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Dirichlet Eigenvalue Ratios for the p-sub-Laplacian in the Carnot Group

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Abstract. We prove some new Hardy type inequalities on the bounded domain with smooth boundary in the Carnot group. Several estimates of the first and second Dirichlet eigenvalues for the p-sub-Laplacian are established.

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1 Introduction

Various boundary value problems on bounded domains in the Euclidean space for the Laplacian and p-Laplacian and their applications in nonlinear problems have been studied extensively, see [1,2] and references therein. Boundary value problems (including the Dirichlet eigenvalue problem) for the sub-Laplacian in the Heisenberg group and Carnot groups have also received some attention in recent years, see, e.g., [3, 4] and references therein. However, we have not seen the results for the Dirichlet eigenvalue problem of the p-sub-Laplacian ($p>1$) in the Carnot group.

In this paper, we consider the ratio of the first and second eigenvalues for the Dirichlet problem,

$$\begin{cases} -\Delta_{G,p}u=\lambda|u|^{p-2}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in the Carnot group G, $\Delta_{G,p}$ is the p-sub-Laplacian in G with the form

$$\Delta_{G,p}u=\nabla_G\cdot(|\nabla_G u|^{p-2}\nabla_G u),\ \ p>1.$$

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Here $\nabla_G u = (X_1 u,\cdots,X_m u)$, $\{X_j\}_{j=1}^m$is a left-invariant basis of the first floor of the Lie algebra corresponding to the Carnot group.

Definition 1.1. *A pair* $(u,\lambda)\in W_0^{1,p}(\Omega)\times\mathbb{R}$ *is a weak solution of the Dirichlet problem* (1.1) *provided that*

$$\int_\Omega |\nabla_G u|^{p-2}\langle\nabla_G u,\nabla_G v\rangle \mathrm{d}x=\lambda\int_\Omega |u|^{p-2}uv\mathrm{d}x, \tag{1.2}$$

for any $v\in W_0^{1,p}(\Omega)$, *where such a pair*(u,λ), *with* u *nontrivial, is called an eigenpair;* λ *is an eigenvalue and* u *is called an associated eigenfunction. By choosing* $v=u$ *in* (1.2), *it follows that all eigenvalues* λ *are nonnegative.*

The arguments as in the Euclidean space show easily that the existence of eigenvalues, simplicity of the first eigenvalue in (1.1) and the variational characterization of the second eigenvalue are true. The authors in [2] provided the fundamental eigenvalue ratio of the p-Laplacian in the Euclidean space. We hope to give such estimates for the p-sub-Laplacian in the Carnot group G. In Section 2, some relevant facts on the Carnot group are presented. Nevertheless, when the method in [2] is used to our case, some new difficulties appear. For our purpose, in Section 3, several Hardy-type inequalities are established. Note that D'Ambrosio [5] obtained the following inequality on bounded domains in G

$$c\int_\Omega \frac{|u|^p}{\phi^p}|\nabla_G\phi|^p\mathrm{d}x\le\int_\Omega|\nabla_G u|^p\mathrm{d}x,$$

where $u\in W_0^{1,p}(\Omega)$, ϕ is some weight function such that

$$-\Delta_{G,p}\phi=\nabla_G(|\nabla_G\phi|^{p-2}\cdot\nabla_G\phi)\ge 0.$$

Because of the appearance of the weight $|\nabla_G\phi|^p$, we find that such class of inequalities cannot be applied to estimate eigenvalues in our case. We also relate a useful property of the Sobolev space $W_0^{1,p}(\Omega)$ in this section. The final section is devoted to the estimates for the first and second eigenvalues based on the results above.

2 Preliminaries

We collect some notations and properties for the Carnot group (see, e.g., [6–8]).

