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Remark on Exponential Decay of Ground States for N-Laplacian Equations

ZHAO Chunshan*

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, USA.

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Abstract. We study exponential decay property of radial ground states to a class of N-Laplacian elliptic equations in the whole space $\mathbb{R}^N$. Their decay rates as $|x| \to \infty$ are obtained explicitly.

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1 Introduction

We consider the exponential decay of radial ground states of a class of N-Laplacian elliptic equations as follows:

$$\Delta_N u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad N \geq 2, \tag{1.1}$$

where $\Delta_N u = \operatorname{div}(|Du|^{N-2} Du)$ is the degenerate N-Laplacian. Here by a ground state we mean a non-negative non-trivial C^1 distribution solution of (1.1) which tends to zero at infinity. The particular interest in this problem is that the order of the Laplacian is the same as the dimension of the underlying space. For the classical case of this problem, i.e., $N=2$, (1.1) reduces to

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^2, \tag{1.2}$$

Under suitable conditions on f it was shown in [1] that the ground state for (1.2) satisfies

$$u(x) = u(|x|) = u(r) > 0, \quad u(0) = \max_{x \in \mathbb{R}^2} u(x),$$
$$u'(0) = 0 \quad \text{and} \quad u'(r) < 0 \quad \text{for} \quad r \in (0, \infty).$$

*Corresponding author. *Email address:* `czhao@GeorgiaSouthern.edu` (C. Zhao)

Moreover,

$$\lim_{r\to\infty} u(r)\sqrt{r}e^r = C \tag{1.3}$$

with some constant $0<C<\infty$. Such precise estimates of asymptotes as $|x|\to\infty$ have been proved to be very useful. For applications of such estimates, readers can refer to [1–5] etc. In an earlier work [6], we considered the exponential decay of ground states of the m-Laplacian equation

$$\Delta_m u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad N>m>1$$

under certain assumptions on f. It has been known that when $1<m<N$, Pohozaev-type restrictions on the nonlinear term f are needed to show the existence of ground states [7], and for $m>N$ no growth conditions are required [8]. So we can regard (1.1) as a transition or borderline case.

Throughout this paper we make the following assumptions on f:

(f_1): f: $[0,\infty)\to\mathbb{R}$ is locally Lipschtiz continuous and there exist positive constants K and α such that

$$f(t)+Kt^{N-1} = \mathcal{O}(t^{N-1+\alpha}) \quad \text{as } t\downarrow 0.$$

(f_2): There exists $\beta>0$ such that

$$F(t)=\int_0^t f(s)\mathrm{d}s<0 \quad \text{on } (0,\beta), \quad F(\beta)=0, \quad f(t)>0 \quad \text{for } t\geq\beta.$$

(f_3): For some $\gamma\geq\beta$ we have $f\in C^1[\gamma,\infty)$ and $f'(t)\geq 0$ for $t>\gamma$.

Note that an example of f satisfying all above assumptions is $f(t) = -t^{N-1}+t^p$ with $N-1<p<\infty$. Also note that under assumptions (f_1)-(f_3), the existence of a radial ground state is guaranteed and moreover if $r=|x|$ the function $u=u(r)$ satisfying $u'(r)<0$ for all $r>0$ such that $u(r)>0$ (see [9]). Our result can be stated as follows:

Theorem 1.1. *Let $u(x)=u(r)$ be a positive radial ground state for (1.1) and $f(t)$ satisfies assumptions (f_1)-(f_3). Then there exists a sequence of constants $\{C_i\}$ $(i=1, 2,\cdots)$ such that for any $l=1, 2,\cdots$,*

$$\Big(-\frac{u'}{u}\Big)^{N-1} = \Big(\frac{K}{N-1}\Big)^{\frac{N-1}{N}} + \frac{C_1}{r} + \frac{C_2}{r^2} + \cdots + \frac{C_l}{r^l} + \mathcal{O}\Big(\frac{1}{r^{l+1}}\Big) \quad \text{as} \quad r\to\infty,$$

where $\{C_i\}$ $(i=1, 2,\cdots)$ are determined by

$$C_1 = \frac{N-1}{N}\Big(\frac{K}{N-1}\Big)^{\frac{N-2}{N}}, \quad C_2 = \frac{(N-2)C_1 - \frac{N}{2(N-1)}\big(\frac{K}{N-1}\big)^{\frac{2-N}{N}}C_1^2}{N\big(\frac{K}{N-1}\big)^{\frac{N-1}{N^2}}};$$

