in [5], Theorem 3.3 in [3], and Theorem 3.1 in [9], we can prove the following interior estimate.

Theorem 3.2 *Let L, p, q_1, q, λ, and η be as in Theorem 3.1. Let $f \in M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$, $u \in W^2 M\dot{K}_{p,q_1}^{\alpha,\lambda}(\Omega) \cap W_0^{1,q_1}(\Omega)$, and $Lu = f$ a.e. in Ω. Then $u \in W_{loc}^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$. Moreover, given any $\Omega' \subset\subset \Omega$, there exists a constant $C = C(n,p,q,\alpha,\lambda,\Lambda,dist(\partial\Omega',\Omega),\eta)$ such that*

$$\|u\|_{W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega')} \le C\{\|u\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)} + \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)}\}.$$

See [3] or [9] for the proof of Theorem 3.2, and we omit the details here. To finish the proof of Theorem 3.1, we still need to establish the following boundary estimate.

Theorem 3.3 *Let L, p, q_1, q, λ, and η be as in Theorem 3.1. Let $f \in M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$, $u \in W^2 M\dot{K}_{p,q_1}^{\alpha,\lambda}(\Omega) \cap W_0^{1,q_1}(\Omega)$, and $Lu = f$ a.e. in Ω. Then $u \in W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)$ and there exists a constant $C = C(n,p,q,\alpha,\lambda,\Lambda,\partial\Omega,\eta)$ such that*

$$\|u\|_{W^2 M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)} \le C\{\|u\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)} + \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\Omega)}\}. \tag{3.4}$$

To prove Theorem 3.3, we need to introduce more notation. Let $B_\sigma = \{x = (x', x_n) : x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbf{R}^{n-1},\ x_n > 0,\ |x| < \sigma\}$. Define $W_{\gamma_0}^{2,q}(B_\sigma)$ to be the closure in $W^{2,q}$ of the subspace
$C_{\gamma_0} = \{u : u$ is the restriction to B_σ of a function belonging to $C_0^\infty(B_\sigma),\ u(x', 0) = 0\}$.
We will make the following assumption:

$$\begin{cases} \text{Let } n \ge 3, \quad b_{i,j} \in VMO,\ i,j = 1,2,\cdots,n, \\ b_{i,j} = b_{j,i}, \quad i,j = 1,2,\cdots,n, \text{ a.e. in } B_\sigma, \\ \tilde{L} = \displaystyle\sum_{i,j=1}^n b_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \\ \exists\Lambda > 0 :\ \Lambda^{-1}|\xi|^2 \le \displaystyle\sum_{i,j=1}^n b_{i,j}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ a.e. in } B_\sigma, \forall\xi \in \mathbf{R}^n. \end{cases} \tag{3.5}$$

Let

$$\Gamma(x,t)=\frac{1}{(n-2)\omega_n(\det b_{i,j})^{1/2}}\left(\sum_{i,j=1}^{n}B_{i,j}(x)t_it_j\right)^{(2-n)/2},$$

$$\Gamma_i(x,t)=\frac{\partial}{\partial t_i}\Gamma(x,t),\quad \Gamma_{ij}(x,t)=\frac{\partial^2}{\partial t_i\partial t_j}\Gamma(x,t),$$

for a.e. $x\in B_\sigma$ and $\forall t\in \mathbf{R}^n\backslash\{0\}$, where the $B_{i,j}$ are the entries of the inverse of the matrix $(b_{i,j})_{i,j=1,\cdots,n}$. We also set

$$b(x)=(b_{in}(x))_{i,j=1,\cdots,n},\quad T(x,y)=x-\frac{2x_n}{b_{nn}(y)}b(y),$$

$T(x)=T(x,x)$, and $B(y)=T(e_n,y)$, where, as usual, $e_n=(0,0,\cdots,1)$. We have the following key lemma established in [6, Theorem 3.2].