The Carnot group $G=(\mathbb{R}^n,\cdot)$ is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ possesses a stratification, i.e., there exist linear subspaces $V_1,\cdots,V_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g}=V_1\oplus\cdots\cdots\oplus V_k,\quad [V_1,V_i]=V_{i+1},\quad i=1,\cdots,k-1,\ \text{and}\ [V_1,V_k]=0,$$

where $[V_1,V_i]$ is the subspace of $\mathfrak{g}$ generated by the elements $[X,Y]$ with $X\in V_1, Y\in V_i$. In this way we get a Carnot group of step k and the integer $k\ge 1$ is the step of G.

The integer

$$Q=\sum_{j=1}^{n}\alpha_j=\sum_{j=1}^{k}j\,\dim(V_j)$$

is the homogeneous dimension of the group . Let $X_1,\cdots,X_m$ be a system of left invariant vector fields with $X_j=-X_j^*$. Let d be the C-C metric induced on $\mathbb{R}^n$ by the system.

The horizontal gradient on G is denoted by

$$\nabla_G=(X_1,\cdots,X_m).$$

The p-sub-Laplacian on G is

$$\Delta_{G,p}u=\nabla_G\cdot(|\nabla_G u|^{p-2}\nabla_G u),\quad p>1 \tag{2.1}$$

and the sub-Laplacian is

$$\Delta_G u=\sum_{j=1}^{m}X_j^2 u,$$

which forms a second-order partial differential operator.

For elements x,y in G, we define the quasi-distance as $\rho(x,y)=|y^{-1}\cdot x|$ and indicate by

$$B_G(x,R)=\{y\in G|\rho(x,y)<R\}$$

an open ball of center x with radius R.

We denote the set of all functions satisfying $X_ju,Y_ju\in L^p(\Omega)$ by $W^{1,p}(\Omega)$ and the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|=\Big(\int_\Omega|\nabla_G u|^p\Big)^{\frac{1}{p}}$$

by $W_0^{1,p}(\Omega)$.The notation $(W_0^{1,p}(\Omega))^*$ means the conjugate space of $W_0^{1,p}(\Omega)$.

To depict a useful property of the Sobolev space $W_0^{1,p}(\Omega)$, we introduce the following:

Definition 2.1. *If a function $\omega_h(x,y)$ on the Carnot group satisfies*

(i) $\omega_h(x,y)=\omega_h(d(x,y))$, *i.e., $\omega_h(x,y)$ relies only on the distance $d=d(x,y)$ and the parameter h;*

(ii) $\omega_h(x,y)>0$, *if $d<h$; $\omega_h(x,y)=0$, if $d\geq h$;*

(iii) $\omega_h(x,y)$ *is infinitely differentiable for all variables;*

(iv) *if $\int_G\omega_h(x,y)\mathrm{d}x=\int_G\omega_h(y,x)\mathrm{d}x=1$, for all $x\in G$, then we call that $\omega_h(x,y)$ is a Sobolev kernel-function.*

Clearly, the function

$$\omega_h(x,y)=\begin{cases}(\Re h^n)^{-1}\exp\Big(\dfrac{d^2}{d^2-h^2}\Big), & d<h,\\ 0, & d\geq h\end{cases}$$

is a Sobolev kernel-function, where

$$\Re=\int_{B_G(0,1)}\exp\Big(\frac{d^2}{d^2-h^2}\Big)\mathrm{d}x.$$

Definition 2.2. *Given a function $u(x)\in L^p(\Omega)$, $p\geq 1$, we extend $u(x)$ to the outside of Ω by setting $u(x)=0$ if $x\notin\Omega$. Let $\omega_h(x,y)$ be a Sobolev kernel-function. We call*

$$u_h(x)=\int_G u(y)\omega_h(x,y)\mathrm{d}y$$

a ω_h-mean-function with respect to $u(x)$.

