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Traveling Wave Solutions to the Three-Dimensional Nonlinear Viscoelastic System

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Abstract. In this paper, we establish the existence of traveling wave solutions to the nonlinear three-dimensional viscoelastic system exhibiting long range memory. Under certain hypotheses, if the speed of propagation is between the speeds determined by the equilibrium and instantaneous elastic tensors, then the system has nontrivial traveling wave solutions. Moreover, the system has only trivial traveling wave solution in some cases.

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1 Introduction

In this paper, we discuss the existence of nontrivial traveling wave solutions to the three-dimensional nonlinear viscoelastic system exhibiting long range memory:

$$\boldsymbol{u}_{tt}(\boldsymbol{x},t)=\mathrm{div}_x\boldsymbol{\sigma}, \tag{1.1a}$$

where $\boldsymbol{u}(\boldsymbol{x},t)=(u_1(\boldsymbol{x},t),u_2(\boldsymbol{x},t),u_3(\boldsymbol{x},t))$ is the displacement of a material particle $\boldsymbol{x}=(x_1,x_2,x_3)$ at time t, and $\boldsymbol{\sigma}=(\sigma_{ij})$ is the stress tensor. For viscoelastic materials, the stress at time t depends on all the history of the deformation gradient up to time t. Here we discuss only the case when the stress is given by a single integral law (see, [1,2])

$$\boldsymbol{\sigma}(\boldsymbol{x},t)=\boldsymbol{g}(\nabla\boldsymbol{u}(\boldsymbol{x},t))-\int_0^\infty \boldsymbol{h}(\tau,\nabla\boldsymbol{u}(\boldsymbol{x},t-\tau))\mathrm{d}\tau, \tag{1.1b}$$

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where $\boldsymbol{g}(\boldsymbol{\xi})=(g_{ij}(\boldsymbol{\xi})),\boldsymbol{h}(\tau,\boldsymbol{\eta})=(h_{ij}(\tau,\boldsymbol{\eta})),\boldsymbol{\xi}=(\xi_{ij}),\boldsymbol{\eta}=(\eta_{ij}),\ i,j=1,2,3$.

Our interest is to find traveling wave solutions to the nonlinear Volterra integro-differential system (1.1). That is, we are looking for a solution $\boldsymbol{u}(\boldsymbol{x},t)$ depending only on $\xi=\lambda t+\boldsymbol{\omega}\cdot\boldsymbol{x}$, where λ is the speed of propagation, $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ with $|\boldsymbol{\omega}|=1$. Moreover, we require that the solution satisfies the upstream condition:

$$\lim_{\xi\to-\infty}\frac{\mathrm{d}\boldsymbol{u}(\xi)}{\mathrm{d}\xi}=\boldsymbol{v}^- \tag{1.2}$$

for a given constant vector $\boldsymbol{v}^-=(v_1^-,v_2^-,v_3^-)$.

Qin and Ni studied the special case in [3] when

$$\boldsymbol{h}(\tau,\nabla\boldsymbol{u}(\boldsymbol{x},t-\tau))=a(\tau)\boldsymbol{h}(\nabla\boldsymbol{u}(\boldsymbol{x},t-\tau)). \tag{1.3}$$

As pointed out in [3], for the pure elastic case

$$\boldsymbol{\sigma}=\boldsymbol{g}(\nabla\boldsymbol{u}(\boldsymbol{x},t)), \tag{1.4}$$

the problem has no nontrivial traveling wave solution except when λ is the speed of propagation for the wave $\boldsymbol{v}^-$ determined by the elastic tensor. For viscoelastic materials, the instantaneous elastic tensor (1.4) is different from the equilibrium elastic tensor determined by the stress tensor

$$\boldsymbol{p}(\boldsymbol{u}(\boldsymbol{x},t))=\boldsymbol{g}(\nabla\boldsymbol{u}(\boldsymbol{x},t))-\int_0^\infty\boldsymbol{h}(\tau,\nabla\boldsymbol{u}(\boldsymbol{x},t))\mathrm{d}\tau, \tag{1.5}$$

which governs the long time behavior of waves. Thanks to the dissipative effect, in general, the speed of propagation for the wave determined by the equilibrium elastic tensor is less than that determined by the instantaneous elastic tensor. Therefore, we should find nontrivial traveling wave solutions to the problem with the propagation speed λ between the two speeds.

For the one-dimensional case, the system (1.1) is reduced to

$$u_{tt}(x,t)=\frac{\partial}{\partial x}g(u_x(x,t))-\int_0^\infty\frac{\partial}{\partial x}h(\tau,u_x(x,t-\tau))\mathrm{d}\tau, \tag{1.6}$$

and the corresponding instantaneous and equilibrium elastic modulus are $g'(u_x)$ and $p'(u_x)$, respectively. The authors of [4] and [5] proved that if

$$p'(v^-)<\lambda^2<g'(v^-), \tag{1.7}$$

then there exist nontrivial traveling wave solutions to (1.6).

All the methods used in the one-dimensional case depend strongly on the monotonicity of both traveling wave solutions and iterative sequences. Therefore, they cannot be applied to the three-dimensional case. In order to overcome this difficulty, we apply the higher-order iterative process, introduced by Qin and Ni in [3], to show the local existence of traveling wave solutions near $\xi=-\infty$, and then prove the corresponding global existence.

In the next section, we state the hypotheses and the main results of this paper. In Section 3, we give the proof of the existence of nontrivial traveling wave solutions. Finally, the uniqueness of trivial traveling wave solution is established.

