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THE CAUCHY PROBLEM OF THE HARTREE EQUATION*

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Dedicated to Professor Li Daqian on the occasion of his seventieth birthday

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Abstract In this paper, we systematically study the wellposedness, illposedness of the Hartree equation, and obtain the sharp local wellposedness, the global existence in H^s, $s \geq 1$ and the small scattering result in H^s for $2 < \gamma < n$ and $s \geq \frac{\gamma}{2} - 1$. In addition, we study the nonexistence of nontrivial asymptotically free solutions of the Hartree equation.

Key Words Hartree equation; well-posedness; illposedness; Galilean invariance; dispersion analysis; scattering; asymptotically free solutions.

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1. Introduction

In this paper, we study the Cauchy problem for the Hartree equation

$$\begin{cases} i\dot{u} + \Delta u = f(u), \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \quad n \geq 1, \\ u(0) = \varphi(x), \quad \text{in } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here the dot denotes the time derivative, Δ is the Laplacian in $\mathbb{R}^n$, $f(u)$ is a nonlinear function of Hartree type such as $f(u) = \lambda\big(V * |u|^2\big)u$ for some fixed constant $\lambda \in \mathbb{R}$ and $0 < \gamma < n$, where $*$ denotes the convolution in $\mathbb{R}^n$ and V is a real valued radial function defined in $\mathbb{R}^n$, here $V(x) = |x|^{-\gamma}$. In practice, we use the integral formulation of (1.1)

$$u(t) = U(t)\varphi - i\int_0^t U(t-s)f(u(s))\mathrm{d}s, \quad U(t) = e^{it\Delta}. \tag{1.2}$$

If the solution u of (1.1) has sufficient decay at infinity and smoothness, it satisfies two conservation laws in [1]:

$$M(u(t)) = \big\|u(t)\big\|_{L^2} = \big\|\varphi\big\|_{L^2},$$

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$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{4}\int\int \frac{1}{|x-y|^\gamma}|u(t,x)|^2|u(t,y)|^2 \, \mathrm{d}x\mathrm{d}y = E(\varphi). \quad (1.3)$$

There is a lot of works on the Cauchy problem and (small data) scattering theory of the Hartree equation. we refer to [1–10]. They all studied in the energy space $H^1(\mathbb{R}^n)$ or some weighted spaces. In this paper, we prove the local wellposedness in H^s, where $s \geq \max(0, s_c)$, $s_c = \frac{\gamma}{2} - 1$. Note that s_c is indicated by the scaling analysis. In addition, we prove some illposedness results for $s < \max(0, s_c)$ in Section 4. Therefore we obtain the sharp local results in this sense.

If we formally rewrite the equation (1.1) as

$$i\dot{u} + \Delta u = \lambda\big((-\Delta)^{-\frac{n-\gamma}{2}}|u|^2\big)u,$$

by the scaling analysis

$$u_\lambda(t,x) = \lambda^{\frac{n+2-\gamma}{2}} u(\lambda^2 t, \lambda x),$$

we obtain the critical exponent

$$s_c = \frac{\gamma}{2} - 1. \quad (1.4)$$

The paper is organized as follows.

In Section 2, we consider the case $s \geq \frac{\gamma}{2}$. We prove the local wellposedness (Theorem 2.1) of the equation (1.1) in H^s, and the global wellposedness of the energy solution (Theorem 2.2). Since $s \geq \frac{\gamma}{2}$, it is enough to obtain the solution by the contraction mapping argument in $C([0,T];H^s)$.

In Section 3, we consider the case $\max(0, \frac{\gamma}{2} - 1) \leq s < \frac{\gamma}{2}$. It is not enough to obtain the solution by the contraction mapping argument only in $C([0,T];H^s)$. Here we make use of the Strichartz estimates and prove the local wellposedness (Theorem 3.1) in $C\big([0,T], H^s\big) \cap L^q_T(H^s_r)$, where (q,r) is defined by (3.1), the global wellposedness of the energy solution (Corollary 3.1) and the small data scattering result (Theorem 3.2).

In Section 4, By the small dispersion analysis, scale and Galilean invariance, we obtain some illposedness results (Theorem 4.1 for $s < \max(0, s_c)$ and Theorem 4.2 for $s < -\frac{n}{2}$ or $0 < s < s_c$). The techniques to be used originated from [11].

Last in Section 5, we give the nonexistence result (Theorem 5.1) of the nontrivial asymptotically free solutions.

We conclude this introduction by giving some notation which will be used freely throughout this paper. $A \lesssim B, A \gtrsim B$ denote $A \leq CB, A \geq C^{-1}B$, respectively. For any $r, 1 \leq r \leq \infty$, we denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^n)$ and by r' the conjugate exponent defined by $\frac{1}{r} + \frac{1}{r'} = 1$. We denote the Schwartz space by $\mathcal{S}(\mathbb{R}^n)$. For any s, we denote by $H^s_r = (1-\Delta)^{-s/2}L^r$ the usual Sobolev spaces and $H^s = H^s_2$. Moreover, we define the $H^{k,k}$ norm:

$$\|u\|_{H^{k,k}} = \sum_{j=0}^{k} \|(1+|x|)^{k-j}\partial_x^j u\|_{L^2}.$$

Note that if $k > n/2$, $H^{k,k}$ norm controls both L^∞ and L^1 norm. We denote $V(y)\chi_{|y|\le 1}$ by $V_\le(y)$, and $V(y)\chi_{|y|\ge 1}$ by $V_\ge(y)$. We associate the variables $\alpha(r)$ and $\delta(r)$ defined by

$$\alpha(r) = \frac{\delta(r)}{n} = \frac{1}{2} - \frac{1}{r}.$$

Last we denote by $< \cdot, \cdot >$ the scalar product in L^2.

2. The Local and Global Existence in H^s, $s \ge \frac{\gamma}{2}$

In this section, we study the local existence in H^s, $s \ge \frac{\gamma}{2}$ based on the contraction mapping argument, and the global existence of the energy solution. To do so, we should use the following generalized Leibniz's rule [12].

Lemma 2.1 *For any $s \ge 0$, we have*

$$\big\|D^s(uv)\big\|_{L^r} \lesssim \big\|D^s u\big\|_{L^{r_1}}\big\|v\big\|_{L^{q_2}} + \big\|u\big\|_{L^{q_1}}\big\|D^s v\big\|_{L^{r_2}},$$

where $D^s = (-\Delta)^{\frac{s}{2}}$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{r_2}$, $r_i \in (1,\infty), q_i \in (1,\infty], i = 1,2$.

In addition, we also need the following maximal estimate [13] which is a direct consequence of the sharp Hardy inequality.

Lemma 2.2 *Let $0 < \gamma < n$, we have*

$$\big\|I_{n-\gamma}(|u|^2)\big\|_{L^\infty} \le C(n,\gamma)\big\|u\big\|^2_{\dot H^{\frac{\gamma}{2}}}.$$

Based on the above estimate, we can use $C(I; H^{\frac{\gamma}{2}})$ alone to study (1.1), but we can not work in $C(I; H^{\frac{n}{2}})$ alone for the nonlinear Schrödinger equations, because $\|u\|_\infty \lesssim \|u\|_{H^{\frac{n}{2}}}$ is not valid.

Let us first introduce the following local existence result.

Theorem 2.1 *Let $0 < \gamma < n$ and $n \ge 1$, $\varphi \in H^s(\mathbb{R}^n)$ with $s \ge \frac{\gamma}{2}$. Then there exists a positive time T such that (1.2) has a unique solution $u \in C\big([0,T], H^s\big)$ with $\big\|u\big\|_{L^\infty_T H^s} \le C\big\|\varphi\big\|_{H^s}$.*

Proof We apply the same approach in [14] to deal with (1.1). Let $\big(X^s_{T,\rho}, d\big)$ be a complete metric space with metric d defined by

$$X^s_{T,\rho} = \big\{u \in L^\infty_T(H^s(\mathbb{R}^n)); \big\|u\big\|_{L^\infty_T H^s} \le \rho\big\}, \quad d(u,v) = \big\|u - v\big\|_{L^\infty_T L^2}.$$

Our strategy is to prove that the following mapping

$$N(u)(t) = U(t)\varphi - i\int_0^t U(t-s)f(u)(s)\mathrm{d}s, \tag{2.1}$$

is a contraction map on $X^s_{T,\rho}$ for sufficiently small T.

First for all $u \in X^s_{T,\rho}$, by Lemma 2.1, Lemma 2.2 and the usual Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
\|N(u)\|_{L^\infty_T H^s} &\le \|\varphi\|_{H^s} + T\|f(u)\|_{L^\infty_T H^s} \\
&\lesssim \|\varphi\|_{H^s} + T\big(\|I_{n-\gamma}(|u|^2)\|_{L^\infty_T L^\infty}\|u\|_{L^\infty_T H^s} \\
&\quad + \|I_{n-\gamma}(|u|^2)\|_{L^\infty_T H^s_{\frac{2n}{\gamma}}}\|u\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}}\big) \\
&\lesssim \|\varphi\|_{H^s} + T\big(\|u\|^2_{L^\infty_T \dot H^{\frac{\gamma}{2}}}\|u\|_{L^\infty_T H^s} + \||u|^2\|_{L^\infty_T H^s_{\frac{2n}{2n-\gamma}}}\|u\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}}\big) \\
&\lesssim \|\varphi\|_{H^s} + T\big(\|u\|^2_{L^\infty_T \dot H^{\frac{\gamma}{2}}}\|u\|_{L^\infty_T H^s} + \|u\|_{L^\infty_T H^s}\|u\|^2_{L^\infty_T L^{\frac{2n}{n-\gamma}}}\big) \\
&\lesssim \|\varphi\|_{H^s} + T\|u\|^2_{L^\infty_T \dot H^{\frac{\gamma}{2}}}\|u\|_{L^\infty_T H^s} \lesssim \|\varphi\|_{H^s} + T\rho^3. \qquad (2.2)
\end{aligned}$$

If we choose ρ and T such that

$$\|\varphi\|_{H^s} \le \frac{\rho}{2}, \quad CT\rho^3 \le \frac{\rho}{2}.$$

Then N maps $X^s_{T,\rho}$ to itself.