and for $i>2$, C_i can be uniquely determined by

$$(N-i)C_{i-1}-N\left(\frac{K}{N-1}\right)^{\frac{1}{N}}C_i=\sum_{j=2}^{i}\frac{F^{(j)}(0)}{j!}\left(\sum_{\substack{j_1+\cdots+j_j=i\\ j_1,\cdots j_j>0}}C_{j_1}C_{j_2}\cdots C_{j_j}\right),$$

where

$$F(\rho)=(N-1)\left(\left(\frac{K}{N-1}\right)^{\frac{N-1}{N}}+\rho\right)^{\frac{N}{N-1}}.$$

Particularly we have

$$\lim_{r\to\infty}u(r)r^{\frac{1}{N}}e^{\left(\frac{K}{N-1}\right)^{\frac{1}{N}}r}=C_*$$

for some constant $C_*>0$.

2 Two lemmas

In this section, we will prove Theorem 1.1. First, let $u(x)=u(r)$ be a radial ground state as stated in Theorem 1.1. It follows from (1.1) that

$$\left(|u'|^{N-2}u'\right)'+\frac{N-1}{r}|u'|^{N-2}u'+f(u)=0. \tag{2.1}$$

Denote

$$\varphi(r)=-\frac{|u'|^{N-2}u'}{u^{N-1}}. \tag{2.2}$$

Then

$$|u'|^{N-2}u'=-\varphi u^{N-1},\quad \frac{u'}{u}=-\varphi^{\frac{1}{N-1}}.$$

Substituting them into (2.1) yields

$$\varphi'-(N-1)\varphi^{\frac{N}{N-1}}+\frac{N-1}{r}\varphi-\frac{f(u)}{u^{N-1}}=0. \tag{2.3}$$

Lemma 2.1. *For φ defined by (2.2), we have*

$$\limsup_{r\to\infty}\varphi(r)<\infty.$$

Proof. It follows from assumption (f_1) and $\lim_{r\to\infty}u(r)=0$ that

$$\kappa=\sup_r\frac{|f(u(r))|}{u^{N-1}(r)}<\infty.$$

It follows from $\varphi(r)\geq 0$ and (2.3) that as long as

$$\varphi(r)\geq\left(\frac{4\kappa}{N-1}\right)^{\frac{N-1}{N}},\quad r\geq 4\left(\frac{N-1}{4\kappa}\right)^{\frac{N-1}{N}}$$

we have

$$\varphi'=(N-1)\varphi^{\frac{N}{N-1}}-\frac{N-1}{r}\varphi+\frac{f(u)}{u^{N-1}}\geq\frac{N-1}{2}\varphi^{\frac{N}{N-1}}. \tag{2.4}$$

Now, suppose to the contrary that $\limsup_{r\to\infty}\varphi(r)=\infty$. Let

$$r_1=\inf\left\{r\geq 4\left(\frac{N-1}{4\kappa}\right)^{\frac{N-1}{N}}:\varphi(r)\geq\left(\frac{4\kappa}{N-1}\right)^{\frac{N-1}{N}}\right\}.$$

Since $\varphi(0)=0$ and $\varphi(r)>0$ for $r>0$, it follows that $0<r_1\leq\infty$. If $r_1=\infty$ we are done. Next we suppose $r_1<\infty$. Then

$$\varphi(r_1)=\left(\frac{4\kappa}{N-1}\right)^{\frac{N-1}{N}},\quad \varphi'(r)\geq\frac{N-1}{2}\varphi(r)^{\frac{N}{N-1}}\quad\text{for all}\ \ r\geq r_1,$$

which blows up before or at

$$r_2=r_1+2\left(\frac{N-1}{4\kappa}\right)^{\frac{1}{N}}.$$

This contradicts the well-definedness of φ. The proof of this lemma is complete. □