2 Hypotheses and main results

Set $\boldsymbol{v}(\xi)=\mathrm{d}\boldsymbol{u}(\xi)/\mathrm{d}\xi$ and

$$a_{ijkl}(\boldsymbol{F})=\frac{\partial g_{ij}(\boldsymbol{F})}{\partial f_{kl}},\quad b_{ijkl}(\eta,\boldsymbol{F})=\frac{\partial h_{ij}(\eta,\boldsymbol{F})}{\partial f_{kl}},\quad c_{ijkl}(\boldsymbol{F})=\frac{\partial p_{ij}(\boldsymbol{F})}{\partial f_{kl}}$$

with $\boldsymbol{F}=(f_{ij})$. For traveling wave solutions, the system (1.1) is reduced to

$$\begin{aligned}&-\lambda^2\frac{\mathrm{d}v_i(\xi)}{\mathrm{d}\xi}+\sum_{j,k,l=1}^{3}\omega_j\omega_l a_{ijkl}(\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\frac{\mathrm{d}v_k(\xi)}{\mathrm{d}\xi}\\=&\int_0^\infty\sum_{j,k,l=1}^{3}\omega_j\omega_l b_{ijkl}(\eta,\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})\frac{\mathrm{d}v_k(\xi-\lambda\eta)}{\mathrm{d}\xi}\mathrm{d}\eta,\quad i=1,\,2,\,3\end{aligned}\tag{2.1}$$

with $\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega}=(v_i(\xi)\omega_j)$. The upstream condition (1.2) is written as

$$\lim_{\xi\to-\infty}\boldsymbol{v}(\xi)=\boldsymbol{v}^-.\tag{2.2}$$

It is clear that $\boldsymbol{v}(\xi)\equiv\boldsymbol{v}^-$ is a solution to (2.1)-(2.2), which we refer to as the trivial solution. Integrating (2.1) with respect to ξ from $-\infty$ to ξ and using (2.2), we have

$$\begin{aligned}&-\lambda^2 v_i(\xi)+\sum_{j=1}^{3}\omega_j g_{ij}(\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\\=&\int_0^\infty\sum_{j=1}^{3}\omega_j h_{ij}(\eta,\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})\mathrm{d}\eta-A_i,\quad i=1,\,2,\,3,\end{aligned}\tag{2.3}$$

where

$$A_i=\lambda^2 v_i^- -\sum_{j=1}^{3}\omega_j g_{ij}(\boldsymbol{v}^-\otimes\boldsymbol{\omega})+\int_0^\infty\sum_{j=1}^{3}\omega_j h_{ij}(\eta,\boldsymbol{v}^-\otimes\boldsymbol{\omega})\mathrm{d}\eta.\tag{2.4}$$

For convenience, we refer to λ satisfying

$$\det\Big(-\lambda^2\boldsymbol{I}+\sum_{j,l=1}^{3}\omega_j\omega_l a_{ijkl}(\boldsymbol{v}\otimes\boldsymbol{\omega})\Big)=0,\tag{2.5}$$

where (a_{ijkl}) stands for elastic tensor, as the speed of propagation for the wave $\boldsymbol{v}$ associated with the direction $\boldsymbol{\omega}$.

Without loss of generality, we assume that

$$\boldsymbol{v}^- \equiv \mathbf{0}. \tag{2.6}$$

In this paper, for a tensor a_{ijkl}, the symmetry means that

$$a_{ijkl} = a_{klij}, \quad \forall\, i,\, j,\, k,\, l = 1,\, 2,\, 3. \tag{2.7}$$

Definition 2.1. *If there exists a constant $\alpha > 0$ such that*

$$\sum_{i,j,k,l=1}^{3} a_{ijkl} \xi_i \xi_k \eta_j \eta_l \geq \alpha |\boldsymbol{\xi}|^2 |\boldsymbol{\eta}|^2, \ \forall\, \boldsymbol{\xi},\, \boldsymbol{\eta} \in \mathbb{R}^3, \tag{2.8}$$

then the tensor $\boldsymbol{A} = (a_{ijkl})$ is said to satisfy the strongly elliptic condition.

Now we state the hypotheses for the system (2.3).

(H) $(a_{ijkl}(\boldsymbol{F}))$, $(b_{ijkl}(\eta,\boldsymbol{F}))$ and $(c_{ijkl}(\boldsymbol{F}))$ are sufficiently smooth and symmetric. Moreover, $(a_{ijkl}(\mathbf{0}))$ and $(c_{ijkl}(\mathbf{0}))$ satisfy the strongly elliptic condition.

Let

$$\boldsymbol{A}(\boldsymbol{F}) = \Big(\sum_{j,l=1}^{3} \omega_j \omega_l a_{ijkl}(\boldsymbol{F}) \Big), \qquad \boldsymbol{B}(\eta,\boldsymbol{F}) = \Big(\sum_{j,l=1}^{3} \omega_j \omega_l b_{ijkl}(\eta,\boldsymbol{F}) \Big),$$
$$\boldsymbol{C}(\boldsymbol{F}) = \Big(\sum_{j,l=1}^{3} \omega_j \omega_l c_{ijkl}(\boldsymbol{F}) \Big).$$

From the hypothesis (**H**), it is easy to see that $\boldsymbol{A}(\mathbf{0})$ and $\boldsymbol{C}(\mathbf{0})$ are positive definite matrices. Let $\rho(\boldsymbol{A})$ and $r(\boldsymbol{A})$ be the spectral radius and the least eigenvalue of a symmetric matrix $\boldsymbol{A}$, respectively. For definiteness, we consider only the case $\lambda > 0$; the discussion for the case $\lambda < 0$ is similar.

Now the problem is reduced to find a traveling wave solution to the nonlinear Volterra integral system (2.3) such that the upstream condition (2.2) holds. In the proof of either existence or uniqueness of traveling wave solutions, the key step is to solve the problem near $\xi = -\infty$.