Second, we need to show that N is a Lipschitz map for sufficiently small T. Let $u, v \in X^s_{T,\rho}$, we have

$$\begin{aligned}
d\big(N(u), N(v)\big) &\lesssim T\|I_{n-\gamma}(|u|^2)u - I_{n-\gamma}(|v|^2)v\|_{L^\infty_T L^2} \\
&\lesssim T\big(\|I_{n-\gamma}(|u|^2)(u-v)\|_{L^\infty_T L^2} + \|I_{n-\gamma}(|u|^2-|v|^2)v\|_{L^\infty_T L^2}\big) \\
&\lesssim T\big(\|u\|^2_{L^\infty_T H^{\frac{\gamma}{2}}} d(u,v) + \|I_{n-\gamma}(|u|^2-|v|^2)\|_{L^\infty_T L^{\frac{2n}{\gamma}}}\|v\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}}\big) \\
&\lesssim T\big(\rho^2 d(u,v) + \rho\||u|^2-|v|^2\|_{L^\infty_T L^{\frac{2n}{2n-\gamma}}}\big) \\
&\lesssim T\big(\rho^2 d(u,v) + \rho\|u+v\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}} d(u,v)\big) \\
&\lesssim T\rho^2 d(u,v).
\end{aligned}$$

Then N is a contraction on $X^s_{T,\rho}$ if T is sufficiently small.

From (1.2) and the contraction mapping argument, we can obtain the continuity in time and the uniqueness of solution. This completes the proof.

By the regularized argument [15, 16], we can show that the conservation laws (1.3) hold for the $H^s(\mathbb{R}^n)$-regularity solution, $s \ge 1$.

From the conservation laws (1.3), we obtain the following global wellposedness.

Theorem 2.2 *Let $s = 1$, $0 < \gamma < n$ and either one of the conditions holds*

(1) $\lambda \ge 0$, $0 < \gamma \le 2$;

(2) $\lambda < 0$, *and* $0 < \gamma < 2$;

(3) $\lambda < 0$, $\gamma = 2$, $n \geq 3$ *and* $\|\varphi\|_{L^2}$ *is sufficiently small.*

Let T^* *be the maximal existence time of the solution* u *as in Theorem* 2.1. *Then* $T^* = \infty$. *Moreover,*

$$\|u(t)\|_{H^1} \leq C\|\varphi\|_{H^1} e^{C_0 t},$$

where C_0 *depends on* $E(\varphi)$ *and* $\|\varphi\|_{L^2}$.

Remark 2.1 (1) As for the case $\lambda \geq 0$, Theorem 2.1 ensures the local existence in H^1 for $0 < \gamma \leq 2$, so it can also ensure the global existence in H^1 by energy conservation law. We also show that there exists the global existence (Theorem 3.1) in H^1 for the case $2 < \gamma < 4$ in next section. For the critical case, i.e. $n \geq 5$ and $\gamma = 4$, we have established the global well-posedness and scattering result in [17, 18]

(2) As for the case $\lambda < 0$, this theorem shows that there exists the global wellposedness in H^1 for the L^2 subcritical case $(0 < \gamma < 2)$, but for the L^2 critical case $(\gamma = 2)$, there exists the global existence of the small solution in H^1 . In addition, the small $\|\varphi\|_{L^2}$ condition can also ensure the global existence (Corollary 3.1) in H^1 for the H^1 subcritical case $\big(2 < \gamma < \min(4, n)\big)$. As for the case $\lambda < 0$, $\gamma = 4$, $n > 4$, Theorem 3.2 shows that the global existence and scattering in H^1 under the small $\|\varphi\|_{\dot{H}^1}$ condition.

(3) As for the case $\lambda < 0$ and $\gamma = 2$, Kurata and Ogawa [19] gave the sharp global existence and the blow-up results in H^1 as do the papers [2], [20] and [21].

Proof As for $\lambda \geq 0$, by (1.3), we have

$$\|u(t)\|^2_{H^{\frac{\gamma}{2}}} \leq \|u(t)\|^2_{H^1} \leq C\big(E(u) + \|\varphi\|^2_{L^2}\big) \leq C\big(E(\varphi) + \|\varphi\|^2_{L^2}\big).$$

As for $\lambda < 0$, and $0 < \gamma < \min(2, n)$, by the same estimate as in (2.1) and Young's inequality, we have

$$\begin{aligned}\Big|\frac{\lambda}{4}\int\int \frac{1}{|x-y|^\gamma}|u(t,x)|^2|u(t,y)|^2\ \mathrm{d}x\mathrm{d}y\Big| &\leq \frac{|\lambda|}{4}\big\|I_{n-\gamma}(|u|^2)\big\|_{L^\infty}\|u\|^2_{L^2} \\ &\leq C\|u\|^2_{\dot{H}^{\frac{\gamma}{2}}}\|\varphi\|^2_{L^2} \leq C\|u\|^\gamma_{\dot{H}^1}\|\varphi\|^{4-\gamma}_{L^2} \\ &\leq \varepsilon\|u\|^2_{\dot{H}^1} + C(\varepsilon)\|\varphi\|^{\frac{8-2\gamma}{2-\gamma}}_{L^2},\end{aligned}$$

which together with (1.3) implies that

$$\|u(t)\|^2_{H^{\frac{\gamma}{2}}} \leq \|u(t)\|^2_{H^1} \leq C\big(E(\varphi) + \|\varphi\|^2_{L^2} + \|\varphi\|^{\frac{8-2\gamma}{2-\gamma}}_{L^2}\big);$$

As for $\lambda < 0$, $\gamma = 2$, $n \geq 3$ and $\|\varphi\|_{L^2}$ is sufficiently small, we have

$$\Big|\frac{\lambda}{4}\int\int \frac{1}{|x-y|^2}\big|u(t,x)\big|^2\big|u(t,y)\big|^2\mathrm{d}x\mathrm{d}y\Big| \leq C\|u\|^2_{\dot{H}^1}\|\varphi\|^2_{L^2} \leq \frac{1}{4}\|u\|^2_{\dot{H}^1},$$

where we use that $\|\varphi\|_{L^2}$ is so sufficiently small that $C\|\varphi\|^2_{L^2} \leq \frac{1}{4}$. Therefore

$$\|u(t)\|^2_{H^{\frac{\gamma}{2}}} \leq C\big(E(\varphi) + \|\varphi\|^2_{L^2}\big).$$

On the other hand, we have the same as in (2.2)

$$\begin{aligned}\|u(t)\|_{H^1} &\lesssim \|\varphi\|_{H^1} + \int_0^t \|u(\tau)\|^2_{\dot{H}^{\frac{\gamma}{2}}} \|u(\tau)\|_{H^1} \mathrm{d}\tau \\ &\lesssim \|\varphi\|_{H^1} + C\int_0^t \|u(\tau)\|_{H^1} \mathrm{d}\tau, \end{aligned} \tag{2.3}$$

Gronwall's inequality implies that

$$\|u(t)\|_{H^1} \leq C\|\varphi\|_{H^1} e^{C_0 t}, \quad C_0 = C\big(E(\varphi), \|\varphi\|_2\big).$$

This completes the proof.

Because the regularity is a local property, and the estimate

$$\|u(t)\|_{H^s} \lesssim \|\varphi\|_{H^s} + C\int_0^t \|u(\tau)\|_{H^s} \mathrm{d}\tau,$$

holds for $s \geq 1$ as in (2.3), we have the following corollary.

Corollary 2.1 *Let $s \geq 1$, $0 < \gamma < n$ and either one of the conditions holds*

(1) $\lambda \geq 0$, $0 < \gamma \leq 2$;

(2) $\lambda < 0$, *and* $0 < \gamma < 2$;

(3) $\lambda < 0$, $\gamma = 2$, $n \geq 3$ *and* $\|\varphi\|_{L^2}$ *is sufficiently small.*

Let T^* be the maximal existence time of the solution u as in Theorem 2.1. *Then $T^* = \infty$. Moreover,*

$$\|u(t)\|_{H^s} \leq C\|\varphi\|_{H^s} e^{C_0 t}, \quad C_0 = C\big(E(\varphi), \|\varphi\|_2\big).$$

3. The Local and Global Existence in H^s, $\max(0, \frac{\gamma}{2} - 1) \leq s < \frac{\gamma}{2}$

In this section, we study the local existence in the lower regularity space H^s, $\max(0, \frac{\gamma}{2} - 1) \leq s < \frac{\gamma}{2}$, and the corresponding global existence. To do so, we should introduce the Strichartz estimate [15].

Definition 3.1 *We say that a pair (q, r) is admissible if*

$$\frac{2}{q} = n\Big(\frac{1}{2} - \frac{2}{r}\Big)$$

and

$$2 \leq r \begin{cases} \leq & \infty, \quad n = 1 \\ < & \infty, \quad n = 2; \\ \leq & \frac{2n}{n-2}, \quad n \geq 3. \end{cases}$$

Proposition 3.1 *The following properties hold:*

1. *For every $\varphi \in L^2(\mathbb{R}^n)$, the function $t \mapsto U(t)\varphi$ belongs to*

$$L^q(\mathbb{R}; L^r(\mathbb{R}^n)) \cap C(\mathbb{R}; L^2(\mathbb{R}^n))$$

for every admissible pair (q,r). Furthermore, there exists a constant C such that

$$\big\|U(t)\varphi\big\|_{L^q(\mathbb{R},L^r)} \leq C\big\|\varphi\big\|_{L^2},$$

for every $\varphi \in L^2(\mathbb{R}^n)$.