Lemma 2.2. *For φ defined by* (2.2)*, we have*

$$\lim_{r\to\infty}\varphi=\left(\frac{K}{N-1}\right)^{\frac{N-1}{N}}, \tag{2.5}$$

where the constant K is the one stated in (f_1)*.*

Proof. Since $\varphi(0)=0$ and $\varphi(r)>0$ for $r>0$ it follows from (2.3) and Lemma 2.1 that

$$0\leq\liminf_{r\to\infty}\ \varphi(r)=\omega_1<\infty$$

and

$$0<\limsup_{r\to\infty}\ \varphi(r)=\omega_2<\infty.$$

Next we use the contradiction arguments to prove (2.5). Suppose

$$\omega_1=\liminf_{r\to\infty}\ \varphi(r)<\limsup_{r\to\infty}\ \varphi(r)=\omega_2.$$

Then we may choose two sequences $\{\eta_i\}$ and $\{\zeta_i\}$ going to ∞ as $i\to\infty$ such that

$$\{\eta_i\}\text{ are local minima of }\varphi,\quad\{\zeta_i\}\text{ are local maxima of }\varphi,$$

and $\zeta_i \in (\eta_i, \eta_{i+1})$, and

$$\liminf_{r\to\infty} \varphi(r) = \lim_{i\to\infty} \varphi(\eta_i) = \omega_1, \quad \limsup_{r\to\infty} \varphi(r) = \lim_{i\to\infty} \varphi(\zeta_i) = \omega_2.$$

Then at η_i $(i=1, 2, \cdots)$ we know $u'(\eta_i)=0$ and thus

$$-(N-1)\varphi(\eta_i)^{\frac{N}{N-1}} + \frac{N-1}{\eta_i}\varphi(\eta_i) - \frac{f(u(\eta_i))}{u(\eta_i)^{N-1}} = 0. \tag{2.6}$$

Similarly at ζ_i $(i=1, 2, \cdots)$ we get

$$-(N-1)\varphi(\zeta_i)^{\frac{N}{N-1}} + \frac{N-1}{\zeta_i}\varphi(\zeta_i) - \frac{f(u(\zeta_i))}{u(\zeta_i)^{N-1}} = 0. \tag{2.7}$$

For any given $\epsilon > 0$, since

$$\lim_{t\downarrow 0} \frac{f(t)}{t^{N-1}} = -K, \quad \lim_{r\to\infty} u(r) = 0,$$

we can take r_3 sufficiently large such that

$$\left| \frac{f(u(r))}{u(r)^{N-1}} + K \right| < \epsilon \quad \text{for all } r > r_3.$$

Next we take i_0 sufficiently large such that $\eta_i > r_3$ for $i > i_0$. It follows from (2.6) and (2.7) that for all $i > i_0$:

$$K-\epsilon < (N-1)\varphi(\eta_i)^{\frac{N}{N-1}} - \frac{N-1}{\eta_i}\varphi(\eta_i) < K+\epsilon,$$
$$K-\epsilon < (N-1)\varphi(\zeta_i)^{\frac{N}{N-1}} - \frac{N-1}{\zeta_i}\varphi(\zeta_i) < K+\epsilon.$$

Letting $i \to \infty$ and noticing the arbitrariness of ϵ, we obtain

$$(N-1)\omega_1^{\frac{N}{N-1}} = K, \qquad (N-1)\omega_2^{\frac{N}{N-1}} = K$$

which yields $\omega_1 = \omega_2$, and contradicts to $\omega_1 < \omega_2$. Consequently,

$$\lim_{r\to\infty} \varphi(r) = \left(\frac{K}{N-1} \right)^{\frac{N-1}{N}}.$$

The proof of this lemma is complete. □

3 Proof of Theorem 1.1

With the help of the two lemmas above, next we give a proof of Theorem 1.1. It follows from Lemma 2.2 that

$$\lim_{r\to\infty}\frac{u'}{u}=\lim_{r\to\infty}-\varphi^{\frac{1}{N-1}}=-\left(\frac{K}{N-1}\right)^{\frac{1}{N}}.$$

Thus $\forall\epsilon>0$, there exists a constant $0<\tilde{C}=\tilde{C}(\epsilon)<\infty$ such that

$$u(r)\le\tilde{C}e^{-(\frac{K-\epsilon}{N-1})^{\frac{1}{N}}r}.$$

Especially taking $\epsilon=K/2$ we have

$$u(r)\le\tilde{C}(K)e^{-(\frac{K}{2(N-1)})^{\frac{1}{N}}r}.$$

For convenience, let

$$\left(\frac{K}{2(N-1)}\right)^{\frac{1}{N}}=\mu,\quad C_\infty=\left(\frac{K}{N-1}\right)^{\frac{N-1}{N}}.$$