Theorem 2.1. *Suppose that the hypothesis* $(\mathbf{H})$ *holds and there exists $a(\eta) \in L^1(0,\infty)$ such that*

$$|b_{ijkl}(\eta,\mathbf{0})| \leq a(\eta), \quad \forall\, \eta > 0, \ i,\, j,\, k,\, l = 1,\, 2,\, 3. \tag{2.9}$$

If $\rho(\boldsymbol{C}(\mathbf{0})) < \lambda^2 < r(\boldsymbol{A}(\mathbf{0}))$, then the system (2.3) *admits a nontrivial continuous solution satisfying the upstream condition* (2.2) *near $\xi = -\infty$.*

If we get the solution on $(-\infty,\xi_0]$ for a certain value of ξ_0, then the system (2.3) can be rewritten as

$$\begin{aligned}&-\lambda^2 v_i(\xi)+\sum_{j=1}^{3}\omega_j g_{ij}(\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\\&=\int_0^{\frac{1}{\lambda}(\xi-\xi_0)}\sum_{j=1}^{3}\omega_j h_{ij}(\eta,\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})\mathrm{d}\eta+f_i(\xi),\ \ i=1,\,2,\,3,\end{aligned}\tag{2.10}$$

where

$$f_i(\xi)=\int_{\frac{1}{\lambda}(\xi-\xi_0)}^{\infty}\sum_{j=1}^{3}\omega_j h_{ij}(\eta,\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})\mathrm{d}\eta-A_i$$

are given functions. Eq. (2.10) is a nonlinear Volterra integral system. Under some assumptions on (g_{ij}) and (h_{ij}), it is not difficult to get the global existence through the Schauder's fixed point theorem.

Theorem 2.2. *Assume the hypotheses of Theorem* 2.1 *hold. Furthermore, we assume that*

$$\lambda^2<\bar{r}<r(\boldsymbol{A}(\boldsymbol{F})),\ \ \forall\,\boldsymbol{F}\in\mathbb{R}^{3\times 3},\tag{2.11}$$

and there exists $b(\eta)\in L^1(0,\infty)$ such that

$$|b_{ijkl}(\eta,\boldsymbol{F})|,\ \ |\partial_\eta h_{ij}(\eta,\boldsymbol{F})|\le b(\eta),\ \ \forall\,\eta>0,\ \boldsymbol{F}\in\mathbb{R}^{3\times 3},\ i,j=1,\,2,\,3,\tag{2.12}$$

where $\bar{r}$ is a constant. Then the system (2.3) *has a nontrivial continuous solution satisfying the upstream condition* (2.2) *on* $\mathbb{R}$.

Theorems 2.1 and 2.2 will be proved in Section 3.

Roughly speaking, for a given direction $\boldsymbol{\omega}$, if λ is less than all the speeds of propagation determined by the equilibrium elastic tensor or larger than all ones determined by the instantaneous elastic tensor, the problem has only the trivial solution.

Now we introduce the definition on the positive type of matrix-valued function (see, e.g., p. 492, [6]).

Definition 2.2. *A matrix-valued function $\boldsymbol{M}(t)\in L^1(\mathbb{R}^+;\mathbb{R}^{3\times 3})$ is said to be of positive type if and only if*

$$\int_0^T\langle \boldsymbol{v}(t),(\boldsymbol{M}*\boldsymbol{v})(t)\rangle\mathrm{d}t=\int_0^T\Big\langle \boldsymbol{v}(t),\int_0^t\boldsymbol{M}(t-\tau)\boldsymbol{v}(\tau)\mathrm{d}\tau\Big\rangle\mathrm{d}t\ge 0,\ \ \forall\,T>0\tag{2.13}$$

for every $\boldsymbol{v}\in C(\mathbb{R}^+;\mathbb{R}^3)$.

Theorem 2.3. *Suppose that the hypothesis* (**H**) *holds and there exists a constant* $\delta_0>0$ *and* $c(\eta)\geq 0$ *such that*

$$|b_{ijkl}(\eta,\mathbf{0})|\leq c(\eta),\ \ \forall\,\eta>0,\ \ i,\,j,\,k,\,l=1,\,2,\,3. \tag{2.14}$$

Moreover, $e^{\delta_0\eta}c(\eta)\in L^1(0,\infty)$*, and* $e^{\delta_0\eta}\boldsymbol{B}(\eta,\mathbf{0})$ *is of positive type. If*

$$\lambda^2>\rho(\boldsymbol{A}(\mathbf{0}))\quad or\quad \lambda^2<r(\boldsymbol{A}(\mathbf{0}))-\rho\Big(\int_0^\infty \boldsymbol{B}(\eta,\mathbf{0})\mathrm{d}\eta\Big),$$

then the problem (2.1)-(2.2) *admits only the trivial solution.*

Theorem 2.3 will be proved in Section 4.

3 Proof of the existence of nontrivial solution

Following the idea in [3], we first construct a nontrivial continuous traveling wave solution near $\xi=-\infty$. Then, by the Schauder's fixed point theorem, we get the global behavior of the traveling wave solution.

3.1 Local existence near $\xi=-\infty$

We are looking for a solution to the system (2.3) in the following form

$$\boldsymbol{v}(\xi)=\boldsymbol{a}^1e^{\delta\xi}+\cdots+\boldsymbol{a}^Ne^{N\delta\xi}+\boldsymbol{p}(\xi), \tag{3.1}$$

where $\boldsymbol{a}^j=(a_1^j,a_2^j,a_3^j)^T(j=1,\cdots,N),\boldsymbol{p}=(p_1,p_2,p_3)^T$, $|\boldsymbol{p}|=\mathcal{O}(e^{(N+1)\delta\xi})$, and N is a sufficiently large integer to be determined later.

Similarly to the proof of Lemma 3.1 in [3], we can show the following lemma.

