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RICCI FLOW ON SURFACES WITH DEGENERATE INITIAL METRICS

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Abstract It is proved that given a conformal metric $e^{u_0}g_0$, with $e^{u_0} \in L^\infty$, on a 2-dim closed Riemannian manfold (M, g_0), there exists a unique smooth solution $u(t)$ of the Ricci flow such that $u(t) \to u_0$ in L^2 as $t \to 0$.

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In Kähler geometry, one of the central problems is the existence problem of the Calabi's extremal Kähler metric ([1],[2]). The well-known Kähler-Einstein metrics are special extremal metrics for which the existence problem can be reduced to a second order complex Monge-Ampere equation. For the general extremal metric, the equation is either 4th or 6th order depending on whether the scalar curvature function is constant. However, there is a variational structure associated to the extremal metric, which, according to E. Calabi[2], is local minimizer of the so called "Calabi enegy" (i.e., the L^2 norm of the scalar curvature). More recently, S. K. Donaldson[3] and the first named author[4] proved that the extremal Kähler metric is the global minimizer of the Calabi energy respectively. A similar result holds with the Calabi energy replaced by the K-energy ([5]), see[3] and [4].

To attack the existence problem, we may pick a minimizing sequence of Kähler metrics for the K energy or the Calabi energy. Suppose that the Kähler potentials of this sequence converges in weak $C^{1,1}$ topology to a limit $C^{1,1}$ Kähler potential. (In general we do not know if such convergence should be true.) Then, we face the problem that the limit Kähler metric might be degenerated. It is a conjecture of the first named

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author that any $C^{1,1}$ minimizer of the K-energy functional must be smooth. The proof of the conjecture seems hopeless with traditional approaches. The conjecture can be proved if the Kähler Ricci flow can be initiated for any $C^{1,1}$ Kähler potentials and is smooth for positive time.

For general closed Kähler manifolds with positive first Chern class, Chen-Tian [6] is able to construct a weak Kähler-Ricci flow initiated from any L^∞ Kähler metric with $C^{1,1}$ potential. They proved that

1. the Kähler potential under the flow remains uniformly bounded in $C^{1,1}$;
2. the volume form of the weak flow converges strongly in L^2 to the volume of the initial Kähler metric as $t \to 0$.

It is natural to ask if the weak Kähler Ricci flow will become smooth immediately after $t > 0$? This question is answered positively in this note in the special case of Riemannian surface. In a forthcoming paper [7], the first named author and G. Tian prove this for all dimensions along with other results. The significance of such results is that, the original Kähler-Ricci flow defined in the open positive cone of Kähler potentials can actually be extended to the boundary of the cone; and the flow initiated at the boundary will leave the boundary right away and enter into the interior of the cone. Thus, the extended (weak) flow not only preserves the cone structure but also has a wonderful effect of regularizing a weak Kähler metric.

The proof in this note is quite different from that in [7], especially the "regularization" part. We believe the techniques used in our proof here may be useful in other nonlinear geometric problems, such as the yamabe flow, the σ_k problems in conformal geometry and the Minkonski problems for convex hypersurfaces in Euclidean spaces.

For convenience of our presentation, we concentreate on the case of S^2. However, the result holds for any closed surfaces. The main theorem in this paper is:

Theorem 1 *Let g_0 be a smooth Riemannian metric on S^2 and $e^{u_0} \in L^\infty(S^2)$ such that*

$$\int_{S^2} e^{u_0} = \int_{S^2} \mathrm{d}A_0.$$

where dA_0 denotes the area element of g_0. Then the Ricci flow with initial metric $g_0' = e^{u_0} g_0$ has a unique solution $g(t) = e^{u(t)} g_0$ such that $u(t)$ is smooth for $t > 0$ and $e^{u(t)} \to e^{u_0}$ in L^2 as $t \to 0$.