2. *Let I be an interval of $\mathbb{R}$ (bounded or not), $J = \overline{I}$, and $t_0 \in J$. If (p,ρ) is an admissible pair and $f \in L^{p'}(I, L^{\rho'}(\mathbb{R}^n))$, then for every admissible pair (q,r), the function*

$$t \mapsto \Phi_f(t) = \int_{t_0}^t U(t-s)f(s)\mathrm{d}s,$$

for $t \in I$ belongs to $L^q(I, L^r(\mathbb{R}^n)) \cap C(J; L^2(\mathbb{R}^n))$. Furthermore, there exists a constant C independent of I such that

$$\big\|\Phi_f(t)\big\|_{L^q(I,L^r)} \leq C\big\|f\big\|_{L^{p'}(I,L^{\rho'})},$$

for every $f \in L^{p'}(I, L^{\rho'}(\mathbb{R}^n))$.

For $0 < \gamma < n$ and $\max(0, \frac{\gamma}{2} - 1) \leq s < \frac{\gamma}{2}$, there is a particular admissible pair (q,r) defined by

$$\frac{1}{q} = \frac{\gamma - 2s}{6}, \quad \frac{1}{r} = \frac{1}{2} + \frac{2s-\gamma}{3n}, \tag{3.1}$$

which will play a crucial role in our estimate and come from scaling relation

$$\frac{1}{r'} + 1 = \frac{1}{2} + 2\Big(\frac{1}{r} - \frac{s}{n}\Big) + \frac{\gamma}{n}.$$

Now we can state the local existence results in the lower regular space H^s.

Theorem 3.1 *Let $0 < \gamma < n$ and $\varphi \in H^s(\mathbb{R}^n)$ with $\max(0, \frac{\gamma}{2} - 1) \leq s < \frac{\gamma}{2}$. Then there exists a positive time T such that (1.2) has a unique solution $u \in C\big([0,T], H^s\big) \cap L^q_T(H^s_r)$, where (q,r) is defined by (3.1). In particular, one have the global wellposedness in $H^1(\mathbb{R}^n)$ for $\lambda \geq 0$, $2 < \gamma \leq 4$, and $\gamma < n$ by the energy conservation laws.*

Note that this result is similar to Proposition 3.1 in [14]. The difference between them is that we can give the sharp local existence in H^s, which is indicated by the scaling analysis, while Proposition 3.1 in [14] didn't give the sharp local existence for the semirelativistic Hartree equation. In addition, we shall use the arguments [22], [11] to obtain some ill-posedness results in Section 4 for $s < \max(0, s_c)$. Hence we obtain the sharp local wellposedness in this sense.

Proof Let $\left(Y^s_{T,\rho}, d\right)$ be a complete metric space with metric d defined by

$$\begin{aligned}
&Y^s_{T,\rho} = \left\{u \in L^\infty_T(H^s(\mathbb{R}^n)) \cap L^q_T(H^s_r(\mathbb{R}^n)); \left\|u\right\|_{L^\infty_T H^s} + \left\|u\right\|_{L^q_T(H^s_r)} \le \rho\right\},\\
&d(u,v) = \left\|u-v\right\|_{L^\infty_T H^s \cap L^q_T(H^s_r)}.
\end{aligned}$$

Our strategy is to prove that the following mapping (2.1) is a contraction map on $Y^s_{T,\rho}$ for sufficiently small T.

First for all $u \in Y^s_{T,\rho}$, by Proposition 3.1, Lemma 2.1 and the usual Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
&\left\|N(u)\right\|_{L^\infty_T H^s} + \left\|N(u)\right\|_{L^q_T(H^s_r)} \le \left\|\varphi\right\|_{H^s} + \left\|f(u)\right\|_{L^{q'}_T H^s_{r'}}\\
&\quad\lesssim \left\|\varphi\right\|_{H^s} + \Big(\left\|I_{n-\gamma}(|u|^2)\right\|_{L^{q'}_T L^{\frac{3n}{\gamma-2s}}}\left\|u\right\|_{L^\infty_T H^s}\\
&\qquad+\left\|I_{n-\gamma}(|u|^2)\right\|_{L_T^{\frac{6}{6+4s-2\gamma}} H^s_{\frac{3n}{2\gamma-s}}}\left\|u\right\|_{L^q_T L^{\frac{6n}{3n-2s-2\gamma}}}\Big)\\
&\quad\lesssim \left\|\varphi\right\|_{H^s} + \Big(\left\||u|^2\right\|_{L^{q'}_T L^{\frac{3n}{3n-2s-2\gamma}}}\left\|u\right\|_{L^\infty_T H^s} + \left\||u|^2\right\|_{L_T^{\frac{6}{6+4s-2\gamma}} H^s_{\frac{3n}{3n-s-\gamma}}}\left\|u\right\|_{L^q_T L^{\frac{6n}{3n-2s-2\gamma}}}\Big)\\
&\quad\lesssim \left\|\varphi\right\|_{H^s} + \Big(\left\|u\right\|^2_{L^{2q'}_T L^{\frac{6n}{3n-2s-2\gamma}}}\left\|u\right\|_{L^\infty_T H^s}\\
&\qquad+\left\|u\right\|_{L_T^{\frac{6}{6+4s-2\gamma}} L^{\frac{6n}{3n-2s-2\gamma}}}\left\|u\right\|_{L^\infty_T H^s}\left\|u\right\|_{L^q_T L^{\frac{6n}{3n-2s-2\gamma}}}\Big)\\
&\quad\lesssim \left\|\varphi\right\|_{H^s} + T^\theta\left\|u\right\|^2_{L^q_T H^s_r}\left\|u\right\|_{L^\infty_T H^s} \lesssim \left\|\varphi\right\|_{H^s} + T^\theta \rho^3
\end{aligned}\tag{3.2}$$

where $\theta = 1 + s - \frac{\gamma}{2}$. Here we use the Sobolev embedding $H^s_r \hookrightarrow L^{\frac{6n}{3n-2s-2\gamma}}$.

If we choose ρ and T such that

$$C\left\|\varphi\right\|_{H^s} \le \frac{\rho}{2}, \quad CT^\theta \rho^3 \le \frac{\rho}{2}.$$

Then N maps $Y^s_{T,\rho}$ to itself.

Second, we need to show that N is a contraction map for sufficiently small T. Let $u, v \in Y^s_{T,\rho}$, we have

$$d\big(N(u), N(v)\big) \lesssim \left\|I_{n-\gamma}(|u|^2)(u-v)\right\|_{L^{q'}_T H^s_{r'}} + \left\|I_{n-\gamma}(|u|^2-|v|^2)v\right\|_{L^{q'}_T H^s_{r'}}. \tag{3.3}$$

By Lemma 2.1, Hölder inequality and Hardy inequality, we have

$$\begin{aligned}
\left\|I_{n-\gamma}(|u|^2)(u-v)\right\|_{L^{q'}_T H^s_{r'}} \lesssim{}& \left\|I_{n-\gamma}(|u|^2)\right\|_{L^{q'}_T L^{\frac{3n}{\gamma-2s}}}\left\|u-v\right\|_{L^\infty_T H^s}\\
&+ \left\|I_{n-\gamma}(|u|^2)\right\|_{L_T^{\frac{6}{6+4s-2\gamma}} H^s_{\frac{3n}{2\gamma-s}}}\left\|u-v\right\|_{L^q_T L^{\frac{6n}{3n-2s-2\gamma}}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|u\|^2_{L_T^{2q'}L^{\frac{6n}{3n-2s-2\gamma}}}\|u-v\|_{L_T^\infty H^s}\\
&\quad+\|u\|_{L_T^{\frac{6}{6+4s-2\gamma}}L^{\frac{6n}{3n-2s-2\gamma}}}\|u\|_{L_T^\infty H^s}\|u-v\|_{L_T^q L^{\frac{6n}{3n-2s-2\gamma}}}\\
&\lesssim T^\theta\big(\|u\|^2_{L_T^q H_r^s}\|u-v\|_{L_T^\infty H^s}\\
&\quad+\|u\|_{L_T^q H_r^s}\|u\|_{L_T^\infty H^s}\|u-v\|_{L_T^q H_r^s}\big)\\
&\lesssim T^\theta\rho^2 d(u,v).
\end{aligned}
$$

Similarly,

$$
\big\|I_{n-\gamma}(|u|^2-|v|^2)v\big\|_{L_T^{q'}H^s_{r'}}\lesssim T^\theta\rho^2 d(u,v).
$$

Substituting the above estimates into (3.3), we can conclude N is a contraction on $Y^s_{T,\rho}$ if T is sufficiently small. This completes the proof.

Now for $\lambda<0$, we can also show that the local energy solutions can be extended globally in time by using the energy conservation law.

Corollary 3.1 *Let $\lambda<0$, $2<\gamma<\min(4,n), n\geq 3$, $\varphi\in H^1(\mathbb{R}^n)$ and $\|\varphi\|_{L^2}$ is sufficiently small. Then* (1.2) *has a unique solution $u\in C\big([0,\infty);H^1\big)\cap L^q_{loc}(H^1_r)$, where (q,r) is defined by* (3.1).

Note that one can obtain the sharp condition for the global wellposedness in $H^1(\mathbb{R}^n)$ as does the paper [23].

Proof Let T^* be the maximal existence time. We prove that T^* is infinite by contradiction.

Suppose that $T^*<\infty$, then Theorem 3.1 implies that $\|u(T^*)\|_{H^1_r}=\infty$.