Proposition 2.1. *Let $u\in C(\overline{\Omega})\cap W^{1,p}(\Omega)(p\geq 1)$ and $u|_{\partial\Omega}=0$. Then $u\in W_0^{1,p}(\Omega)$.*

Proof. The proof is similar to the one in the Euclidean space, see [9] or [10]. □

3 Hardy-type inequalities

Let K be a closed set in the Carnot group G. The Carnot-Carathéodory distance between $x\in G$ and K is defined as

$$d_K(x)=\inf_{y\in K}d(x,y).$$

Monti and Serra Cassano [7] have proved that the C-C distance between x and K satisfies the eikonal equation with respect to the generalized gradient

$$|\nabla_G d_k(x)|=1,\quad \text{a.e. } x\in G\backslash K.$$

This implies that, if $S=\partial\Omega$ is the boundary of Ω, and S is piecewise C^2, then

$$|\nabla_G\delta(x)|=1,\quad \text{a.e. } x\in\Omega, \tag{3.1}$$

where we denote by $\delta(x)$ the C-C distance between x and S.

Lemma 3.1. *Let ζ_1,ζ_2 be two nonnegative real numbers. Then*

$$(p-1)\zeta_2^p-p\zeta_2^{p-1}\zeta_1\geq-\zeta_1^p.$$

The proof of Lemma 3.1 is elementary. Now we give a Hardy-type inequality on the domain Ω.

Theorem 3.1. *Let $\delta=\delta(x)$ be as above. Then*

$$\int_\Omega \frac{|u|^p}{\delta^p}\mathrm{d}x \le \Big(\frac{p}{p-1}\Big)^p \int_\Omega |\nabla_G u|^p \mathrm{d}x + \frac{p}{p-1}\int_\Omega \frac{|u|^p \Delta_G \delta}{\delta^{p-1}}\mathrm{d}x, \tag{3.2}$$

for any $u\in W_0^{1,p}(\Omega)$ such that the integrand in the last term above belong to $L^1(\Omega)$.

Proof. Let $u\not\equiv 0$ and consider the integral

$$I=\int_\Omega \frac{\nabla_G\delta}{\delta^{p-1}}\cdot\nabla_G(|u|^p)\mathrm{d}x.$$

By the Hölder inequality and (3.1), we get

$$\begin{aligned}|I| &= \Big|p\int_\Omega \frac{\nabla_G\delta}{\delta^{p-1}}\cdot|u|^{p-2}u\cdot\nabla_G u\mathrm{d}x\Big|\\ &\le p\int_\Omega \frac{|u|^{p-1}}{\delta^{p-1}}|\nabla_G u|\mathrm{d}x\\ &\le p\Big(\int_\Omega |\nabla_G u|^p\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_\Omega \frac{|u|^p}{\delta^p}\mathrm{d}x\Big)^{\frac{1}{p'}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. On the other hand, the integration by parts and (3.1) yield

$$\begin{aligned}|I| &= \Big|-\int_\Omega |u|^p\Big(\frac{\Delta_G\delta}{\delta^{p-1}}-(p-1)\frac{|\nabla_G\delta|^2}{\delta^p}\Big)\mathrm{d}x\Big|\\ &=(p-1)\Big|\int_\Omega |u|^p\Big(\frac{1}{\delta^p}-\frac{\Delta_G\delta}{(p-1)\delta^{p-1}}\Big)\mathrm{d}x\Big|.\end{aligned}$$

Consequently,

$$\frac{p}{p-1}\Big(\int_\Omega|\nabla_G u|^p\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_\Omega\frac{|u|^p}{\delta^p}\mathrm{d}x\Big)^{\frac{p-1}{p}} \ge \Big|\int_\Omega |u|^p\Big(\frac{1}{\delta^p}-\frac{\Delta_G\delta}{(p-1)\delta^{p-1}}\Big)\mathrm{d}x\Big|,$$

which leads to

$$\Big(\frac{p}{p-1}\Big)^p\Big(\int_\Omega|\nabla_G u|^p\mathrm{d}x\Big) \ge \frac{\Big|\int_\Omega |u|^p\cdot\Big(\frac{1}{\delta^p}-\frac{\Delta_G\delta}{(p-1)\delta^{p-1}}\Big)\mathrm{d}x\Big|^p}{\Big(\int_\Omega\frac{|u|^p}{\delta^p}\mathrm{d}x\Big)^{p-1}}.$$