Lemma 3.1. *If* $\rho(\boldsymbol{C}(\mathbf{0}))<\lambda^2<r(\boldsymbol{A}(\mathbf{0}))$*, then there exists a constant* $\sigma>0$ *such that*

$$\det\Big(-\lambda^2\boldsymbol{I}+\boldsymbol{A}(\mathbf{0})-\int_0^\infty\boldsymbol{B}(\eta,\mathbf{0})e^{-\lambda\sigma\eta}\mathrm{d}\eta\Big)=0. \tag{3.2}$$

Let

$$\delta=\max\Big\{\sigma\in\mathbb{R}^1,\det\Big(-\lambda^2\boldsymbol{I}+\boldsymbol{A}(\mathbf{0})-\int_0^\infty\boldsymbol{B}(\eta,\mathbf{0})e^{-\lambda\sigma\eta}\mathrm{d}\eta\Big)=0\Big\}.$$

There exists a nontrivial solution $\boldsymbol{a}^1=(a_1^1,a_2^1,a_3^1)^T$ such that

$$\Big(-\lambda^2\boldsymbol{I}+\boldsymbol{A}(\mathbf{0})-\int_0^\infty\boldsymbol{B}(\eta,\mathbf{0})e^{-\lambda\sigma\eta}\mathrm{d}\eta\Big)\boldsymbol{a}^1=0. \tag{3.3}$$

Inserting (3.1) into the system (2.3), we have, for $i=1,2,3$,

$$
\begin{aligned}
&-\lambda^2\Big(a_i^1 e^{\delta\xi}+\cdots+a_i^N e^{N\delta\xi}+p_i(\xi)\Big)+\sum_{j=1}^{3}\omega_j g_{ij}\Big((\boldsymbol{a}^1 e^{\delta\xi}+\cdots+\boldsymbol{a}^N e^{N\delta\xi}+\boldsymbol{p}(\xi))\otimes\boldsymbol{\omega}\Big)\\
&=\int_0^\infty\sum_{j=1}^{3}\omega_j h_{ij}\big(\eta,(\boldsymbol{a}^1 e^{\delta(\xi-\lambda\eta)}+\cdots+\boldsymbol{a}^N e^{N\delta(\xi-\lambda\eta)}+\boldsymbol{p}(\xi-\lambda\eta))\otimes\boldsymbol{\omega}\big)\mathrm{d}\eta-A_i. \qquad (3.4)
\end{aligned}
$$

Comparing the coefficients of $e^{2\delta\xi},\cdots,e^{N\delta\xi}$ in both sides of the system (3.4), we can determine $\boldsymbol{a}^2,\cdots,\boldsymbol{a}^N$, successively. In fact, once $\boldsymbol{a}^1,\cdots,\boldsymbol{a}^{j-1}$ have been determined, comparing the coefficients of $e^{j\delta\xi}$ in both sides of (3.4), it is not difficult to see that $\boldsymbol{a}^j$ satisfies

$$
\Big(-\lambda^2\boldsymbol{I}+\boldsymbol{A}(\boldsymbol{0})-\int_0^\infty \boldsymbol{B}(\eta,\boldsymbol{0})e^{-j\lambda\delta\eta}\mathrm{d}\eta\Big)\boldsymbol{a}^j=\boldsymbol{b}^j(\boldsymbol{a}^1,\cdots,\boldsymbol{a}^{j-1}), \qquad (3.5)
$$

where $\boldsymbol{b}^j=(b_1^j,b_2^j,b_3^j)$ is a given polynomial of $\boldsymbol{a}^1,\cdots,\boldsymbol{a}^{j-1}$. Note that the choice of δ and $\boldsymbol{a}^j$ is determined uniquely by the system (3.5).

We next employ the contraction mapping principle to determine $\boldsymbol{p}(\xi)$ in (3.1), such that $\boldsymbol{v}(\xi)$ is a solution to the system (2.3). Let

$$
S_K=\Big\{\boldsymbol{p}\in\boldsymbol{C}((-\infty,\xi_0];\mathbb{R}^3),\ \|\boldsymbol{p}\|\le K\Big\},
$$

where K is a fixed positive constant and

$$
\|\boldsymbol{p}\|=\sup_{\xi\in(-\infty,\xi_0]} e^{-N\delta\xi}|\boldsymbol{p}(\xi)|.
$$

We consider a map T: $\boldsymbol{q}=T\boldsymbol{p}$, $\forall\ \boldsymbol{p}\in S_K$, determined by

$$
\begin{aligned}
-\lambda^2 q_i+\sum_{j,k,l=1}^{3}\omega_j\omega_l a_{ijkl}(\boldsymbol{0})q_k=&\lambda^2(a_i^1 e^{\delta\xi}+\cdots+a_i^N e^{N\delta\xi})\\
&+\sum_{j,k,l=1}^{3}\omega_j\omega_l a_{ijkl}(\boldsymbol{0})p_k-\sum_{j=1}^{3}\omega_j g_{ij}\Big((\boldsymbol{a}^1 e^{\delta\xi}+\cdots+\boldsymbol{a}^N e^{N\delta\xi}+\boldsymbol{p}(\xi))\otimes\boldsymbol{\omega}\Big)\\
&+\int_0^\infty\sum_{j=1}^{3}\omega_j h_{ij}\Big(\eta,(\boldsymbol{a}^1 e^{\delta(\xi-\lambda\eta)}+\cdots+\boldsymbol{a}^N e^{N\delta(\xi-\lambda\eta)}\\
&\qquad+\boldsymbol{p}(\xi-\lambda\eta))\otimes\omega\Big)\mathrm{d}\eta-A_i,\quad \forall\ \boldsymbol{p}\in S_K,\ \ i=1,2,3. \qquad (3.6)
\end{aligned}
$$