Then the above result becomes

$$u(r)\le\tilde{C}e^{-\mu r}. \tag{3.1}$$

Next we give more precise expansion of $\varphi(r)$ at ∞. Let $\varphi=C_\infty+\varphi_1$. We know from (2.3) and Lemma 2.2 that $\lim_{r\to\infty}\varphi_1(r)=0$ and $\varphi_1(r)$ satisfies

$$\varphi_1'-(N-1)(C_\infty+\varphi_1)^{\frac{N}{N-1}}+\frac{N-1}{r}(\varphi_\infty+\varphi_1)-\frac{f(u)}{u^{N-1}}=0,$$

or equivalently,

$$\begin{aligned}&\varphi_1'-NC_\infty^{\frac{1}{N-1}}\varphi_1+\frac{N-1}{r}\varphi_1\\=&\frac{f(u)}{u^{N-1}}+(N-1)(C_\infty+\varphi_1)^{\frac{N}{N-1}}-NC_\infty^{\frac{1}{N-1}}\varphi_1-\frac{N-1}{r}C_\infty.\end{aligned} \tag{3.2}$$

Notice that

$$\lim_{r\to\infty}u(r)=0,\quad \frac{f(t)+Kt^{N-1}}{t^{N-1}}=\mathcal{O}(t^\alpha)\quad\text{as }\ t\downarrow 0.$$

We get for r sufficiently large

$$\frac{f(u)}{u^{N-1}}=-K+\mathcal{O}(u^\alpha). \tag{3.3}$$

At the same time for r sufficiently large we have

$$(N-1)(C_\infty+\varphi_1)^{\frac{N}{N-1}}=(N-1)C_\infty^{\frac{N}{N-1}}+NC_\infty^{\frac{1}{N-1}}\varphi_1+\mathcal{O}\left(\varphi_1^2\right). \tag{3.4}$$

Thus it follows from (3.2)-(3.4) that for r sufficiently large

$$\varphi_1' - NC_\infty^{\frac{1}{N-1}}\varphi_1 + \frac{N-1}{r}\varphi_1 = \mathcal{O}(\varphi_1^2) - \frac{N-1}{r}C_\infty + \mathcal{O}(u^\alpha). \tag{3.5}$$

Multiplying both sides of (3.5) by φ_1 and integrating the resulting equation from r to ∞ for r sufficiently large yield

$$\begin{aligned}&\frac{1}{2}\varphi_1^2(r) + \int_r^\infty \left(NC_\infty^{\frac{1}{N-1}} - \frac{N-1}{s} + \mathcal{O}(\varphi_1)\right)\varphi_1^2 \mathrm{d}s\\ =& \int_r^\infty \frac{N-1}{s}C_\infty\varphi_1 \mathrm{d}s - \int_r^\infty \mathcal{O}(u^\alpha)\varphi_1 \mathrm{d}s.\end{aligned} \tag{3.6}$$

We can take r sufficiently large such that

$$NC_\infty^{\frac{1}{N-1}} - \frac{N-1}{s} + \mathcal{O}(\varphi_1) \geq \frac{N}{2}C_\infty^{\frac{1}{N-1}} \quad \text{for } s > r.$$

Therefore for such large r it follows that

$$\varphi_1^2(r) + \int_r^\infty \left(NC_\infty^{\frac{1}{N-1}}\right)\varphi_1^2 \mathrm{d}s \leq 2\int_r^\infty \frac{N-1}{s}C_\infty\varphi_1 \mathrm{d}s - 2\int_r^\infty \mathcal{O}(u^\alpha)\varphi_1 \mathrm{d}s.$$

Note that

$$\begin{aligned}&2\int_r^\infty \frac{N-1}{s}C_\infty\varphi_1 \mathrm{d}s \leq \frac{1}{4}\left(NC_\infty^{\frac{1}{N-1}}\right)\int_r^\infty \varphi_1^2 \mathrm{d}s + \frac{4(N-1)^2C_\infty^2}{NC_\infty^{\frac{1}{N-1}}}\int_r^\infty \frac{1}{s^2}\mathrm{d}s,\\ &2\int_r^\infty \mathcal{O}(u^\alpha)\varphi_1 \mathrm{d}s \leq \frac{1}{4}\left(NC_\infty^{\frac{1}{N-1}}\right)\int_r^\infty \varphi_1^2 \mathrm{d}s + \frac{4}{NC_\infty^{\frac{1}{N-1}}}\int_r^\infty \mathcal{O}(u^{2\delta})\mathrm{d}s.\end{aligned}$$