Lemma 3.1 *Assume* (3.5). *Let* $u\in W^{2,q}_{\gamma_0}(B_\sigma)$ *with* $1<q<\infty$. *Then*

$$u_{x_ix_j}(x)=P.V.\int_{B_\sigma}\Gamma_{ij}(x,x-y)\left\{\sum_{h,k=1}^{n}(b_{hk}(x)-b_{hk}(y))u_{x_hx_k}(y)+\tilde{L}u(y)\right\}dy$$
$$+\tilde{L}u(x)\int_{|t|=1}\Gamma_i(x,t)t_j\,d\sigma_t+I_{ij}(x), \tag{3.6}$$

where for $i,j=1,2,\cdots,n-1$,

$$I_{ij}(x)=\int_{B_\sigma}\Gamma_{ij}(x,T(x)-y)\left\{\sum_{h,k=1}^{n}(b_{hk}(x)-b_{hk}(y))u_{x_hx_k}(y)+\tilde{L}u(y)\right\}dy;$$

for $i=1,2,\cdots,n-1$,

$$I_{in}(x)=I_{ni}(x)=\int_{B_\sigma}\left(\sum_{j=1}^{n}\Gamma_{ij}(x,T(x)-y)B_j(x)\right)\{\cdots\}\,dy;$$

and

$$I_{nn}(x)=\int_{B_\sigma}\left(\sum_{i,j=1}^{n}\Gamma_{ij}(x,T(x)-y)B_i(x)B_j(x)\right)\{\cdots\}\,dy;$$

in the formulas above $B_i(x)$ *is the i-th component of the vector* $B(x)$ *and in the curly brackets there is always the same expression as in the first case.*

By a covering and flattening argument, to finish the proof of Theorem 3.3, we only need to prove the following theorem.

Theorem 3.4 *Assume* (3.5). *Let L satisfy the above assumption, $0 < p < \infty$, $1 < q_1 \le q < \infty$, $\lambda > 0$, $-n/q + \lambda \le \alpha \le n(1-1/q) + \lambda$. Set $\tilde{\eta} = (\sum_{i,j=1}^n \tilde{\eta}_{i,j}^2)^{1/2}$, where $\tilde{\eta}_{i,j}$ is the VMO modulus of $b_{i,j}$, and*

$$M \equiv \max_{i,j=1,\cdots,n} \max_{|\beta| \le 2n} \left\| \frac{\partial^\beta}{\partial t^\beta} \Gamma_{ij}(x,t) \right\|_{L^\infty(B_\sigma \times S^{n-1})}.$$

Then there exists a positive number $\rho_0 = \rho_0(n,p,q,\alpha,\lambda,\Lambda,M,\tilde{\eta})$, $\rho_0 < \sigma$, such that for any $r \in (0,\rho_0)$ and any $u \in W^{2,q_1}_{\gamma_0}(B_r)$ with $\tilde{L}u \in M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)$, we have $u \in W^2 M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)$. Furthermore, there exists a constant $C = C(n,p,q,\alpha,\lambda,\Lambda,M,\tilde{\eta})$ such that

$$\|u_{x_ix_j}\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)} \le C \|\tilde{L}u\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)}. \tag{3.7}$$

Proof Let $0 < r \le \sigma$, $q_1 \le s \le q$, and $f \in M\dot{K}^{\alpha,\lambda}_{p,s}(B_r)$. Set for $i,j,h,k = 1,2,\cdots,n$,

$$S_{ijhk}(f)(x) = P.V.\int_{B_r} \Gamma_{ij}(x, x-y)(b_{hk}(x) - b_{hk}(y)) f(y)\, dy,$$

and for $i,j = 1,2,\cdots,n-1$, $h,k = 1,2,\cdots,n$,

$$\tilde{S}_{ijhk}(f)(x) = \int_{B_r} \Gamma_{ij}(x, T(x)-y)(b_{hk}(x) - b_{hk}(y)) f(y)\, dy,$$