By Lemma 3.1,

$$\begin{aligned}\Big(\frac{p}{p-1}\Big)^p\Big(\int_\Omega|\nabla_G u|^p\mathrm{d}x\Big) &\ge p\int_\Omega\Big(\frac{|u|^p}{\delta^p}-\frac{|u|^p\cdot\Delta_G\delta}{(p-1)\delta^{p-1}}\Big)\mathrm{d}x-(p-1)\int_\Omega\frac{|u|^p}{\delta^p}\mathrm{d}x\\ &=\int_\Omega\frac{|u|^p}{\delta^p}\mathrm{d}x-\frac{p}{p-1}\int_\Omega\frac{|u|^p\Delta_G\delta}{\delta^{p-1}}\mathrm{d}x.\end{aligned}$$

The statement is proved. $\square$

As a consequence of Theorem 3.1, we have:

Theorem 3.2. *If δ satisfies $-\Delta_G\delta\geq 0$, then the following inequality holds:*

$$\int_\Omega \frac{|u|^p}{\delta^p}\mathrm{d}x\leq\Big(\frac{p}{p-1}\Big)^p\int_\Omega|\nabla_G u|^p\mathrm{d}x. \tag{3.3}$$

Remark 3.1. The domains with suitable assumptions in Theorems 3.1 and 3.2 have been considered in some papers (see, e.g., [11]).

4 The estimates of eigenvalues

In this section we offer some estimates for the first and second eigenvalues of (1.1) by using the previous results. The following lemma is elementary.

Lemma 4.1. *There are constants $\hat{m}\geq 1$ and $\hat{k}>0$ such that for all $N\in\mathbb{N}$ and $a\in\mathbb{R}^N, b\in\mathbb{R}^N$,*

$$|a+b|^p\leq\hat{m}^p|a|^p+\hat{k}\big(|b|^p+p|b|^{p-2}b\cdot a\big). \tag{4.1}$$

Moreover, if $N\geq 2$, then

$$\begin{aligned}&\hat{m}=2^{\frac{2-p}{2p}}(p-1),\quad \hat{k}=2^{\frac{p-2}{2}}p^{2-p}(p-1)^{p-1},\ \text{ if } p\geq 2;\\ &\hat{m}=2^{\frac{2-p}{2p}}m_0,\quad \hat{k}=1,\ \text{ if } 1<p<2,\end{aligned}$$

where m_0 is the constant defined by

$$m_0^p=\max_{0\leq x\leq 1}\big((p-x)x^{p-1}+(1-x)^p\big),\quad x\in\mathbb{R}.$$

Lemma 4.2. *With the constants $\hat{m}$ and $\hat{k}$ in Lemma 4.1, for any $\varphi>0$, a.e. $x\in\Omega$ (Ω is a piecewise C^1 domain in G) and any $u\in W_0^{1,p}(\Omega)$ satisfying $\Delta_{G,p}u\in L^{p'}(\Omega)$, we have*

$$\int_\Omega|\nabla_G(\varphi u)|^p\mathrm{d}x\leq\hat{m}^p\int_\Omega|u\cdot\nabla_G\varphi|^p\mathrm{d}x+\hat{k}\int_\Omega u\varphi^p(-\Delta_{G,p}u)\mathrm{d}x. \tag{4.2}$$

Proof. By choosing $a=u\nabla_G\varphi$, $b=\varphi\nabla_G u$ in (4.1), we get

$$\begin{aligned}&\int_\Omega|u\nabla_G\varphi+\varphi\nabla_G u|^p\mathrm{d}x\\ \leq&\hat{m}^p\int_\Omega|u\nabla_G\varphi|^p\mathrm{d}x+\hat{k}\int_\Omega[|\varphi\nabla_G u|^p+p\varphi^{p-1}u|\nabla_G u|^{p-2}\nabla_G u\cdot\nabla_G\varphi]\mathrm{d}x.\end{aligned} \tag{4.3}$$