It is easy to see that this map T can be rewritten as

$$
(-\lambda^2\boldsymbol{I}+\boldsymbol{A}(\boldsymbol{0}))\boldsymbol{q}=\int_0^\infty \boldsymbol{B}(\eta,\boldsymbol{0})\boldsymbol{p}(\xi-\lambda\eta)\mathrm{d}\eta+\mathcal{O}\left(e^{(N+1)\delta\xi}\right)+\mathbf{M}(\boldsymbol{p}),\quad \forall\ \boldsymbol{p}\in S_K, \qquad (3.7)
$$

where $\mathcal{O}(e^{(N+1)\delta\xi})$ is independent of $\boldsymbol{p}$, and $\mathbf{M}(\boldsymbol{p})$ satisfies the following estimate with a positive constant $C(K)$ depending on K:

$$\|\mathbf{M}(\boldsymbol{p})\|\le C(K)e^{\delta\xi_0},\quad \forall\,\boldsymbol{p}\in S_K. \tag{3.8}$$

In view of the assumptions in Theorem 2.1, we can take N so large that

$$\rho\Big(\int_0^\infty \boldsymbol{B}(\eta,\mathbf{0})e^{-N\lambda\delta\eta}\mathrm{d}\eta\Big)<r(\boldsymbol{A}(\mathbf{0}))-\lambda^2. \tag{3.9}$$

Then, we can take ξ_0 sufficiently negative so that

$$\|\mathcal{O}(e^{(N+1)\delta\xi})\|,\ \|\mathbf{M}(\boldsymbol{p})\|<\frac{1}{2}(1-\sigma)(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2)K,\ \forall\,\xi\in(-\infty,\xi_0],\ \boldsymbol{p}\in S_K, \tag{3.10}$$

where

$$\sigma=\rho\Big(\int_0^\infty \boldsymbol{B}(\eta,\mathbf{0})e^{-N\lambda\delta\eta}\mathrm{d}\eta\Big)\Big/\Big(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2\Big). \tag{3.11}$$

Lemma 3.2. *Suppose that the assumptions of Theorem* 2.1 *hold. Let N and ξ_0 satisfy* (3.9) *and* (3.10)*, respectively. Then the map $\boldsymbol{q}=T\boldsymbol{p},\forall\ \boldsymbol{p}\in S_K$, defined by* (3.6)*, is injective.*

Proof. From (3.7) we have

$$\begin{aligned}&(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2)\|\boldsymbol{q}\|\\ \le\;&\rho\Big(\int_0^\infty \boldsymbol{B}(\eta,\mathbf{0})e^{-N\lambda\delta\eta}\mathrm{d}\eta\Big)\|\boldsymbol{p}\|+\|\mathcal{O}(e^{(N+1)\delta\xi})\|+\|\mathbf{M}(\boldsymbol{p})\|,\quad\forall\ \boldsymbol{p}\in S_K.\end{aligned} \tag{3.12}$$

In view of (3.9) and (3.10), we get from (3.12) that $\|\boldsymbol{q}\|\le K$, that is, $\boldsymbol{q}\in S_K$. □

Lemma 3.3. *Suppose the assumptions of Lemma* 3.2 *hold. If ξ_0 is sufficiently negative, then the map $T:S_K\longrightarrow S_K$ is contractive.*

Proof. Suppose that $\boldsymbol{p},\tilde{\boldsymbol{p}}\in S_K$ and $\boldsymbol{q}=T\boldsymbol{p},\tilde{\boldsymbol{q}}=T\tilde{\boldsymbol{p}}$. Then, it follows from the definition (3.6) of the map T and the expression (2.4) of A_i that

$$\begin{aligned}&-\lambda^2(\tilde{q}_i-q_i)+\sum_{j,k,l=1}^3\omega_j\omega_l a_{ijkl}(\mathbf{0})(\tilde{q}_k-q_k)\\ =&\sum_{j,k,l=1}^3\omega_j\omega_l\int_0^1\Big(a_{ijkl}(\mathbf{0})-a_{ijkl}\big((\boldsymbol{a}^1e^{\delta\xi}+\cdots+\boldsymbol{a}^Ne^{N\delta\xi}+\boldsymbol{p}(\xi)\\ &\quad+\theta(\tilde{\boldsymbol{p}}(\xi)-\boldsymbol{p}(\xi)))\otimes\boldsymbol{\omega}\big)\Big)\mathrm{d}\theta(\tilde{p}_k-p_k)\\ &+\int_0^\infty\sum_{j,k,l=1}^3\omega_j\omega_l\int_0^1 b_{ijkl}\Big(\eta,(\boldsymbol{a}^1e^{\delta(\xi-\lambda\eta)}+\cdots+\boldsymbol{a}^Ne^{N\delta(\xi-\lambda\eta)}+\boldsymbol{p}(\xi-\lambda\eta)\\ &\quad+\theta(\tilde{\boldsymbol{p}}(\xi-\lambda\eta)-\boldsymbol{p}(\xi-\lambda\eta)))\otimes\boldsymbol{\omega}\Big)\mathrm{d}\theta(\tilde{p}_k(\xi-\lambda\eta)-p_k(\xi-\lambda\eta))\mathrm{d}\eta,\quad i=1,\,2,\,3.\end{aligned} \tag{3.13}$$

For σ given by (3.11), noting (2.9), we can take $\varepsilon>0$ sufficiently small such that

$$\rho\Big(\sum_{j,l=1}^{3}\omega_j\omega_l\big(a_{ijkl}(\boldsymbol{v}\otimes\boldsymbol{\omega})-a_{ijkl}(\mathbf{0})\big)\Big)\leq\frac{1}{4}(1-\sigma)(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2), \tag{3.14}$$

$$\begin{aligned}&\rho\Big(\int_0^\infty\sum_{j,l=1}^{3}\omega_j\omega_l b_{ijkl}(\eta,\boldsymbol{v}\otimes\boldsymbol{\omega})e^{-N\lambda\delta\eta}\mathrm{d}\eta\Big)\\ \leq&\rho\Big(\int_0^\infty\boldsymbol{B}(\eta,\mathbf{0})e^{-N\lambda\delta\eta}\mathrm{d}\eta\Big)+\frac{1}{4}(1-\sigma)\big(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2\big),\end{aligned} \tag{3.15}$$