For any $t<T^*$, we have by (1.3)

$$
\begin{aligned}
\frac12\|u(t)\|^2_{H^1}&\leq\frac12\|u(t)\|^2_{L^2}+|E(u)|+\Big|\frac{\lambda}{4}\int\int\frac{1}{|x-y|^\gamma}|u(t,x)|^2|u(t,y)|^2\,\mathrm{d}x\mathrm{d}y\Big|\\
&\leq\frac12\|\varphi\|^2_{L^2}+|E(\varphi)|+C\|u(t)\|^2_{L^{\frac{2n}{n-\gamma+2}}}\|u(t)\|^2_{H^1}\\
&\leq\frac12\|\varphi\|^2_{L^2}+|E(\varphi)|+C\|\varphi\|^{4-\gamma}_{L^2}\|u(t)\|^\gamma_{H^1}.
\end{aligned}
$$

The smallness of $\|\varphi\|_{L^2}$ implies that

$$
\|u(t)\|^2_{H^1}\leq C\big(\|\varphi\|^2_{L^2}+|E(\varphi)|\big)<\infty,\qquad\forall\, t<T^*.
$$

This implies that $\|u(T^*)\|_{H^1}<\infty$, Hence we have $T^*=\infty$. This completes the proof.

So far, we have obtained the global existence in H^1 for the case $\lambda>0, 0<\gamma<4$ and $\gamma<n$ or the case $\lambda<0, 0<\gamma<2$; and also show that the small $\|\varphi\|_{L^2}$ condition ensures the global existence in H^1 for the case $\lambda<0,\ 2\leq\gamma<\min(4,n)$.

Finally under the small $\|\varphi\|_{\dot H^{s_c}}$ assumption, we will not only obtain the global existence in H^s, $s\geq\frac{\gamma}{2}-1$ for the case $2<\gamma<n$, $n\geq 3$, but also obtain the scattering

result in H^s. This result is similar to that in [24–28]. Note that the energy scattering result for $2<\gamma<\min(4,n)$ has been obtain by Ginibre and Velo [1] and Nakanishi [9]. The authors recently obtain the energy scattering result for the H^1-critical Hartree equation in [17, 18] by energy deduction in [29–31] and for the Klein-Gordon equation with a cubic convolution nonlinearity [32].

Theorem 3.2 *Let $2<\gamma<n$, $n\geq 3$, and $s\geq \frac{\gamma}{2}-1$. Then there exists ρ such that for any $\varphi\in H^s$ with $\|\varphi\|_{\dot H^{s_c}}\leq\rho$, (1.2) has a unique solution $u\in (C\cap L^\infty)(\mathbb{R},H^s)\cap L^4(\mathbb{R},H^s_{\frac{2n}{n-1}})$. Moreover, there is $\varphi^+\in H^s$ such that*

$$\big\|u(t)-U(t)\varphi^+\big\|_{H^s}\to 0\quad as\quad t\to\infty.$$

Proof Let $\big(\Omega^s_{\rho,R},d\big)$ be a complete metric space with metric d defined by

$$\begin{aligned}
&\Omega^s_{\rho,R}=\big\{u\in L^4(\mathbb{R},H^s_{\frac{2n}{n-1}}),\big\|u\big\|_{L^4(\mathbb{R},\dot H^{s_c}_{\frac{2n}{n-1}})}\leq\rho,\ \big\|u\big\|_{L^4(\mathbb{R},\dot H^s_{\frac{2n}{n-1}})}\leq R\big\},\\
&d(u,v)=\big\|u-v\big\|_{L^4(L^{\frac{2n}{n-1}})}.
\end{aligned}$$

Then we have from Proposition 3.1 and Lemma 2.1

$$\begin{aligned}
\big\|N(u)\big\|_{L^4(\mathbb{R},\dot H^{s_c}_{\frac{2n}{n-1}})}&\leq C\big\|\varphi\big\|_{\dot H^{s_c}}+C\big\|f(u)\big\|_{L^{\frac{4}{3}}(\dot H^{s_c}_{\frac{2n}{n+1}})}\\
&\leq C\big\|\varphi\big\|_{\dot H^{s_c}}+C\big\|u\big\|^2_{L^4(L^{\frac{2n}{n-\gamma+1}})}\big\|u\big\|_{L^4(\dot H^{s_c}_{\frac{2n}{n-1}})}\\
&\leq C\big\|\varphi\big\|_{\dot H^{s_c}}+C\big\|u\big\|^3_{L^4(\dot H^{s_c}_{\frac{2n}{n-1}})};\\
\big\|N(u)\big\|_{L^4(\mathbb{R},\dot H^{s}_{\frac{2n}{n-1}})}&\leq C\big\|\varphi\big\|_{\dot H^{s}}+C\big\|u\big\|^2_{L^4(\dot H^{s_c}_{\frac{2n}{n-1}})}\big\|u\big\|_{L^4(\dot H^{s}_{\frac{2n}{n-1}})}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\big\|N(u)-N(v)\big\|_{L^4(\mathbb{R},L^{\frac{2n}{n-1}})}&\leq C\big\|f(u)-f(v)\big\|_{L^{\frac{4}{3}}(L^{\frac{2n}{n+1}})}\\
&\leq C\big(\big\|u\big\|^2_{L^4(L^{\frac{2n}{n-\gamma+1}})}+\big\|v\big\|^2_{L^4(L^{\frac{2n}{n-\gamma+1}})}\big)\big\|u-v\big\|_{L^4(L^{\frac{2n}{n-1}})}\\
&\leq C\big(\big\|u\big\|^2_{L^4(\dot H^{s_c}_{\frac{2n}{n-1}})}+\big\|v\big\|^2_{L^4(\dot H^{s_c}_{\frac{2n}{n-1}})}\big)d(u,v).
\end{aligned}$$

If we choose R and sufficiently small ρ such that

$$C\big\|\varphi\big\|_{\dot H^{s_c}}\leq\frac{\rho}{2},\quad C\big\|\varphi\big\|_{\dot H^{s}}\leq\frac{R}{2},\quad 2C\rho^2\leq\frac{1}{2}.$$

then N maps $\Omega^s_{\rho,R}$ to itself and is a contraction map. According to Proposition 3.1, we obtain $u\in(C\cap L^\infty)(\mathbb{R},H^s)$. This proves the existence part.

To prove the scattering, let us define a function φ^+ by

$$\varphi^+ = \varphi - i\int_0^\infty U(-s)f(u)(s)\mathrm{d}s.$$

Then since the solution u is in $\Omega^s_{\rho,R}$, $\varphi^+ \in H^s$, therefore it holds that

$$\begin{aligned}\big\|u(t) - u^+(t)\big\|_{H^s} &\lesssim C\big\|u\big\|^2_{L^4(t,\infty;\dot{H}^{s_c}_{\frac{2n}{n-1}})}\big\|u\big\|_{L^4(t,\infty;H^s_{\frac{2n}{n-1}})}\\ &\longrightarrow 0 \quad \text{as} \quad t\to\infty.\end{aligned}$$

This completes the proof.

4. Ill-Posedness in H^s, $s < \max(0, s_c)$

In this section, we prove the ill-posedness in some sense for (1.1). Our proof relies heavily on the methods of small dispersion analysis and scale and Galilean invariance, which are initiated in [11] by M. Christ, J. Colliander, T. Tao. The main difficulty here lies in small dispersion analysis due to the non-local nonlinearity.

The main results are the following.

Theorem 4.1 *For any $s < \max(0, s_c)$, the Cauchy problem (1.1) fails to be well-posed in H^s in the following sense: for any $0 < \delta, \varepsilon < 1$ and for any $t > 0$ there exist solutions u_1, u_2 of (1.1) with initial data $u_1(0)$, $u_2(0) \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$\begin{aligned}&\|u_1(0)\|_{H^s}, \|u_2(0)\|_{H^s} \le C\varepsilon;\\ &\|u_1(0) - u_2(0)\|_{H^s} < C\delta;\\ &\|u_1(t) - u_2(t)\|_{H^s} > c\varepsilon,\end{aligned}$$

for some $C \gg 1$ as well as some $0 < c \ll 1$.

Theorem 4.2 *Suppose either $0 < s < s_c = \frac{\gamma}{2} - 1$ or $s \le -\frac{n}{2}$, for any $\varepsilon > 0$, there exist a solution of (1.1) and $t > 0$ such that $u(0) \in \mathcal{S}(\mathbb{R}^n)$,*

$$\begin{aligned}&\|u(0)\|_{H^s} < C\varepsilon;\\ &\|u(t)\|_{H^s} > C\varepsilon^{-1}, \quad 0 < t < \varepsilon.\end{aligned}$$

For $n = 1, 2$, the ill-posedness results are relatively simple, see Figure 1. because $\gamma < n$, $s_c = \gamma/2 - 1 \le 0$, we need not consider the case $0 < s < s_c$. For $n > 2$, $s_c > 0$, so the case $0 < s < s_c$ come up and it seems more complicated, see Figure 2. As for the case $s \le -\frac{n}{2}$, the solution of (4.1) transfers its energy to decreasingly lower frequencies; while as for the case $0 < s < s_c, n \ge 3$, the solution of (4.1) transfers its energy to increasingly higer frequencies. But for the case $-\frac{n}{2} < s \le 0$, the interaction is more complicated, only weak illposedness can be obtained. We hope to prove the local wellposedness for $s \ge s_c$, $0 < \gamma < 2$ by the Fourier truncation norm method.