Recall that if $g = e^u g_0$ (with smooth function u) and R_0 is the Gaussian curvature of g_0, then the Gaussian curvature R of g given by

$$K = e^{-u}(K_0 - \frac{1}{2}\Delta u),$$

where Δ denotes the Laplacian with respect to the metric g_0. The Ricci flow equation for g can be reduced to an equation for u as follows

$$\frac{\partial u}{\partial t} = \bar{K} - K(u(t) \tag{1}$$

where $\bar{K}$ denotes the mean value of K, i.e.

$$\bar{K} = \frac{1}{A(t)} \int K(u(t))e^{u(t)},$$

where $A(t)$ is the area for $g(t) = e^{u(t)}g_0$ given by

$$A(t) = \int e^{u(t)}.$$

It is easy to check that $A(t)$ is a constant independent of t, and we may assume that the area equals 4π. Then it follows from the Gauss-Bonnet theorem and that the Euler characteristic of S^2 is 2, that $\bar{K} \equiv 1$. Thus, equation (1) can be rewritten as

$$\frac{\partial u}{\partial t} = 1 - K(u(t))$$

or equivalently

$$\frac{\partial u}{\partial t} = e^{-u}\left(\frac{1}{2}\Delta u - K_0\right) + 1. \tag{2}$$

To prove Theorem 1, we approximate the non-smooth initial function u_0 by a one parameter family of smooth functions $u_0(s)$, $s \in (0, 1]$ such that

$$\lim_{s\to 0} u_0(s) = u_0 \quad \text{strongly in } L^\infty.$$

We will denote by $u(s, t)$ the solution of (2) with initial value $u(s, 0) = u_0(s)$. It is known that $u(s, t)$ $(s > 0)$ exists for all time $t > 0$.

Lemma 2 *There exists a constant $C > 0$ such that*

$$u(s, t) \leq t + C.$$

Proof By a direct computation using equation (2), we have for $u = u(s, t)$

$$\left(\frac{\partial}{\partial t} - e^{-u}\triangle\right) e^u = e^u - K_0 - |\nabla u|^2.$$

By the maximum principle, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e^u\|_{L^\infty} \leq \|e^u\|_{L^\infty} + C.$$

This implies $e^u \leq Ce^t$, and the lemma follows.

Next, we continue our estimates assuming the so-called Liouville energy of $u(s, t)$ is uniformly bounded at some positive time. By definition, this energy is given by

$$E_1(u) = \int \left(\frac{1}{2}|\nabla u|^2 + K_0 u\right).$$

Lemma 3 *Suppose that for some constant* $C > 0$ *and all* $s \in (0, 1]$ *there exist* $t_1(s) > 0$ *such that*

$$E_1(u(s, t_1(s)) \le C.$$

Then for any $\delta > 0$ *there exists* $C(\delta) > 0$ *and for any* $B \ge t_1(s)$ *there exists* $t_2(s) \in [B, B + \delta]$ *such that*

$$\int (K(u(s, t_2(s)) - 1)^2 \, \mathrm{d}A(t_2(s)) \le C(\delta),$$

where $\mathrm{d}A(t) = e^{u(t)}\mathrm{d}A_0$.

Proof For any solution u of the Ricci flow, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_1(u) &= \int \frac{\partial u}{\partial t}\,(-\frac{1}{2}\Delta u + K_0)\,\mathrm{d}A_0 \\ &= \int (1 - K(u))K(u)\,\mathrm{d}A(t) = -\int (K(u) - 1)^2\,\mathrm{d}A(t). \end{aligned}$$

In the surface case, we know that E_1 has a lower bound $c_0 > -\infty$. Therefore,

$$\int_B^{B+\delta} \int (K(u) - 1)^2 \mathrm{d}A(t) \le C - c_0.$$

The lemma then follows from the mean value theorem and we may take

$$C(\delta) = \delta^{-1}(C - c_0).$$

Lemma 4 *Let* δ, B *and* $t_2(s)$ *be as in Lemma* 3. *Then there exists a constant* $C(B, \delta)$ *such that*

$$|u(s, t_2(s))| \le C(B, \delta).$$

Proof Note that we have the L^2-bound of curvature by Lemma 3. This together with the upper bound of the solution given by Lemma 1 implies that the L^∞ bounded, as proved in [1].