Observe that

$$\begin{aligned}\int_\Omega|\varphi\nabla_G u|^p\mathrm{d}x=&\int_\Omega\varphi^p|\nabla_G u|^{p-2}\nabla_G u\cdot\nabla_G u\mathrm{d}x\\ =&-p\int_\Omega\varphi^{p-1}u|\nabla_G u|^{p-2}\nabla_G u\cdot\nabla_G\varphi\mathrm{d}x-\int_\Omega u\varphi^p(\Delta_{G,p}u)\mathrm{d}x.\end{aligned}$$

This result, together with (4.3), leads to the desired estimate (4.2). □

Theorem 4.1. *Let u_1 be the eigenfunction associated with λ_1 such that*

$$\int_\Omega u_1^p \mathrm{d}x = 1, \quad \frac{|u_1|^p \Delta_G \delta}{\delta^{p-1}} \in L^1(\Omega).$$

Then

$$\lambda_2 \le \Big(\hat{k} + \hat{m}^p \big(\frac{p}{p-1}\big)^p\Big)\lambda_1 + \frac{p\hat{m}^p}{p-1}\int_\Omega \frac{|u_1|^p \Delta_G \delta}{\delta^{p-1}} \mathrm{d}x. \tag{4.4}$$

Proof. Set $\sigma = \{x = (x_1, \cdots, x_n) \in \Omega | x_k < y_k,\ y = (y_1, \cdots, y_n) \in \partial\Omega\}$ and $\Gamma = \lambda_2 - \hat{k}\lambda_1$. Assume that a point $y \in \partial\Omega$ is chosen so that $\mathrm{meas}(\sigma) > 0$ and $\mathrm{meas}(\Omega \setminus \sigma) > 0$. Note that $\delta(x) \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ and $\delta(x)|_{\partial\Omega} = 0$. Then $\delta(x) \in W_0^{1,p}(\Omega)$ by Proposition 2.1. We define a set

$$D = \{u_1 \cdot G_{\alpha,\beta} : \alpha \in \mathbb{R},\ \beta \in \mathbb{R};\ |\alpha|^p + |\beta|^p = 1\},$$

where

$$G_{\alpha,\beta} = \delta(x)(\alpha\chi_\sigma + \beta\chi_{\Omega\setminus\sigma}).$$

Similar to the proof of Theorem 1.5 of [12], we get

$$\lambda_2 \le \max_{|\alpha|^p + |\beta|^p = 1} \frac{\displaystyle\int_\Omega |\nabla_G (u_1 G_{\alpha,\beta})|^p \mathrm{d}x}{\displaystyle\int_\Omega |u_1 G_{\alpha.\beta}|^p \mathrm{d}x}$$

and from Lemma 4.2,

$$\begin{aligned}\int_\Omega |\nabla_G (u_1 G_{\alpha,\beta})|^p \mathrm{d}x &\le \hat{m}^p \int_\Omega |\nabla_G (G_{\alpha,\beta}) u_1|^p \mathrm{d}x + \hat{k}\int_\Omega u_1 |G_{\alpha,\beta}|^p (-\Delta_{G,p} u_1) \mathrm{d}x \\ &= \hat{m}^p \int_\Omega |\nabla_G (G_{\alpha,\beta}) u_1|^p \mathrm{d}x + \hat{k}\lambda_1 \int_\Omega |u_1 G_{\alpha,\beta}|^p \mathrm{d}x.\end{aligned}$$