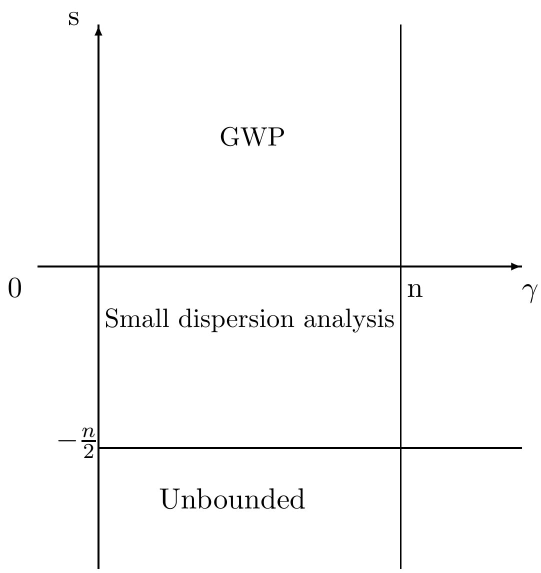

Figure 1. (n=1,2)

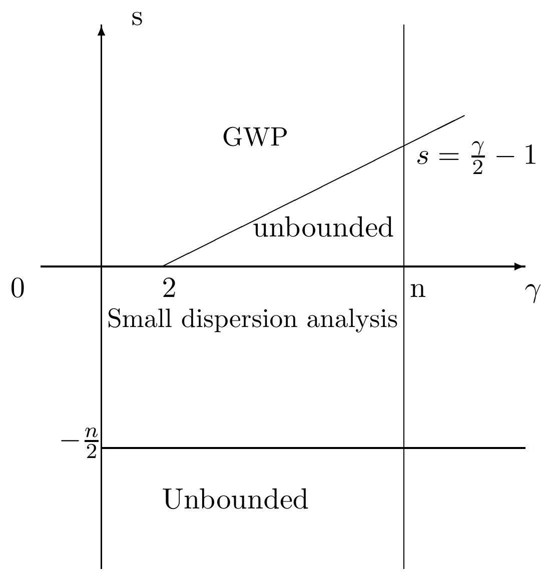

Figure 2. $(n > 2)$

In order to analyze the behavior of equation (1.1), we begin to analyze the small dispersion version

$$iu_t + \nu^2 \triangle u = (V * |u|^2)u, \quad u(0,x) = \phi_0(x), \tag{4.1}$$

and the corresponding ordinary differential equation

$$iv_t = (V * |v|^2)v, \quad v(0,x) = \phi_0(x). \tag{4.2}$$

In fact, the solutions of (4.2) can be written as following

$$v = \phi_0 e^{-itV*|\phi_0|^2}. \tag{4.3}$$

If the solution of (4.1) can be approximated by that of (4.2) in some sense, we can learn much information because the solution of (4.2) is well understood.

Lemma 4.1 *Let $n \geq 1$, $k > n/2$ be an integer, let ϕ_0 be a Schwartz function. Then there exists C, c depending on all the above parameters, such that if $0 < \nu \leq c$ is a sufficiently small real number, then for $T = c|\log \nu|^c$ there exists a solution $u(t,x) \in C^1([-T,T], H^{k,k})$ of (4.1) satisfying*

$$\|u(t) - v(t)\|_{H^{k,k}} \leq C\nu \quad \textit{for all} \quad |t| \leq c|\log \nu|^c.$$

Proof We define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) := (V * |z|^2)z.$$

Let $w = u - v$, where u and v are solutions of (4.1) and (4.2), respectively. Then w solves

$$\begin{cases} iw_t + \nu^2 \Delta w = -\nu^2 \Delta v + (V * |w+v|^2)(w+v) - (V * |v|^2)v \\ \qquad\qquad\quad \doteq -\nu^2 \Delta v + F(v+w) - F(v), \\ w(0,x) = 0. \end{cases}$$

Thus it suffices to prove

$$\sup_{|t|<T} \|w\|_{H^{k,k}} \leq C\nu.$$

We have the energy inequality

$$\partial_t \|w\|_{H^{k,k}} \leq C\| - \nu^2 \Delta v + (F(v+w) - F(v))\|_{H^{k,k}} + C\|w\|_{H^{k,k}}.$$

Since ϕ_0 is Schwartz, $\|\Delta v(t)\|_{H^{k,k}} \leq C(1+|t|)^{k+2}$. We consider the pointwise bound for $\|(V * |w+v|^2)(w+v) - (V * |v|^2)v\|_{H^{k,k}}$.

$$\begin{aligned} &F(v+w) - F(v) \\ &\quad = \int V(y)\big(|w+v|^2(x-y)(w+v)(x) - |v|^2(x-y)v(x)\big)\mathrm{d}y \\ &\quad = \int V(y)\big(2\mathrm{Re}(w\overline{v}) + |w|^2\big)(x-y)v(x)\mathrm{d}y \\ &\qquad + \int V(y)\big(|v|^2 + 2\mathrm{Re}w\overline{v} + |w|^2\big)(x-y)w(x)\mathrm{d}y. \end{aligned}$$

We take one term $2\int V(y)\mathrm{Re}\big(\overline{v}(x-y)w(x-y)\big)v(x)\mathrm{d}y$ as an example to estimate, the other terms can be similarly estimated.

$$\begin{aligned} &\left\|2\int V(y)\mathrm{Re}(\overline{v}(x-y)w(x-y))v(x)\mathrm{d}y\right\|_{H^{k,k}} \\ &\quad \lesssim \sum_{j=1}^{k} \left\|(1+|x|)^{k-j}\partial_x^j \int V(y)\overline{v}(x-y)w(x-y)v(x)\mathrm{d}y\right\|_{L^2} \\ &\quad \lesssim \sum_{j=1}^{k} \left\|(1+|x|)^{k-j} \int V(y)\overline{v}(x-y)w(x-y)\partial_x^j v(x)\mathrm{d}y\right\|_{L^2} \\ &\qquad + \sum_{j=1}^{k} \left\|(1+|x|)^{k-j} \int V(y)\overline{v}(x-y)\partial_x^j w(x-y)v(x)\mathrm{d}y\right\|_{L^2} \\ &\qquad + \sum_{j=1}^{k} \left\|(1+|x|)^{k-j} \int V(y)\partial_x^j \overline{v}(x-y)w(x-y)v(x)\mathrm{d}y\right\|_{L^2} \\ &\quad := I_1 + I_2 + I_3. \end{aligned}$$

We estimate I_1 first. By Young inequality and $\gamma < n$,

$$\|V * (\overline{v}w)\|_{L^\infty} \lesssim \|V_\geq * (\overline{v}w)\|_{L^\infty} + \|V_\leq * (\overline{v}w)\|_{L^\infty}$$

$$\lesssim \|V_{\geq}\|_{L^\infty}\|\overline{v}\|_{L^2}\|w\|_{L^2} + \|V_{\leq}\|_{L^1}\|\overline{v}\|_{L^\infty}\|w\|_{L^\infty}$$
$$\lesssim \|v\|_{H^{k,k}}\|w\|_{H^{k,k}}.$$

So

$$I_1 \lesssim \|(1+|x|)^{k-j}\partial_x^j\overline{v}\|_{L^2}\|V*(\overline{v}w)\|_{L^\infty} \lesssim \|v\|_{H^{k,k}}^2\|w\|_{H^{k,k}}.$$

We estimate I_2 next.

$$I_2 \lesssim \Big\| \int_{|x-y|>|x|} V_{\leq}(y)\overline{v}(x-y)(1+|x|)^{k-j}\partial_x^j w(x-y)v(x)\mathrm{d}y \Big\|_{L^2} \tag{a}$$
$$+ \Big\| \int_{|x-y|>|x|} V_{\geq}(y)\overline{v}(x-y)(1+|x|)^{k-j}\partial_x^j w(x-y)v(x)\mathrm{d}y \Big\|_{L^2} \tag{b}$$
$$+ \Big\| \int_{|x-y|<|x|} V_{\leq}(y)\overline{v}(x-y)(1+|x|)^{k-j}\partial_x^j w(x-y)v(x)\mathrm{d}y \Big\|_{L^2} \tag{c}$$
$$+ \Big\| \int_{|x-y|<|x|} V_{\geq}(y)\overline{v}(x-y)(1+|x|)^{k-j}\partial_x^j w(x-y)v(x)\mathrm{d}y \Big\|_{L^2}. \tag{d}$$

By Young inequality, Sobolev imbedding and Hölder inequality,

$$(a) \lesssim \|V_{\leq}\|_{L^1}\big\|(1+|\cdot|)^{k-j}\partial_x^j w\big\|_{L^2}\|v\|_{L^\infty}^2 \lesssim \|v\|_{H^{k,k}}^2\|w\|_{H^{k,k}};$$
$$(b) \lesssim \|V_{\geq}\|_{L^\infty}\big\|(1+|\cdot|)^{k-j}\partial_x^j w\big\|_{L^2}\|v\|_{L^1}\|v\|_{L^\infty} \lesssim \|v\|_{H^{k,k}}^2\|w\|_{H^{k,k}};$$
$$\begin{aligned}(c) &\lesssim \Big\| \int_{|x-y|<|x|} V_{\leq}(y)\overline{v}(x-y)(1+|x-y|)^{k-j}(1+|y|)^{k-j}\partial_x^j w(x-y)v(x)\mathrm{d}y \Big\|_{L^2}\\
&\lesssim \Big\| \int V_{\leq}(y)\overline{v}(x-y)(1+|x-y|)^{k-j}\partial_x^j w(x-y)\mathrm{d}y \Big\|_{L^2}\|v\|_{L^\infty}\\
&\lesssim \|V_{\leq}\|_{L^1}\big\|(1+|\cdot|)^{k-j}\partial_x^j w\big\|_{L^2}\|v\|_{L^\infty}^2\\
&\lesssim \|v\|_{H^{k,k}}^2\|w\|_{H^{k,k}};\end{aligned}$$
$$\begin{aligned}(d) &\lesssim \Big\| \int_{|x-y|<|x|} V_{\geq}(y)\overline{v}(x-y)(1+|x|)^{-j}\partial_x^j w(x-y)\mathrm{d}y \Big\|_{L^\infty}\|(1+|\cdot|)^k v\|_{L^2}\\
&\lesssim \Big\| \int_{|y|<|2x|} V_{\geq}(y)\overline{v}(x-y)(1+|y|)^{-j}\partial_x^j w(x-y)\mathrm{d}y \Big\|_{L^\infty}\|v\|_{H^{k,k}}\\
&\lesssim \|(1+|\cdot|)^{-j}V_{\geq}\|_{L^\infty}\|\partial_x^j w\|_{L^2}\|v\|_{L^2}\|v\|_{H^{k,k}}\\
&\lesssim \|v\|_{H^{k,k}}^2\|w\|_{H^{k,k}}.\end{aligned}$$

We can estimate I_3 in the same way, so we get

$$\partial_t\|w\|_{H^{k,k}} \leq C\nu^2(1+|t|)^C + C(1+|t|)^C(\|w\|_{H^{k,k}} + \|w\|_{H^{k,k}}^3).$$

Under a priori assumption that $w(t)$ is bounded in $H^{k,k}$, e.g. $\|w\|_{H^{k,k}} \leq 1$, we get

$$\|w\|_{H^{k,k}} \leq C\nu^2 e^{C(1+|t|)^C},$$

if $t \le c|\log\nu|^c$ for suitably small c and ν.