for $i = 1,2,\cdots,n-1$, $h,k = 1,2,\cdots,n$,

$$\tilde{S}_{inhk}(f)(x) = \int_{B_r} \left(\sum_{j=1}^n \Gamma_{ij}(x, T(x)-y) B_j(x) \right) (b_{hk}(x) - b_{hk}(y)) f(y)\, dy,$$

and, finally, for $h,k = 1,2,\cdots,n$,

$$\tilde{S}_{nnhk}(f)(x) = \int_{B_r} \left(\sum_{i,j=1}^n \Gamma_{ij}(x, T(x)-y) B_i(x) B_j(x) \right) (b_{hk}(x) - b_{hk}(y)) f(y)\, dy.$$

By the estimates (2.8), (2.11), and the result in [6, Lemma 3.1], we can fix ρ_0 so small that

$$\sum_{i,j,h,k=1}^n \|S_{ijhk} + \tilde{S}_{ijhk}\| < 1,$$

where the norm of the operators $S_{ijhk} + \tilde{S}_{ijhk}$ is the norm in the space of linear operators from $M\dot{K}^{\alpha,\lambda}_{p,s}(B_r)$ into itself if $0 < r < \rho_0$, $q_1 \le s \le q$.

Consider $u \in W^{2,q}_{\gamma_0}(B_r)$ with $\tilde{L}u \in M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)$, $0 < r < \rho_0$, and set

$$h_{ij} = P.V.\int_{B_r} \Gamma_{ij}(x, x-y) \tilde{L}u(y)\, dy + \tilde{L}u(x) \int_{|t|=1} \Gamma_i(x,t) t_j\, d\sigma_t + \tilde{I}_{ij}(x),$$

where

$$\tilde{I}_{ij}(x)=\begin{cases}\displaystyle\int_{B_r}\Gamma_{ij}(x,T(x)-y)\tilde{L}u(y)\,dy, \text{ for } i,j=1,2,\cdots,n-1,\\ \displaystyle\int_{B_r}\left(\sum_{l=1}^{n}\Gamma_{il}(x,T(x)-y)B_l(x)\right)\tilde{L}u(y)\,dy, \text{ for } i=1,2,\cdots,n-1, j=n,\\ \displaystyle\int_{B_r}\left(\sum_{l,m=1}^{n}\Gamma_{lm}(x,T(x)-y)B_l(x)B_m(x)\right)\tilde{L}u(y)\,dy, \text{ for } i=j=n.\end{cases}$$

By the estimate (2.10), we can obtain that $h_{ij}\in M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)$.

Now, consider $w\in[M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)]^{n^2}$ and define $Tw:[M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)]^{n^2}\longrightarrow[M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)]^{n^2}$ by setting

$$Tw=(Tw)_{ij})_{i,j=1,\cdots,n}\equiv\left(\sum_{h,k=1}^{n}(S_{ijhk}+\tilde{S}_{ijhk})+h_{ij}\right)_{i,j=1,\cdots,n}.$$

Then the operator T is a contraction on $[M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)]^{n^2}$ and thus has a unique fixed point $\tilde{w}$. Since, by Lemma 3.1, $\{u_{x_ix_j}\}_{i,j=1,\cdots,n}$ is also a fixed point in $[M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)]^{n^2}$ and $M\dot{K}^{\alpha,\lambda}_{p,q}(B_r)\subseteq M\dot{K}^{\alpha,\lambda}_{p,q_1}(B_r)$ if $q_1\le q$, the uniqueness of the fixed point implies

$$u_{x_ix_j}=\tilde{w}_{ij}\in M\dot{K}^{\alpha,\lambda}_{p,q}(B_r),\quad\forall\, i,j=1,2,\cdots,n.$$

Then the estimate (3.7) can be deduced from Corollaries 2.1, 2.3, 2.5, 2.6, and Lemma 3.1 .

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