Hence,

$$\begin{aligned}\Gamma &\le \max_{|\alpha|^p + |\beta|^p = 1} \frac{\displaystyle\int_\Omega |\nabla_G (u_1 G_{\alpha,\beta})|^p \mathrm{d}x}{\displaystyle\int_\Omega |u_1 G_{\alpha,\beta}|^p \mathrm{d}x} - \hat{k}\lambda_1 \\ &\le \max_{|\alpha|^p + |\beta|^p = 1} \frac{\hat{m}^p \displaystyle\int_\Omega |\nabla_G (G_{\alpha,\beta}) u_1|^p \mathrm{d}x + \hat{k}\lambda_1 \int_\Omega |G_{\alpha,\beta}|^p u_1 \mathrm{d}x}{\displaystyle\int_\Omega |u_1 G_{\alpha,\beta}|^p \mathrm{d}x} - \hat{k}\lambda_1 \\ &= \max_{|\alpha|^p + |\beta|^p = 1} \hat{m}^p \frac{\displaystyle\int_\Omega |\nabla_G (G_{\alpha,\beta}) u_1|^p \mathrm{d}x}{\displaystyle\int_\Omega |u_1 G_{\alpha,\beta}|^p \mathrm{d}x} \\ &= \hat{m}^p \max_{|\alpha|^p + |\beta|^p = 1} \frac{|\alpha|^p \displaystyle\int_\sigma |u_1 \nabla_G \delta|^p \mathrm{d}x + |\beta|^p \int_{\Omega\setminus\sigma} |u_1 \nabla_G \delta|^p \mathrm{d}x}{|\alpha|^p \displaystyle\int_\sigma |u_1 \delta(x)|^p \mathrm{d}x + |\beta|^p \int_{\Omega\setminus\sigma} |u_1 \delta(x)|^p \mathrm{d}x}.\end{aligned}$$

Let $s=|\alpha|^p$. The problem above is turned to maximize an expression

$$\theta(s)=\frac{as+b(1-s)}{es+f(1-s)} \quad \text{for } 0\le s\le 1.$$

If $\theta(s)$ is a constant or equals to

$$\frac{b}{f}=\frac{a}{e}=\frac{a+b}{e+f},$$

then its derivative is always zero. On the other hand, if $\theta(s)$ is not a constant, then it achieves the maximum b/f when $s=0$, or a/e when $s=1$. It concludes that

$$\Gamma\le\hat{m}^p\max_{|\alpha|^p+|\beta|^p=1}\left\{\frac{\int_\sigma u_1^p\mathrm{d}x}{\int_\sigma u_1^p\delta^p\mathrm{d}x},\frac{\int_{\Omega\backslash\sigma}u_1^p\mathrm{d}x}{\int_{\Omega\backslash\sigma}u_1^p\delta^p\mathrm{d}x}\right\}.$$

By continuity, there is a point y on the boundary such that

$$\frac{\int_\sigma u_1^p\mathrm{d}x}{\int_\sigma u_1^p\delta^p\mathrm{d}x}=\frac{\int_{\Omega\backslash\sigma}u_1^p\mathrm{d}x}{\int_{\Omega\backslash\sigma}u_1^p\delta^p\mathrm{d}x}=\frac{\int_\Omega u_1^p\mathrm{d}x}{\int_\Omega u_1^p\delta^p\mathrm{d}x}.$$

Therefore, we have

$$\Gamma\le\frac{\hat{m}^p}{\int_\Omega u_1^p\delta^p\mathrm{d}x}. \tag{4.5}$$

It follows from (3.2) that

$$\begin{aligned}1=&\Big(\int_\Omega u_1^p\mathrm{d}x\Big)^2\\ \le&\int_\Omega u_1^p\delta^p\mathrm{d}x\cdot\int_\Omega u_1^p\delta^{-p}\mathrm{d}x\\ \le&\Big((\frac{p}{p-1})^p\int_\Omega|\nabla_G u_1|^p\mathrm{d}x+\frac{p}{p-1}\int_\Omega\frac{|u|^p\Delta_G\delta}{\delta^{p-1}}\mathrm{d}x\Big)\int_\Omega u_1^p\delta^p\mathrm{d}x,\end{aligned}$$

which leads to

$$\begin{aligned}\frac{1}{\int_\Omega u_1^p\delta^p\mathrm{d}x}\le&\Big(\frac{p}{p-1}\Big)^p\int_\Omega|\nabla_G u_1|^p\mathrm{d}x+\frac{p}{p-1}\int_\Omega\frac{|u|^p\Delta_G\delta}{\delta^{p-1}}\mathrm{d}x\\ =&\Big(\frac{p}{p-1}\Big)^p\lambda_1+\frac{p}{p-1}\int_\Omega\frac{|u|^p\Delta_G\delta}{\delta^{p-1}}\mathrm{d}x.\end{aligned}$$