By virtue of the above estimates and (3.1) we get that for r sufficiently large

$$\begin{aligned}&\varphi_1^2(r) + \frac{1}{2}\left(NC_\infty^{\frac{1}{N-1}}\right)\int_r^\infty \varphi_1^2(s)\mathrm{d}s\\ \leq& \frac{4(N-1)^2C_\infty^2}{NC_\infty^{\frac{1}{N-1}}}\frac{1}{r} + \frac{4\bar{C}}{2\alpha NC_\infty^{\frac{1}{N-1}}}e^{-2\delta\mu r} \leq \frac{8(N-1)^2C_\infty^2}{NC_\infty^{\frac{1}{N-1}}}\frac{1}{r},\end{aligned} \tag{3.7}$$

where $\bar{C} > 0$ is a constant independent of r. Thus we have

$$\varphi_1^2(r) = \mathcal{O}\left(r^{-1}\right) \quad \text{as } r \to \infty. \tag{3.8}$$

By this estimate and (3.5) it follows that as $r \to \infty$

$$\varphi_1' - NC_\infty^{\frac{1}{N-1}}\varphi_1 + \frac{N-1}{r}\varphi_1 = \mathcal{O}\left(r^{-1}\right). \tag{3.9}$$

For convenience let $\alpha_0 = NC_\infty^{\frac{1}{N-1}}$. Then we get from (3.9) as $r\to\infty$

$$\left(r^{N-1}e^{-\alpha_0 r}\varphi_1\right)' = r^{N-1}e^{-\alpha_0 r}\mathcal{O}\left(r^{-1}\right). \tag{3.10}$$

Integrating both sides of (3.10) from r to ∞ yields, as $r\to\infty$,

$$\begin{aligned}\varphi_1(r) &= \frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-1}\mathcal{O}\left(s^{-1}\right)e^{-\alpha_0 s}\mathrm{d}s\\ &= \mathcal{O}\left(\frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-2}e^{-\alpha_0 s}\mathrm{d}s\right).\end{aligned} \tag{3.11}$$

Applying integration by parts as many steps as we want we arrive at that there exists a sequence of constants $\{a_i\}(i=1,2,\cdots)$ such that

$$\begin{aligned}\int_r^\infty s^{N-2}e^{-\alpha_0 s}\mathrm{d}s &= a_1 r^{N-2}e^{-\alpha_0 r} + a_2 r^{N-3}e^{-\alpha_0 r} + \cdots\\ &= e^{-\alpha_0 r}\left(a_1 r^{N-2} + a_2 r^{N-3} + \cdots + a_l r^{N-l-1} + \cdots\right),\end{aligned} \tag{3.12}$$

where $a_1 = 1/\alpha_0$. Thus it follows from (3.10) that

$$\varphi_1(r) = \mathcal{O}\left(r^{-1}\right), \quad \text{as } r\to\infty, \tag{3.13}$$

which is an improvement of (3.8). Using (3.13) and (3.5) we obtain

$$\left(r^{N-1}e^{-\alpha_0 r}\varphi_1\right)' = -r^{N-1}e^{-\alpha_0 r}\left(\frac{N-1}{r}C_\infty + \mathcal{O}\left(r^{-2}\right)\right). \tag{3.14}$$

Similar to (3.10) and (3.11), we arrive at that

$$\begin{aligned}\varphi_1(r) &= \frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-1}e^{-\alpha_0 s}\left(\frac{N-1}{s}C_\infty + \mathcal{O}\left(s^{-2}\right)\right)\mathrm{d}s\\ &= \frac{(N-1)C_\infty a_1}{r} + \mathcal{O}\left(r^{-2}\right).\end{aligned}$$