provided that

$$|\boldsymbol{v}\otimes\boldsymbol{\omega}|\leq\varepsilon.$$

Taking ξ_0 so negative that

$$\big|(\boldsymbol{a}^1e^{\delta\xi}+\cdots+\boldsymbol{a}^Ne^{N\delta\xi}+\boldsymbol{p}(\xi))\otimes\boldsymbol{\omega}\big|\leq\varepsilon,\quad\forall\ \boldsymbol{p}\in S_K,\ \ \xi\in(-\infty,\xi_0], \tag{3.16}$$

by the above hypotheses, we obtain from (3.13) that

$$\begin{aligned}&\big(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2\big)\|\tilde{\boldsymbol{q}}-\boldsymbol{q}\|\\ \leq&\frac{1}{4}(1-\sigma)(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2)\|\tilde{\boldsymbol{p}}-\boldsymbol{p}\|+\Big(\rho\Big(\int_0^\infty\boldsymbol{B}(\eta,\mathbf{0})e^{-N\lambda\delta\eta}\mathrm{d}\eta\Big)\\ &+\frac{1}{4}(1-\sigma)(r(\boldsymbol{A}(\mathbf{0}))-\lambda^2)\Big)\|\tilde{\boldsymbol{p}}-\boldsymbol{p}\|.\end{aligned} \tag{3.17}$$

Consequently,

$$\|\tilde{\boldsymbol{q}}-\boldsymbol{q}\|\leq\frac{1}{2}(1+\sigma)\|\tilde{\boldsymbol{p}}-\boldsymbol{p}\|<\|\tilde{\boldsymbol{p}}-\boldsymbol{p}\|,$$

provided that $(1+\sigma)/2<1$. □

Proof of Theorem 2.1. From Lemmas 3.2 and 3.3, we can reach the conclusion of Theorem 2.1 immediately. □

3.2 Global existence

First, for any fixed $T_0>\xi_0$, we give a *prior* estimate for the solution on $[\xi_0,T_0]$. Suppose that there is a solution $\boldsymbol{v}(\xi)$ to the system (2.10) on $[\xi_0,T_0]$. Then we have

$$\begin{aligned}&-\lambda^2v_i(\xi)+\sum_{j,k,l=1}^{3}\omega_j\omega_l\int_0^1a_{ijkl}(\theta\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\mathrm{d}\theta\ v_k(\xi)\\ =&\int_0^{\frac{1}{\lambda}(\xi-\xi_0)}\sum_{j,k,l=1}^{3}\omega_j\omega_l\int_0^1b_{ijkl}(\eta,\theta\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})\mathrm{d}\theta v_k(\xi-\lambda\eta)\mathrm{d}\eta+\mathscr{F}_i(\xi),\end{aligned}$$

with

$$\mathscr{F}_i(\xi)=\int_{\frac{1}{\lambda}(\xi-\xi_0)}^{\infty}\sum_{j=1}^{3}\omega_j h_{ij}(\eta,\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})\mathrm{d}\eta-\int_{\frac{1}{\lambda}(\xi-\xi_0)}^{\infty}\sum_{j=1}^{3}\omega_j h_{ij}(\eta,\mathbf{0})\mathrm{d}\eta,\quad i=1,\,2,\,3.$$

That is,

$$\begin{aligned}&\Big(-\lambda^2\boldsymbol{I}+\int_0^1 A(\theta\boldsymbol{v})\mathrm{d}\theta\Big)\cdot\boldsymbol{v}(\xi)\\ =&\int_0^{\frac{1}{\lambda}(\xi-\xi_0)}\int_0^1\boldsymbol{B}(\eta,\theta\boldsymbol{v}(\xi-\lambda\eta))\mathrm{d}\theta\cdot\boldsymbol{v}(\xi-\lambda\eta)\mathrm{d}\eta+\mathscr{F}(\xi),\end{aligned}\tag{3.18}$$

where $\mathscr{F}(\xi)=(\mathscr{F}_1(\xi),\mathscr{F}_2(\xi),\mathscr{F}_3(\xi))^T$. Now, by (2.11) and (2.12), we get

$$\begin{aligned}\big(-\lambda^2+\bar{r}\big)|\boldsymbol{v}(\xi)|&\le\int_0^{\frac{1}{\lambda}(\xi-\xi_0)}b(\eta)|\boldsymbol{v}(\xi-\lambda\eta)|\mathrm{d}\eta+\sup_{\xi\in[\xi_0,T_0]}|\mathscr{F}|\\ &\le\tilde{C}_1\int_{\xi_0}^{\xi}|\boldsymbol{v}(\tau)|\mathrm{d}\tau+\tilde{C}_2.\end{aligned}$$

Then, by Gronwall's inequality, we obtain the estimate

$$|\boldsymbol{v}(\xi)|\le C_2+C_3e^{C_1(T_0-\xi_0)}\triangleq M,\ \ \forall\,\xi\in[\xi_0,T_0].\tag{3.19}$$

Proof of the Theorem 2.2. According to Theorem 2.1, there exists a nontrivial solution on $(-\infty,\xi_0]$. To obtain the existence on $[\xi_0,\infty)$, we need to show only that the system (2.10) admits a solution $\boldsymbol{v}(\xi)$ on $[\xi_0,T_0]$ for any $T_0>\xi_0$. For this purpose, we define an operator T by

$$\begin{aligned}&-\lambda^2(Tv)_i(\xi)+\sum_{j=1}^{3}\omega_j g_{ij}\big((T\boldsymbol{v})(\xi)\otimes\boldsymbol{\omega}\big)\\ =&\int_0^{\frac{1}{\lambda}(\xi-\xi_0)}\sum_{j=1}^{3}\omega_j h_{ij}(\eta,\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})\mathrm{d}\eta+f_i(\xi),\quad i=1,\,2,\,3.\end{aligned}\tag{3.20}$$

It is not difficult to show that (3.20) defines uniquely a continuous function $(T\boldsymbol{v})(\xi)$. The function $\boldsymbol{v}$ satisfying

$$(T\boldsymbol{v})(\xi)=\boldsymbol{v}(\xi)\quad\text{for all }\ \xi\in[\xi_0,T_0]$$

is, of course, a solution to the system (2.1).