Consequently,

$$\begin{aligned}\Gamma=\lambda_2-\hat{k}\lambda_1 &\le \hat{m}^p\frac{1}{\displaystyle\int_\Omega u_1^p\delta^p\mathrm{d}x}\\ &\le \hat{m}^p\Big(\big(\frac{p}{p-1}\big)^p\lambda_1+\frac{p}{p-1}\int_\Omega\frac{|u|^p\Delta_G\delta}{\delta^{p-1}}\mathrm{d}x\Big),\end{aligned}$$

which leads to the desired estimate (4.4). □

Theorem 4.2. *If the C-C distance δ satisfies $-\Delta_G\delta\ge 0$, then*

$$\Gamma=\lambda_2-\hat{k}\lambda_1\le\hat{m}^p\Big(\frac{p}{p-1}\Big)^p\lambda_1, \tag{4.6}$$

or

$$\frac{\lambda_2}{\lambda_1}\le\hat{k}+\hat{m}^p\Big(\frac{p}{p-1}\Big)^p. \tag{4.7}$$

Proof. By Theorem 3.2, we see that

$$\begin{aligned}1=\Big(\int_\Omega u_1^p\mathrm{d}x\Big)^2&\le\int_\Omega u_1^p\delta^p\mathrm{d}x\cdot\int_\Omega u_1^p\delta^{-p}\mathrm{d}x\\ &\le\Big(\frac{p}{p-1}\Big)^p\int_\Omega|\nabla_G u_1|^p\mathrm{d}x\cdot\int_\Omega u_1^p\delta^p\mathrm{d}x,\end{aligned}$$

and so

$$\frac{1}{\displaystyle\int_\Omega u_1^p\delta^p\mathrm{d}x}\le\Big(\frac{p}{p-1}\Big)^p\int_\Omega|\nabla_G u_1|^p\mathrm{d}x=\Big(\frac{p}{p-1}\Big)^p\lambda_1. \tag{4.8}$$

This, together with (4.5), completes the proof. □

Remark 4.1. We note that the estimates in Theorem 4.2 are not dependent on the homogeneous dimension Q of the Carnot group. As a consequence, we obtain immediately the estimate for the eigenvalue ratio of the sub-Laplacian.

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References

[1] Lê A., Eigenvalue problems for the p-Laplacian. *Nonlinear Anal.*, 2006, **64**: 1057–1099.

[2] Fleckinger J., Harrell II E. M. and de Thélin F., On the fundamental eigenvalue ratio of the p-Laplacian. *Bull. Sci. Math.*, 2007, **131**(7): 613–619.

[3] Niu P. C. and Zhang H. Q., Payne-Polya-Weinberger type inequalities for eigenvalues of nonelliptic operators. *Pacific J. Math.*, 2003, **208**(2): 325–345.

[4] Luo X. B. and Niu P. C., Eigenvalue problems for square sum operators consisting of vector fields. *Math. Appl.*, 1997, **10**(4): 101–104.

[5] D'Ambrosio L., Hardy type inequalities related to degenerate elliptic differential operators. *Annali della Scuola Normale Superiore di Pisa*, 2005, **4**(3): 451–486.

[6] Folland G. B., Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Math.*, 1975, **13**: 161–207.

[7] Monti R. and Serra Cassano F., Surface measures in Carnot-Carathé spaces. *Calc. Var. Partial Diff. Eqs.*, 2001, **13**(3): 339–376.

[8] Stein E. M., Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton Univ. Press, 1993.

[9] Brézis H., Analyse Fonctionnelle: Théorie et Application. Paris: Masson, 1983.

[10] Wang X. D., Liang J. T. and Rong H., Sobolev Space Theory. Beijing: Science Press, 2004.

[11] Arcozzi N. and Ferrari F., Metric normal and distance function in the Heisenberg group. *Math. Z.*, 2007, **256**(3): 661-684.

[12] Bonder J. F. and Rossi J. D., A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding. *Publ. Mat.*, 2002, **46**: 221-235.