The next two lemmas show that the solutions $u(s, t)$ are actually smooth for $t > t_2(s)$, and we have uniform bounds for their derivatives of any order uniformly in s.

Lemma 5 *Let* δ, B *and* $t_2(s)$ *be as in Lemma* 3. *We have for* $t \in (t_2(s), T)$, *where* T *is any number greater than* $t_2(s)$,

$$|u(s, t)| \le C(B, \delta, T) \quad \forall\, t \in [B + \delta,\, T].$$

Proof We need to introduce two functions v and f defined by the following equations.

$$\Delta v = e^u - 1,$$

and

$$\Delta f = (1 - K(u))e^u.$$

The function v is actually the Kähler potential, and f the so-called Calabi-Futaki potential. We normalize these functions by requiring

$$\int v \, \mathrm{d}A_0 = \int f \, \mathrm{d}A_0 = 0.$$

Then we have (using equation (1))

$$\begin{aligned} \triangle \frac{\partial v}{\partial t} &= e^u \frac{\partial u}{\partial t} \\ &= (1 - K(u))e^u = \Delta f. \end{aligned}$$

On the other hand, we also have using equation (2)

$$\begin{aligned} \triangle \frac{\partial v}{\partial t} &= e^u - K_0 + \frac{1}{2}\Delta u \\ &= e^u - 1 + 1 - K_0 + \frac{1}{2}\Delta u \\ &= \Delta v + \Delta h + \frac{1}{2}\Delta u, \end{aligned}$$

where h is a solution to $\Delta h = 1 - K_0$ satisfying

$$\int h \, \mathrm{d}A_0 = -\frac{1}{2} \int u \, \mathrm{d}A_0.$$

Then we see

$$\Delta f = \Delta(v + h + \frac{1}{2}u)$$

and the normalizations for v, f and h implies that we must have

$$\frac{\partial v}{\partial t} = f = \frac{1}{2}u + v + h. \tag{3}$$

Differentiating (3) leads to

$$\frac{\partial f}{\partial t} = \frac{1}{2}e^{-u}\Delta f + f. \tag{4}$$

Note that $|h|$ is bounded by $\frac{1}{2}\|u\|_{L^\infty}$ plus a constant and $\|v\|_{L^\infty}$ is bounded via the bound of e^u given by Lemma 2. Therefore, we can get the L^∞ bound of u if only such bound has been established for f. However, (4) is a linear equation for f and we have (similar to Lemma 2)

$$\|f(s,t)\|_{L^\infty} \leq \|f(s, t_2(s))\|_{L^\infty} \, e^t \tag{5}$$

for $t > t_2(s)$. But at $t = t_2(s)$ we know $u(s, t_2(s)$ has a L^∞ bound by Lemma 4, and hence f is also bounded at $t_2(s)$. The desired bound then follows from (5).

Lemma 6 *We have for any integer $k \geq 1$*

$$\|u(s,t)\|_{C^k} \leq C(k, B, \delta, T) \quad \forall\, t \in [B+\delta,\, T].$$

Proof Let f be the Calabi-Futaki potential function as in Lemma 5. Using equation (5), we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{2}e^{-u}\Delta\right) f^2 &= 2f(\frac{1}{2}e^{-u}\Delta f + f) - e^{-u}(|\nabla f|^2 + f\Delta f)\\ &= 2f^2 - e^{-u}|\nabla f|^2. \end{aligned} \tag{6}$$

Next, we claim that

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}e^{-u}\Delta\right) e^{-u}|\nabla f|^2 = e^{-u}|\nabla f|^2 - e^{-2u}|\nabla\nabla f|^2. \tag{7}$$