Now we can exploit Lemma 4.1 to prove Theorem 4.1 and Theorem 4.2.

In order to prove Theorem 4.1, it suffices to consider $-\frac{n}{2} < s < 0$ only. In fact, if $0 < s < s_c$ or $s < -\frac{n}{2}$, Theorem 4.1 follows from Theorem 4.2. Let $w(x)$ be an arbitrary nonzero Schwartz function. Let $a \in [1/2, 1]$, $\nu \in (0, 1]$ be parameters. If $\phi_0 = aw(x)$, $v^{(a,\nu)}(t,x) = v(t,x)$ is the solution of (4.1) with initial data $v^{(a,\nu)}(0,x) = aw(x)$. From Lemma 4.1,

$$\left\|v^{(a,\nu)}(t) - v^{(a,0)}(t)\right\|_{H^{k,k}} \le C\nu,$$

for all $|t| \le c|\log\nu|^c$, where

$$v^{(a,0)}(t) = aw(x)e^{(-itV*|aw|^2(x))}.$$

By scaling and Galilean invariance of (1.1), we obtain a family of solutions $u = u^{(a,\nu,\rho,\mu)}$, where $a \in [1/2, 1]$, $0 < \nu \ll 1$, $0 < \rho \ll 1$, $\mu \in \mathbb{R}^n$, and

$$u = u^{(a,\nu,\rho,\mu)} = \rho^{-\frac{n}{2}+\frac{\gamma}{2}-1}e^{i\mu\cdot x/2}e^{-i|\mu|^2t/4}v^{(a,\nu)}(\rho^{-2}t, \rho^{-1}\nu(x-\mu t)).$$

Lemma 4.2 *Let $0 \ne w \in \mathcal{S}(\mathbb{R}^n)$, $s < 0$, and suppose a,a' are in $[1/2, 1]$, $0 < \nu < \rho \ll 1$. Then*

$$\left\|u^{(a,\nu,\rho,\mu)}(0)\right\|_{H^s} \le C\rho^{-\frac{n}{2}+\frac{\gamma}{2}-1}|\mu|^s(\rho/\nu)^{n/2},$$

and

$$\left\|u^{(a,\nu,\rho,\mu)}(0) - u^{(a',\nu,\rho,\mu)}(0)\right\|_{H^s} \le C\rho^{-\frac{n}{2}+\frac{\gamma}{2}-1}|\mu|^s(\rho/\nu)^{n/2}|a-a'|.$$

Moreover,

$$\begin{aligned}\left\|u^{(a,\nu,\rho,\mu)}(t) - u^{(a',\nu,\rho,\mu)}(t)\right\|_{H^s} \ge & C\rho^{-\frac{n}{2}+\frac{\gamma}{2}-1}|\mu|^s\Big(\frac{\rho}{\nu}\Big)^{\frac{n}{2}}\bigg(\Big\|v^{(a,\nu)}\Big(\frac{t}{\rho^2}\Big) - v^{(a',\nu)}\Big(\frac{t}{\rho^2}\Big)\Big\|_2 \\ & - C|\log\nu|^C\Big(\frac{\rho}{\nu}\Big)^{-k}|\mu|^{-s-k}\bigg),\end{aligned} \tag{4.4}$$

whenever $|t| \le c|\log\nu|^c\rho^2$.

Proof The proof of the lemma is similar with that of Lemma 3.1 of [11] except that $\rho^{-\frac{2}{p-1}}$ is replaced by $\rho^{-\frac{n}{2}+\frac{\gamma}{2}-1}$.

Proof of Theorem 4.1 Now we set $\rho = \nu^\sigma$, where σ is a small positive number to determined. Then we choose any vector μ such that

$$\rho^{-\frac{n}{2}+\frac{\gamma}{2}-1}|\mu|^s(\rho/\nu)^{n/2} = \varepsilon.$$

Thus

$$|\mu| = \nu^{\frac{1}{s}[\frac{n}{2}-\sigma(\frac{\gamma}{2}-1)]}\varepsilon^{\frac{1}{s}}.$$

We may choose σ sufficiently small such that $\frac{n}{2} - \sigma(\frac{\gamma}{2} - 1)$ is positive. Note that $s < 0$, so the power to ν is negative. As a consequence, $|\mu|$ grows faster than any power of $|\log\nu|$ as $\nu\to 0$.

Now we can construct u satisfying Theorem 4.1. From Lemma 4.2,

$$\left\|u^{(a,\nu,\rho,\mu)}(0)\right\|_{H^s}+\left\|u^{(a',\nu,\rho,\mu)}(0)\right\|_{H^s}\le C\varepsilon,$$

and

$$\left\|u^{(a,\nu,\rho,\mu)}(0)-u^{(a',\nu,\rho,\mu)}(0)\right\|_{H^s}\le C\varepsilon|a-a'|.$$

Meanwhile, there exists a time $T=T(a,a')>0$ such that

$$\left\|v^{(a,\nu)}(T)-v^{(a',\nu)}(T)\right\|_{L^2}>c,$$

for ν sufficiently small and $T\le c|\log\nu|^c$. So we have

$$\left\|u^{(a,\nu,\rho,\mu)}(\rho^2T)-u^{(a',\nu,\rho,\mu)}(\rho^2T)\right\|_{H^s}\ge c\varepsilon-C(\rho/\nu)^{-k}|\mu|^{-s-k}|\log\nu|^C,$$

whenever $T<c|\log\nu|^c$.

Because $-\frac{n}{2}<s<0$ and $k>\frac{n}{2}$,

$$(\rho/\nu)^{-k}|\mu|^{-s-k}=\nu^{k-\frac{n}{2}(1+\frac{k}{s})-\sigma k+\sigma(1+\frac{k}{s})(\frac{\gamma}{2}-1)}\varepsilon^{-\frac{s+k}{s}}\to 0$$

as long as σ is chose sufficiently small. So we get that

$$\left\|u^{(a,\nu,\rho,\mu)}(\rho^2T)-u^{(a',\nu,\rho,\mu)}(\rho^2T)\right\|_{H^s}\ge c\varepsilon.$$

Finally, letting $\nu\to 0$, and $\rho^2T\to 0$. Theorem 4.1 is proved, which shows that the solution map is not uniformly continuous.

For $0=s<s_c$, we defer to the end of the section. This completes the proof.

Proof of Theorem 4.2

Case 1: $0<s<s_c$**.** We still apply the family $u^{(a,\nu,\rho,0)}$. Suppose that $a\in[1/2,2]$. We have

$$u^{(a,\nu,\rho,0)}(0,x)=\rho^{-n/2+\gamma/2-1}aw(\nu x/\rho).$$

One can compute that

$$\begin{aligned}\left\|u^{(a,\nu,\rho,0)}(0)\right\|_{H^s}^2=&a^2\rho^{-n+\gamma-2}(\rho/\nu)^{2n}\int|\widehat{w}(\rho\nu^{-1}\xi)|^2(1+|\xi|^2)^s\mathrm{d}\xi\\ \sim&\rho^{-n+\gamma-2}(\rho/\nu)^{n-2s}\int_{|\eta|\ge\rho\nu^{-1}}|\widehat{w}(\eta)|^2|\eta|^{2s}\mathrm{d}\eta\\ &+\rho^{-n+\gamma-2}(\rho/\nu)^{n}\int_{|\eta|\le\rho\nu^{-1}}|\widehat{w}(\eta)|^2\mathrm{d}\eta\\ =&\rho^{-n+\gamma-2}(\rho/\nu)^{n-2s}\int_{\mathbb{R}^n}|\widehat{w}(\eta)|^2|\eta|^{2s}\mathrm{d}\eta\\ &-\rho^{-n+\gamma-2}(\rho/\nu)^{n-2s}\int_{|\eta|\le\rho\nu^{-1}}|\widehat{w}(\eta)|^2\big((\rho/\nu)^{2s}-|\eta|^{2s}\big)\mathrm{d}\eta.\end{aligned}$$

For any $s > -n/2$,

$$\left\|u^{(a,\nu,\rho,0)}(0)\right\|_{H^s} = c\rho^{-n/2+\gamma/2-1}(\rho/\nu)^{n/2-s}\big(1+O((\rho/\nu)^{s+n/2})\big),$$

where $c \neq 0$. In particular, if $s > -n/2$ and $\rho < \nu$

$$\left\|u^{(a,\nu,\rho,0)}(0)\right\|_{H^s} \lesssim \rho^{-n/2+\gamma/2-1}(\rho/\nu)^{n/2-s} = \rho^{s_c-s}\nu^{s-n/2}.$$