Again if we let

$$\varphi_1 = \frac{(N-1)C_\infty a_1}{r} + \varphi_2$$

such that $\varphi_2(r) = \mathcal{O}(r^{-2})$, we obtain from (3.4) and (3.5) that

$$\varphi_2' - \alpha_0\varphi_2 + \frac{N-1}{r}\varphi_2 = \frac{(5-2N)(N-1)}{2N}C_\infty^{\frac{N-2}{N-1}}\frac{1}{r^2} + \mathcal{O}\left(r^{-2}\right).$$

We can then repeat the same process to obtain the expansion as stated in Theorem 1.1 to any polynomial order as we want. Next we need determine C_i $(i=1,2,\cdots)$ in Theorem 1.1. Let

$$F(\rho) = (N-1)(\varphi_\infty + \rho)^{\frac{N}{N-1}}.$$

Then the Taylor expansion of $F(\rho)$ at $\rho=0$ is as follows:

$$
\begin{aligned}
F(\rho)=&(N-1)\varphi_\infty^{\frac{N}{N-1}}+\alpha_0\rho+\frac{N}{2(N-1)}C_\infty^{-\frac{N-2}{N-1}}\rho^2\\
&-\frac{N(N-2)}{3!(N-1)^2}C_\infty^{-\frac{2N-3}{N-1}}\rho^3+\cdots+\frac{F^{(n)}(0)}{n!}\rho^n+\cdots,
\end{aligned}
\tag{3.15}
$$

where

$$
F^{(n)}(0)=\frac{(-1)^nN(N-2)(2N-3)\cdots(lN-l-1)\cdots[(n-2)N-n+1]}{(N-1)^{n-1}}
$$

for $n\geq 4$. Thus from (3.2), (3.3) and (3.15) we get

$$
\begin{aligned}
&\varphi_1'-\alpha_0\varphi_1+\frac{N-1}{r}\varphi_1\\
=&\mathcal{O}(u^\delta)-\frac{N-1}{r}C_\infty+\frac{N}{2(N-1)}C_\infty^{-\frac{N-2}{N-1}}\varphi_1^2\\
&-\frac{N(N-2)}{3!(N-1)^2}C_\infty^{-\frac{2N}{N-1}}\varphi_1^3+\cdots+\frac{F^{(n)}(0)}{n!}\varphi_1^n+\cdots.
\end{aligned}
\tag{3.16}
$$

Substituting $\varphi_1(r)=\sum_{j=1}^\infty C_j/r^j$ into (3.16) we get by comparing the coefficients of $1/r^n$ $(n=1,2,\cdots)$ that

$$
C_1=\frac{N-1}{N}\left(\frac{K}{N-1}\right)^{\frac{N-2}{N}},\quad C_2=\frac{(N-2)C_1-\frac{N}{2(N-1)}C_\infty^{-\frac{N-2}{N-1}}C_1^2}{\alpha_0},
$$

and C_i $(i>2)$ is determined by

$$
(N-i)C_{i-1}-\alpha_0C_i=\sum_{j=2}^i\frac{F^{(j)}(0)}{j!}\left(\sum_{j_1+\cdots+j_j=i}C_{j_1}C_{j_2}\cdots C_{j_j}\right).
$$

Note that

$$
\frac{u'}{u}=-\varphi^{\frac{1}{N-1}}.
$$

We know as $r\rightarrow\infty$,

$$
\begin{aligned}
\frac{u'}{u}=&-\left(C_\infty^{\frac{1}{N-1}}+\frac{1}{N-1}C_\infty^{\frac{2-N}{N-1}}C_1\cdot\frac{1}{r}+\mathcal{O}\left(r^{-2}\right)\right)\\
=&-\left(\frac{K}{N-1}\right)^{\frac{1}{N}}-\frac{1}{N}\cdot\frac{1}{r}+\mathcal{O}\left(r^{-2}\right),
\end{aligned}
$$

which yields that

$$
u(r)=\mathcal{O}\left(r^{-\frac{1}{N}}e^{-(\frac{K}{N-1})^{\frac{1}{N}}r}\right)\quad\text{as } r\rightarrow\infty.
$$

Consequently,

$$\lim_{r\to\infty} u(r) r^{\frac{1}{N}} e^{\left(\frac{K}{N-1}\right)^{\frac{1}{N}} r} = C_*,$$

for some $0 < C_* < \infty$. The proof of Theorem 1.1 is complete. □

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