It is convenient to compute using the metric $g_u = e^u g_0$. Then

$$|\nabla f|_u^2 = e^{-u}|\nabla f|^2, \quad \Delta_u = e^{-u}\Delta \ \text{ and } R_{ij}^u = K(u)(g_u)_{ij},$$

where R_{ij}^u is the Ricci curvature of g_u. We have (using (1) and (4))

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla f|_u^2 &= 2(\nabla\frac{\partial}{\partial t}f, \nabla f)_u - (1-K)|\nabla f|_u^2\\ &= (\nabla\triangle_u f, \nabla f)_u + |\nabla f|_u^2 + K(u)|\nabla f|_u^2.\end{aligned}$$

On the other hand, in a orthonormal frame of g_u, we have (using Ricci formula)

$$\begin{aligned}\frac{1}{2}\Delta_u|\nabla f|_u^2 &= \frac{1}{2}(f_k^2)_{ii}\\ &= f_{ki}f_{ki} + f_{kii}f_k = |\nabla\nabla f|_u^2 + f_{iik}f_k + R_{lj}^u f_l f_j\\ &= |\nabla\nabla f|_u^2 + (\nabla\triangle_u f, \nabla f)_u + K(u)|\nabla f|_u^2\end{aligned}$$

Combining the above two identities we get (7).

Now for any constant $\tau > 0$, set

$$F_\tau = f^2 + (t-\tau)e^{-u}|\nabla f|^2.$$

Then for $t \in (\tau, \tau+1$ we have by (6) and (7)

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{2}e^{-u}\Delta\right) F_\tau &= 2f^2 - e^{-u}|\nabla f|^2 + (t-\tau)(e^{-u}|\nabla f|^2 - e^{-2u}|\nabla\nabla f|^2)\\ &\leq 2f^2 - (1-t+\tau)e^{-u}|\nabla f|^2 \leq 2f^2 \leq C,\end{aligned}$$

where we have assumed f is bounded on $[\tau, tau+1]$, as asserted by (5). The inequality then implies we have

$$f^2 + (t-\tau)e^{-u}|\nabla f|^2 \leq C(t, \|f(\tau)\|_{L^\infty}).$$

Combining this with Lemma 5, we get the gradient estimate

$$|\nabla f| \leq C(1, B, \delta, T).$$

Now we see that the coefficient e^{-u} in equation (2) is C^1 bounded, so we may apply linear parabolic estimate to complete the proof of this lemma.

Next, we are going to show that the assumption on the boundedness of the E_1 energy used in the proof of Lemma 3 is unnecessary. Note first that by the definition of the function v we have actually

$$g_u = g_0 + \partial\bar{\partial}v.$$

So we may consider the K-energy $E_0(v)$ introduced by Mabuchi[5], which has a lower bound $E(v) \geq c_0$ in the case of complex dimension one. We also know that

$$\frac{\partial}{\partial t}E_0(v) = \int \frac{\partial v}{\partial t}(1 - K(u))e^u \mathrm{d}A_0, \tag{8}$$

and using the facts in the proof of Lemma 5, we then have

$$\frac{\partial}{\partial t}E_0(v) = \int f\Delta f = -\int |\nabla f|^2$$

Integrating this over time and using (3) we get for any $t > 0$

$$\int_0^t \int \left|\nabla(\frac{1}{2}u + v + h)\right|^2 \mathrm{d}A_0 \,\mathrm{d}t \leq E_0(v_0(s)) - c_0.$$

We note that the upper bound of $E_0(v_0(s))$ depends only $\|u_0(s)\|_{L^\infty}$. This can be checked as follows. Let $v_0(t) = v_0(s,t)$ be the Kähler potential for $e^{tu_0(s)}g_0$, and $f_0(t)$ be the Calabi-Futaki potential. Denote their derivative w.r.t. t by $\dot{v}_0$ and $\dot{f}_0$. Then

$$\Delta\dot{v}_0(t) = te^{tu_0}, \quad \Delta f_0(t) = \frac{t}{2}\Delta u_0 - K_0 + e^{tu_0}. \tag{9}$$