Recall now that $s < s_c$ and that ε is assumed to satisfy

$$\rho^{s_c-s}\nu^{s-n/2} = \varepsilon.$$

In other words, we take $\rho = \nu^\sigma$ and $\sigma = \frac{n/2-s}{s_c-s} > 1$, then as $\nu \to 0$, $0 \leq \rho \leq \nu$, and

$$\left\|u^{(a,\nu,\rho,0)}(0)\right\|_{H^s} \leq C\varepsilon.$$

Now the solution of (4.2) $v^{(a,0)}(t,x) = aw(x)e^{-ia^2V*|w|^2t}$ satisfies that

$$\partial_x^j v^{(a,0)}(t,x) = aw(x)t^j[ia^2\partial_x(V*|w|^2)]^j e^{ia^2V*|w|^2t} + O(t^{j-1}).$$

By Lemma 4.1 and the log-convexity of Sobolev norm, we have that if $\nu \ll 1$ and $1 \ll t \ll c|\log\nu|^c$,

$$\left\|v^{(a,\nu)}(t)\right\|_{H^s} \sim t^s. \tag{4.5}$$

Now we verify that $u^{(a,\nu,\rho,0)}$ satisfies Theorem 4.2. As above, we have

$$\left\|u^{(a,\nu,\rho,0)}(\rho^2 t)\right\|_{H^s}^2 \geq c\rho^{-n/2+\gamma/2-1}(\rho/\nu)^{n/2-s}\left\|v^{(a,\nu)}(t)\right\|_{H^s} \geq c\varepsilon t^s.$$

If we choose t depending on ε large enough, ν, ρ sufficiently small depending on ε and t, Theorem 4.2 follows.

Case 2: $s < -n/2$. We assume now that $\hat{w}(\xi) = O(|\xi|^k)$ as $\xi \to 0$, for some $k > -s - n/2$. So if $\rho \leq \nu$,

$$\int_{\mathbb{R}^n} |\hat{w}(\eta)|^2|\eta|^{2s}\mathrm{d}\eta < \infty,$$

and

$$\int_{|\eta|\leq\rho\nu^{-1}} |\widehat{w}(\eta)|^2\big((\rho/\nu)^{2s} - |\eta|^{2s}\big)\mathrm{d}\eta \leq C(\rho\nu^{-1})^{n+2s+2k} \leq C < \infty.$$

So we get that $\left\|u^{(a,\nu,\rho,0)}(0)\right\|_{H^s} \leq C\varepsilon$ as Case 1. Moreover, w and a can be chosen so that

$$\left|\int v^{(a,0)}(1,y)\mathrm{d}y\right| \geq c,$$

for some constant $c > 0$. This means

$$\left|[\widehat{v^{(a,0)}(1)}](0)\right| \geq c.$$

Since $\widehat{v^{(a,0)}(1)}$ is rapidly decreasing, we thus see by continuity that for $|\xi| \le c$,

$$\left|[\widehat{v^{(a,0)}(1)}](\xi)\right| \ge c.$$

By Lemma 4.1, we have

$$\left|[\widehat{v^{(a,\nu)}(1)}](\xi) - [\widehat{v^{(a,0)}(1)}](\xi)\right| \le C\nu.$$

We can get similarly as before,

$$\left\|u^{(a,\nu,\rho,0)}(\rho^2 t)\right\|_{H^s} \ge c\varepsilon(\rho/\nu)^{n/2+s}.$$

As $\nu \to 0$, $\rho/\nu \to 0$, $(\rho/\nu)^{n/2+s} \to \infty$ since $s < -n/2$.

Case 3: $s = -n/2$. We can bound

$$\left\|u^{(a,\nu,\rho,0)}(\rho^2 t)\right\|_{H^{-n/2}} \ge c\varepsilon \log(\rho/\nu).$$

As $\nu \to 0$, $\rho/\nu \to 0$, $c\varepsilon \log(\rho/\nu) \to \infty$.

Finally we prove Theorem 4.1 for $0 = s < s_c$. In this case (4.5) does not hold because of L^2 conservation law. But we can still prove Theorem 4.2 as long as we modify the above procedure slightly. Suppose a, a' are distinct numbers in $[1/2, 2]$, we can find t, $(|a-a'|^{-1} \le t \le c|\log \nu|^c)$, such that

$$\left\|v^{(a,0)}(t) - v^{(a',0)}(t)\right\|_{L^2} \ge c > 0.$$

By Lemma 4.1, this implies that for ν small enough

$$\left\|v^{(a,\nu)}(t) - v^{(a',\nu)}(t)\right\|_{L^2} \ge c > 0.$$

Coming back to $u^{(a,\nu,\rho,0)}$, we can obtain that

$$\left\|u^{(a,\nu,\rho,0)}(\rho^2 t) - u^{(a',\nu,\rho,0)}(\rho^2 t)\right\|_{L^2} \ge c\varepsilon.$$

However a direct computation shows that

$$\left\|u^{(a,\nu,\rho,0)}(\rho^2 t)\right\|_{L^2} \le C\varepsilon,$$

and

$$\left\|u^{(a,\nu,\rho,0)}(\rho^2 t) - u^{(a',\nu,\rho,0)}(\rho^2 t)\right\|_{L^2} \le C\varepsilon|a-a'|.$$

Since $|a-a'|$ can be arbitrary small, this contradicts uniform continuity of the solution map. This completes the proof.

5. Nonexistence of the Nontrivial Asymptotically Free Solutions

Finally, we prove the nonexistence of the nontrivial asymptotically free solution in this section. In do so, we need the following dispersion of L^2-norm of the Schrödinger equation, see [33].

Proposition 5.1 *Let u be a global solution of (1.1) with $E(u) = E(\varphi) < \infty$. Let B be a compact subset of $\mathbb{R}^n$. Then, for any $R > 0$ and $T > 0$, we have*

$$\int_{B(R)} \left|u(T,x)\right|^2 \mathrm{d}x \geq \int_B \left|u(0,x)\right|^2 \mathrm{d}x - C(E(\varphi))\frac{T}{R},$$

where $B(R) := \left\{x \in \mathbb{R}^n; \exists\ y \in B,\ \text{s.t.}\ \ \left|x-y\right| \leq R\right\}$.

Now we can state the nonexistence of the nontrivial asymptotically free solution.

Theorem 5.1 *Assume that $0 < \gamma \leq 1$ for $n \geq 3$ and $0 < \gamma < \frac{n}{2}$ for $n = 1, 2$. Suppose that u is a smooth global solution in $C(0,\infty; H^1) \cap C^1(0,\infty; H^{-1})$ to (1.1) and there exists a smooth function $\varphi^+ \in H^1 \cap L^1$ such that*

$$\left\|u(t) - u^+(t)\right\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty, \tag{5.1}$$

where $u^+(t) = U(t)\varphi^+$. Then $u = u^+ = 0$.

Proof Let us define a function of $H(t)$* by

$$H(t) = \operatorname{sign}(\lambda)\ \operatorname{Im} < u(t), u^+(t) > .$$

Then $H(t)$ is uniformly bounded on t and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H(t) &= \operatorname{sign}(\lambda)\ \operatorname{Im} < u_t(t), u^+(t) > + \operatorname{sign}(\lambda)\ \operatorname{Im} < u(t), (u^+)_t(t) > \\
&= \operatorname{sign}(\lambda)\ \operatorname{Im} < i\Delta u - i\lambda I_{n-\gamma}(|u|^2)u, u^+ > + \operatorname{sign}(\lambda)\ \operatorname{Im} < u, i\Delta u^+ > \\
&= -|\lambda|\ \operatorname{Re} < I_{n-\gamma}(|u|^2)u, u^+ > \\
&= -|\lambda|\ \operatorname{Re}(J_1 + J_2 + J_3),
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= < I_{n-\gamma}(|u^+|^2)u^+, u^+ >; \\
J_2 &= < I_{n-\gamma}(|u|^2 - |u^+|^2)u^+, u^+ >; \\
J_3 &= < I_{n-\gamma}(|u|^2)(u - u^+), u^+ > .
\end{aligned}$$

Suppose $\varphi^+ \neq 0$. Then we will obtain a contradiction to the uniform boundedness of $H(t)$ on t.

*There is a mistake in the definition of $H(t)$ in [14]. We should replace Re with Im .

To estimate each J_i, we need the following time decay estimate.

$$\|U(t)\varphi^+\|_{L^\infty} \lesssim t^{-\frac{n}{2}}\|\varphi^+\|_{L^1}, \quad \text{for} \quad \varphi^+ \in L^1. \tag{5.2}$$

As for J_2, by Lemma 2.2, we have

$$\begin{aligned}
|J_2| &= |<|u|^2 - |u^+|^2, I_{n-\gamma}(|u^+|^2)>| \\
&\le \|u-u^+\|_{L^2}(\|u\|_{L^2}+\|u^+\|_{L^2})\|I_{n-\gamma}(|u^+|^2)\|_{L^\infty} \\
&\lesssim \|u-u^+\|_{L^2}(\|u\|_{L^2}+\|u^+\|_{L^2})\|u^+\|^2_{L^{\frac{2n}{n-\gamma}}} \\
&\lesssim \|u-u^+\|_{L^2}(\|u\|_{L^2}+\|u^+\|_{L^2})\|u^+\|_{L^2}^{2-\frac{2\gamma}{n}}\|u^+\|_{L^\infty}^{\frac{2\gamma}{n}}.
\end{aligned}$$

From (5.1) and (5.2), we have

$$|J_2| = o(|t|^{-\gamma}). \tag{5.3}$$

As for J_3, we have

$$\begin{aligned}
|J_3(t)| &= |<|u|^2, I_{n-\gamma}((\overline{u}-\overline{u^+})u^+)>| \\
&\le \|u\|^2_{L^2}\|(\overline{u}-\overline{u^+})u^+\|_{L^{\frac{n}{n-\gamma}}} \lesssim \|u\|^2_{L^2}\|u-u^+\|_{L^2}\|u^+\|_{L^{\frac{2n}{n-2\gamma}}} \\
&\lesssim \|u\|^2_{L^2}\|u-u^+\|_{L^2}\|u^+\|_{L^2}^{1-\frac{2\gamma}{n}}\|u^+\|_{L^\infty}^{\frac{2\gamma}{n}}.
\end{aligned}$$

From (5.1) and (5.2), we also have

$$|J_3| = o(|t|^{-\gamma}). \tag{5.4}$$

As for J_1, if $|x| \le At$ for some $A > 1$ which will be determined later, then for any $t > 0$

$$I_{n-\gamma}(|u^+|^2)(x) \ge \int_{|y|\le At} \frac{|u^+(y)|^2}{|x-y|^\gamma}\,\mathrm{d}y \ge \frac{1}{(2At)^\gamma}\int_{|y|\le At}|u^+(y)|^2\,\mathrm{d}y. \tag{5.5}$$

Now we prove that

$$\int_{|y|\le At}|u^+(y)|^2\,\mathrm{d}y \gtrsim \|\varphi^+\|^2_{L^2}, \tag{5.6}$$

for large t.