We take the domain of T to be

$$K=\Big\{\boldsymbol{v}(\xi)\in C\big([\xi_0,T_0];\mathbb{R}^3\big)\,\Big|\,\boldsymbol{v}(\xi_0)=\boldsymbol{v}^0,\,|\boldsymbol{v}(\xi)|\le M\Big\},\tag{3.21}$$

where M is defined by (3.19), $\boldsymbol{v}^0$ is the value at ξ_0 of the local solution which has been constructed on $(-\infty,\xi_0]$.

It is clear that the function $\xi \mapsto (T\boldsymbol{v})(\xi)$ is continuous. Now we prove that T maps K into itself. To this end, taking $\xi = \xi_0$ in the system (3.20), we get

$$-\lambda^2 (Tv)_i(\xi_0) + \sum_{j=1}^{3} \omega_j g_{ij}((T\boldsymbol{v})(\xi_0) \otimes \boldsymbol{\omega}) = f_i(\xi_0).$$

On the other hand, from the definition of $\boldsymbol{v}^0$, we have

$$-\lambda^2 v_i^0 + \sum_{j=1}^{3} \omega_j g_{ij}(\boldsymbol{v}^0 \otimes \boldsymbol{\omega}) = f_i(\xi_0).$$

Recalling now (2.11), we obtain $(T\boldsymbol{v})(\xi_0) = \boldsymbol{v}^0$ immediately. Similarly to getting a *prior* estimate (3.19), we can obtain that

$$|(T\boldsymbol{v})(\xi)| \le M \quad \text{for any } \ \xi \in [\xi_0, T_0].$$

Therefore, T maps K into itself.

The set K is a bounded, closed and convex set. Now we show that T is a compact map of K into itself. By Arzelà-Ascoli Theorem and what we have already proved, this suffices to show that the set $\{T\boldsymbol{v} | \boldsymbol{v} \in K\}$ is equicontinuous. Let $\xi_0 \le \xi_1 \le \xi_2 \le T_0$. Noting (2.11), we have

$$\begin{aligned}
&(-\lambda^2 + \bar{r})|(T\boldsymbol{v})(\xi_2) - (T\boldsymbol{v})(\xi_1)| \\
\le& |\boldsymbol{f}(\xi_2) - \boldsymbol{f}(\xi_1)| + \int_{\frac{1}{\lambda}(\xi_1 - \xi_0)}^{\frac{1}{\lambda}(\xi_2 - \xi_0)} \Big| \boldsymbol{h}(\eta, \boldsymbol{v}(\xi_2 - \lambda\eta) \otimes \boldsymbol{\omega}) \Big| \mathrm{d}\eta \\
&+ \Big| \int_0^{\frac{1}{\lambda}(\xi_1 - \xi_0)} \boldsymbol{h}(\eta, \boldsymbol{v}(\xi_2 - \lambda\eta) \otimes \boldsymbol{\omega}) - \boldsymbol{h}(\eta, \boldsymbol{v}(\xi_1 - \lambda\eta) \otimes \boldsymbol{\omega}) \mathrm{d}\eta \Big|,
\end{aligned} \tag{3.22}$$

where

$$\boldsymbol{f}(\xi) = (f_1(\xi), f_2(\xi), f_3(\xi))^T, \ \boldsymbol{h} = \Big(\sum_{j=1}^{3} \omega_j h_{1j}, \sum_{j=1}^{3} \omega_j h_{2j}, \sum_{j=1}^{3} \omega_j h_{3j} \Big)^T.$$

Then in view of (2.12), from (3.22) we conclude immediately that the set $\{T\boldsymbol{v} \mid \boldsymbol{v} \in K\}$ is equicontinuous on $[\xi_0, T_0]$.

An application of the Schauder's fixed point theorem shows that there exists a fixed point $\boldsymbol{v}$ of T. The proof of Theorem 2.2 is complete. □

4 Proof of the uniqueness of trivial solution

Lemma 4.1. *Let $\boldsymbol{M}(t) \in L^1(\mathbb{R}^+; \mathbb{R}^{3\times 3})$ be of positive type. For any bounded function $\boldsymbol{v} \in C((-\infty, \xi_0]; \mathbb{R}^3)$, we have*

$$\int_{-\infty}^{\xi_0} \int_0^{\infty} \boldsymbol{M}(\eta) \boldsymbol{v}(\xi - \lambda\eta) \mathrm{d}\eta \cdot \boldsymbol{v}(\xi) \mathrm{d}\xi \ge 0, \quad \forall \ \lambda > 0. \tag{4.1}$$

Proof. The result can be obtained by using a way similar to that of Lemma 3.4 in [3]. □

In the same way we can prove the following result.