Then, similar to (8), we have

$$\begin{aligned} E(v_0) &= \int_0^1 \int \dot{v}_0(t)\Delta f_0(t)\,\mathrm{d}A_0\,\mathrm{d}t \\ &= \int_0^1 \int \Delta\dot{v}_0(t)\,f_0(t)\,\mathrm{d}A_0\,\mathrm{d}t = \int_0^1 \int te^{tu_0}\,f_0(t)\,\mathrm{d}A_0\,\mathrm{d}t \end{aligned}$$

Now, e^{u_0} is bounded; and from the second equation of (9) we see that $|f_0(t)|$ has a uniform bound. Therefore, the above integral has a bound depending only on $\|e^{u_0(s)}\|_{L^\infty}$. It follows that we can find $\tilde{t}(s) \in [0, B]$ such that

$$\int \left|\nabla(\frac{1}{2}u + v + h)\right|^2_{t=\tilde{t}(s)} \leq C(B).$$

Since it is easily seen from the defining equations that ∇v and ∇h are uniformly bounded in L^2, we get

$$\int |\nabla u|^2_{t=\tilde{t}(s)} \leq C(B).$$

Therefore, to prove $E_1(u(\tilde{t}(s)))$ is bounded above, we need only to show $\int K_0 u$ in the definition of E_1 has a upper bound. This is simple because

$$\int K_0 u = \int K_0(u - \bar{u}) + \int u,$$

where $\bar{u} = \frac{1}{4\pi}\int u$ is the mean value of u. The first integral is bounded since $u - \bar{u}$ is bounded in L^2 (by Poicarè's inequality), and for the second integral we have

$$\frac{1}{4\pi}\int u \leq \log\left(\frac{1}{4\pi}\int e^u\right) = 0.$$

Since E_1 decreases along the Ricci flow, we get the upper bound of $E_1(u(s,t))$ for $t > B$.

By Lemma 5, we may choose sequences $t_i \to 0$, $B_i, \delta_i \to 0$, and show there exists a subsequence $u_j = u(s_j)$ of $u(s)$ which converges in C^k for any $k \geq 1$ to a smooth solution u of the Ricci flow on $(0,\infty) \times S^2$. We still need to show this solution satisfies the initial condition, i.e. $u(0) = u_0$. This is shown by the following

Lemma 7 *The solution u we have constructed satisfies*

$$e^{u(t)} \to e^{u_0} \quad \text{in } L^2 \qquad \text{as } t \to 0.$$

Proof We have $u(s_j) \to u$ in C^k on $[\delta, 1] \times S^2$ for any $\delta > 0$. By Lemma 2, we know that e^u is uniformly bounded. So for any sequence $0 < t_k \to 0$ we may choose a sequence t_i such that

$$e^{u(t_i)} \to F \quad \text{weakly in } L^p \tag{10}$$

for some $F \in L^\infty$ and any $p > 1$. We may also choose a subsequence s_i of s_j such that

$$|u(s_i, t_i) - u(t_i)| \to 0$$

as $i \to \infty$. Then we have

$$e^{u(s_i,t_i)} \to F \quad \text{weakly in } L^p. \tag{11}$$

We want to show that $F = e^{u_0}$.

By equation (2) we have for $u = u(s,t)$ and any $\chi \in C^\infty$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int \chi e^{2u} &= \int \chi e^{u_i}(\Delta u - 2K_0 + 2e^u) \\ &= \int \chi(\Delta e^u - e^u|\nabla u|^2 - 2K_0 + e^u) \\ &\leq \int e^u \Delta\chi e^u + C \leq C. \end{aligned}$$