Choose a large R such that $\|\eta_R\varphi^+\|^2_{L^2} \ge \frac{2}{3}\|\varphi^+\|^2_{L^2}$ and $\|\nabla(\eta_R\varphi^+)\|^2_{L^2} \ge \frac{1}{2}\|\nabla\varphi^+\|^2_{L^2}$, where η_R is a smooth cut-off function supported in the ball of radius $2R$ with center at the origin. Then

$$\begin{aligned}
\|u^+\|^2_{L^2(|x|\le At)} &\ge \|U(t)(\eta_R\varphi^+)\|^2_{L^2(|x|\le At)} - \|U(t)((1-\eta_R)\varphi^+)\|^2_{L^2(|x|\le At)} \\
&\ge \|U(t)(\eta_R\varphi^+)\|^2_{L^2(|x|\le At)} - \|U(t)((1-\eta_R)\varphi^+)\|^2_{L^2} \\
&\ge \|U(t)(\eta_R\varphi^+)\|^2_{L^2(|x|\le At)} - \|\varphi^+\|^2_{L^2(|x|\ge R)}.
\end{aligned} \tag{5.7}$$

Hence choosing

$$A > 1 + \frac{\|\varphi^+\|_{L^2}^2}{2C(\|\nabla\varphi^+\|_{L^2}^2)},$$

and t is large enough such that

$$\frac{2R}{A-1} < t < \frac{2R\|\varphi^+\|_{L^2}^2}{3C(\|\nabla\varphi^+\|_{L^2}^2)}.$$

then we have from Proposition 5.1

$$\begin{aligned}\|U(t)(\eta_R\varphi^+)\|_{L^2(|x|<At)}^2 &\geq \|\eta_R\varphi^+\|_{L^2(|x|<2R)}^2 - C(\|\nabla(\eta_R\varphi^+)\|_{L^2}^2)\frac{t}{2R} \\ &\geq \|\eta_R\varphi^+\|_{L^2}^2 - C(\|\nabla\varphi^+\|_{L^2}^2)\frac{t}{2R} \\ &\geq \frac{2}{3}\|\varphi^+\|_{L^2}^2 - \frac{1}{3}\|\varphi^+\|_{L^2}^2 = \frac{1}{3}\|\varphi^+\|_{L^2}^2.\end{aligned}$$

Choosing t large enough, this helps us to get (5.6) and

$$J_1(t) \geq t^{-\gamma}. \tag{5.8}$$

by (5.5).

Now from (5.3), (5.4) and (5.4), we obtain that for t sufficient enough

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t) \gtrsim t^{-\gamma}.$$

This is a contradiction to the uniform boundedness of $H(t)$ on t.

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References

[1] Ginibre J, Velo G. Scattering theory in the energy space for a class of Hartree equations[G]. Nonlinear wave equations (Providence, RI, 1998), 29-60, Contemp Math, 263, Providence, RI: Amer Math Soc, 2000.

[2] Fröhlich J, Lenzmann E. Blow-up for nonlinear wave equations describing Boson stars[J]. *Comm Pure Appl Math*, 2007, **60**: 1691-1705.

[3] Ginibre J, Ozawa T. Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$[J]. *Comm Math Phys*, 1993, **151**: 619-645.

[4] Ginibre J, Velo G. On a class of nonlinear Schrödinger equations with nonlocal interactions[J]. *Math Z*, 1980, **170**: 109-136.

[5] Ginibre J, Velo G. Long range scattering and modified wave operators for some Hartree type equations[J]. *Rev Math Phys*, 2000, **12**(3): 361-429.

[6] Ginibre J, Velo G. Long range scattering and modified wave operators for some Hartree type equations II[J]. *Ann Henri Poincaré*, 2000, **1**(4): 753-800.
[7] Ginibre J, Velo G. Long range scattering and modified wave operators for some Hartree type equations. III: Gevrey spaces and low dimensions[J]. *J Differ Equations*, 2001, **175**(2): 415-501.
[8] Miao C. H^m-modified wave operator for nonlinear Hartree equation in the space dimensions $n \geq 2$[J]. *Acta Mathematica Sinica*, 1997, **13**(2): 247-268.
[9] Nakanishi K. Energy scattering for Hartree equations[J]. *Math Res Lett*, 1999, **6**: 107-118.
[10] Nawa H, Ozawa T. Nonlinear scattering with nonlocal interactions[J]. *Comm Math Phys*, 1992, **146**: 259-275.
[11] Christ M, Colliander J, Tao T. Ill-posedness for nonlinear Schrödinger and wave equations[EB]. arXiv:math.AP/0311048. to appear in Annales IHP.
[12] Kato T. On nonlinear Schrödinger equations II. H^s-solutions and unconditional wellposedness[J]. *J Anal Math*, 1995, **67**: 281-306.
[13] Kato T. Pertubation Theory for Linear Operators, 2nd ed[M]. Berlin: Springer-Verlag, 1980.
[14] Cho Y, Ozawa T. On the semirelativistic Hartree-type equation[J]. *SIAM J Math Anal*, 2006, **38**(4): 1060-1074.
[15] Cazenave T. Semilinear Schrödinger Equation[M]. Courant Lecture Notes in Math. 10, New York University, Courant Institute of Mathematical Sciences, New York, Providence, RI: American Mathematical Society, 2003.
[16] Cazenave T, Weissler F. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s[J]. *Nonlinear Anal T M A*, 1990, **14**(10): 807-836.
[17] Miao C, Xu G, Zhao L. Global well-posedness and scattering for the energy-critical, defocusing Hartree equation for radial data[J]. *J Functional Analysis*, 2007, **253**: 605-627.
[18] Miao C, Xu G, Zhao L. Global well-posedness and scattering for the energy-critical, defocusing Hartree equation in $\mathbb{R}^{1+n}$[EB]. arXiv:math.AP/0707.3254.
[19] Kurata K, Ogawa T. Remarks on blowing-up of solutions for some nonlinear Schrödinger equations[J]. *Tokyo J Math*, 1990, **13**(2): 399-419.
[20] Lenzmann E. Well-posedness for semi-relativistic Hartree equations of critical type[J]. *Math Phys Anal Geom*, 2007, **10**: 43-64.
[21] Weinstein M I. Nonlinear Schrödinger equations and sharp interpolation estimates[J]. *Comm Math Phys*, 1983, **87**: 567-576.
[22] Christ M, Colliander J, Tao T. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations[J]. *Amer J Math*, 2003, **125**: 1235-1293.
[23] Zhang Jian. Sharp conditions of global existence for nonlinear Schrödinger and Klein-Gordon equations[J]. *Nonlinear Analysis*, 2002, **48**: 191-207.
[24] Nakamura M, Ozawa T. Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces[J]. *Reviews in Math Physics*, 1997, **9**(3): 397-410.
[25] Pecher H. Nonlinear small data scattering for the wave and Klein-Gordon equation[J]. *Math Z*, 1984, **185**: 261-270.
[26] Pecher H. Low energy scattering for nonlinear Klein-Gordon equations[J]. *J Funct Anal*, 1985, **63**: 101-122.
[27] Strauss W A. Nonlinear scattering theory at low energy[J]. *J Funct Anal*, 1981, **41**: 110-133.

[28] Strauss W A. Nonlinear scattering theory at low energy: Sequel[J]. *J Funct Anal*, 1981, **43**: 281-293.

[29] Colliander J, Keel M, Staffilani G, Takaoka H, Tao T. Global well-posedness and scattering for the energy-cirtical nonlinear Schrödinger equation in $\mathbb{R}^3$[J]. *Ann of Math*, 2007, **166**: 1-100.

[30] Ryckman E, Visan M. Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$[J]. *Amer J Math*, 2007, **129**: 1-60.

[31] Visan M. The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions[J]. *Duke Math J*, 2007, **138**: 281-374.

[32] Miao C, Xu G. Energy scattering for the Klein-Gordon equation with a cubic convolution nonlinearity[EB]. arXiv:math.AP/0612028.

[33] Bourgain J. Scattering in the energy space and below for 3D NLS[J]. *J Anal Math*, 1998, **75**: 267-297.

[34] Hayashi N, Tsutsumi Y. Scattering theory for the Hartree equations[J]. *Ann Inst H Poincare Phys Theorique*, 1987, **61**: 187-213.

[35] Kenig C E, Merle F. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case[J]. *Invent Math*, 2006, **166**: 645-675.

[36] Kenig C E, Merle F. Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation[EB]. arXiv:math.AP/0610801.

[37] Nakanishi K. Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2[J]. *J Funct Anal*, 1999, **169**: 201-225.

[38] Tzvetkov N. Ill-posedness issues for nonlinear dispersive equations[EB]. arXiv:math. AP/0411455.