So we have

$$\int \chi e^{2u(s_i,t_i)} \le \int \chi e^{2u_0(s_i)} + Ct_i.$$

Letting $i \to \infty$, we see

$$\lim \int \chi e^{2u(s_i,t_i)} \le \int \chi e^{u_0}.$$

However,

$$\begin{aligned}\lim \int \chi e^{2u(s_i,t_i)} &= \lim \int \chi[\,(e^{u(s_i,t_i)} - F)^2 + 2Fe^{u(s_i,t_i)} - F^2\,] \\ &= \int \chi F^2 + \lim \int \chi (e^{u(s_i,t_i)} - F)^2 \ge \int \chi F^2.\end{aligned}$$

It follows that $\int \chi F^2 \le \int \chi e^{2u_0}$ for all $\chi \in C^\infty$, hence

$$F^2 \le e^{u_0} \qquad \text{almost everywhere.} \tag{12}$$

On the other hand, we have

$$4\pi = \int e^{u_0} = \int e^{u(s_i,t_i)} \to \int F$$

by (11). This combining with (12) implies $F = e^{u_0}$ almost everywhere, and hence the convergence in (11) is L^2-strong. Since the sequence t_i in (10) can be arbitrary, we see the lemma is true.

Finally, we prove the uniqueness of the solution we have constructed and complete the proof of Theorem 1. Assume that we have two solutions, u_1 and u_2. The corresponding Kähler potentials are denoted by v_1 and v_2, which satisfy at any fixed time $t > 0$

$$\Delta v_i = e^{u_i} - 1, \quad \text{or equivalently } \; u_i = \log(1 + \Delta v_i).$$

The normalization of v_i at $t = 0$ is determined by:

$$\int_M v_i = 0, \qquad \forall i \;=\; 1,\ 2.$$

Then the evolution equation(3) uniquely determined $v_i(t)$ is for $t > 0$.

Letting $w = v_1 - v_2$, we have by (3)

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{1}{2}(u_1 - u_2) + w \\ &= \frac{1}{1 + \triangle \tilde{v}}(v_1 - v_2) + w\end{aligned}$$

Here $\tilde{v}$ is some linear interpolation of v_1 and v_2. Thus, maximal principle holds and we have

$$\max_{t,x} |w| \le e^{t-s} \max_{t=s} |w| \qquad \text{for } t > s > 0.$$

Letting $s \to 0$, and noting that $w(s) \to 0$ in C^α, we must have $w \equiv 0$. So $v_1 \equiv v_2$, and hence $u_1 \equiv u_2$.

References

[1] Calabi E. Extremal Kähler metrics. In Seminar on Differential Geometry, **16**(102): 259–290. Ann. of Math. Studies, Princeton, N. J., Princeton University Press, 1982.

[2] Calabi E. Extremal Kähler metrics, II. In Differential Geometry and Complex Analysis, Berlin, Springer, 1985, 96–114.

[3] Donaldson S K. Lower bounds on the Calabi functional. *J. Diff. Geom.*, 2005, **70**(3): 453-472.

[4] Chen X X. Space of Kähler metrics (III)—On the greatest lower bound of the Calabi energy and the lower bound of geodesic distance. Priprint, 2006.

[5] Mabuchi T. K-energy maps integrating Futaki invariants. *Tohoku Math. J.*, 1986, **38**: 575-593.

[6] Chen X X, Tian G. Foliation by holomorphic discs and its application in Kähler geometry. math.DG/0507148. Submitted.

[7] Chen X X, Tian G. On the weak Kähler Ricci flow. 2006. in preparation.

[8] Chen X X. Weak limits of Riemannian metrics in surfaces with integral curvature bound. *Calc. Var. Partial Differential Equations*, 1998, **6**(3): 189–226.

[9] Donaldson S K. Scalar curvature and projective embeddings. II. *Q. J. Math.*, 2005, **56**(3): 345–356.

[10] Simon M. Deformation of Lipschitz Riemannian metrics in the direction of their Ricci curvature. Differential geometry, Valencia, 2001, 281–296, River Edge, NJ, World Sci. Publishing, 2002.