Lemma 4.2. *Let $\boldsymbol{M}(t)\in L^1(\mathbb{R}^+;\mathbb{R}^{3\times 3})$ be of positive type. For any function $\boldsymbol{v}\in C([\xi^*,\xi^*+h];\mathbb{R}^3)$, we have*

$$\int_{\xi^*}^{\xi^*+h}\int_0^{\frac{1}{\lambda}(\xi-\xi^*)}\boldsymbol{M}(\eta)\boldsymbol{v}(\xi-\lambda\eta)\mathrm{d}\eta\cdot\boldsymbol{v}(\xi)\mathrm{d}\xi\geq 0,\quad\forall\,\lambda>0,\ \xi^*\geq\xi_0,\ h>0. \tag{4.2}$$

Proof. It is easy to verify that

$$\begin{aligned}&\int_{\xi^*}^{\xi^*+h}\int_0^{\frac{1}{\lambda}(\xi-\xi^*)}\boldsymbol{M}(\eta)\boldsymbol{v}(\xi-\lambda\eta)\mathrm{d}\eta\cdot\boldsymbol{v}(\xi)\mathrm{d}\xi\\ =&\frac{1}{\lambda}\int_0^h\int_0^\xi\boldsymbol{M}(\frac{\xi-\eta}{\lambda})\boldsymbol{v}(\eta+\xi^*)\mathrm{d}\eta\cdot\boldsymbol{v}(\xi+\xi^*)\mathrm{d}\xi.\end{aligned}$$

Then from the definition of positive type, we get (4.2) immediately. □

Proof of Theorem 2.3. Set $\lambda^2>\rho(\boldsymbol{A}(0))$. Noting (2.4), the system (2.3) can be written as

$$\begin{aligned}\lambda^2 v_i(\xi)=&\sum_{j,k,l=1}^3\omega_j\omega_l\int_0^1 a_{ijkl}(\theta\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\mathrm{d}\theta\, v_k(\xi)-\int_0^\infty\sum_{j,k,l=1}^3\omega_j\omega_l b_{ijkl}(\eta,\boldsymbol{0})v_k(\xi-\lambda\eta)\mathrm{d}\eta\\ &-\int_0^\infty\sum_{j,k,l=1}^3\omega_j\omega_l\int_0^1 A_{ijkl}\mathrm{d}\theta\, v_k(\xi-\lambda\eta)\mathrm{d}\eta,\end{aligned} \tag{4.3}$$

where

$$A_{ijkl}=b_{ijkl}(\eta,\theta\boldsymbol{v}(\xi-\lambda\eta)\otimes\boldsymbol{\omega})-b_{ijkl}(\eta,\boldsymbol{0}). \tag{4.4}$$

Multiplying both sides of (4.3) by $e^{2\delta\xi}v_i(\xi)$ $(\delta>0)$ and adding the resulting equations for $i=1,\,2,\,3$, we get

$$\begin{aligned}\sum_{i=1}^3\lambda^2 e^{2\delta\xi}v_i^2(\xi)=&\sum_{i,j,k,l=1}^3\omega_j\omega_l\int_0^1 a_{ijkl}(\theta\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\mathrm{d}\theta\, e^{2\delta\xi}v_k(\xi)v_i(\xi)\\ &-\int_0^\infty\sum_{i,j,k,l=1}^3\omega_j\omega_l b_{ijkl}(\eta,\boldsymbol{0})v_k(\xi-\lambda\eta)\mathrm{d}\eta\, e^{2\delta\xi}v_i(\xi)\\ &-\int_0^\infty\sum_{i,j,k,l=1}^3\omega_j\omega_l\int_0^1 A_{ijkl}\mathrm{d}\theta\, v_k(\xi-\lambda\eta)\mathrm{d}\eta\, e^{2\delta\xi}v_i(\xi),\end{aligned} \tag{4.5}$$

where A_{ijkl} is defined by (4.4). Integrating (4.5) with respect to ξ from $-\infty$ to ξ_0 gives

$$\lambda^2\int_{-\infty}^{\xi_0}e^{2\delta\xi}|\boldsymbol{v}(\xi)|^2\mathrm{d}\xi=I_1-I_2-I_3 \tag{4.6}$$

with

$$I_1=\int_{-\infty}^{\xi_0}\sum_{i,j,k,l=1}^{3}\omega_j\omega_l\int_0^1 a_{ijkl}(\theta\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\mathrm{d}\theta\ e^{2\delta\xi}v_k(\xi)v_i(\xi)\mathrm{d}\xi,$$

$$I_2=\int_{-\infty}^{\xi_0}\int_0^{\infty}\sum_{i,j,k,l=1}^{3}\omega_j\omega_l b_{ijkl}(\eta,\mathbf{0})v_k(\xi-\lambda\eta)\ \mathrm{d}\eta\ e^{2\delta\xi}v_i(\xi)\mathrm{d}\xi,$$

$$I_3=\int_{-\infty}^{\xi_0}\int_0^{\infty}\sum_{i,j,k,l=1}^{3}\omega_j\omega_l\int_0^1 A_{ijkl}\mathrm{d}\theta\ e^{2\delta\xi}v_k(\xi-\lambda\eta)\mathrm{d}\eta\ v_i(\xi)\mathrm{d}\xi.$$

Set $\varepsilon=\lambda^2-\rho(\boldsymbol{A}(0))$. Taking ξ_0 sufficiently negative, we have

$$\rho\Big(\sum_{j,l=1}^{3}\omega_j\omega_l\int_0^1 a_{ijkl}(\theta\boldsymbol{v}(\xi)\otimes\boldsymbol{\omega})\mathrm{d}\theta\Big)<\rho(\boldsymbol{A}(\mathbf{0}))+\frac{\varepsilon}{4}. \tag{4.7}$$

Then

$$|I_1|\le\Big(\rho(\boldsymbol{A}(\mathbf{0}))+\frac{\varepsilon}{4}\Big)\int_{-\infty}^{\xi_0}e^{2\delta\xi}|\boldsymbol{v}(\xi)|^2\mathrm{d}\xi. \tag{4.8}$$

Let $\delta=\delta_0/\lambda$, $\boldsymbol{w}(\xi)=e^{\delta\xi}\boldsymbol{v}(\xi)$. By Lemma 4.1, we have

$$\begin{aligned}I_2&=\int_{-\infty}^{\xi_0}\int_0^{\infty}\sum_{i,j,k,l=1}^{3}\omega_j\omega_l b_{ijkl}(\eta,\mathbf{0})e^{\lambda\delta\eta}e^{\delta(\xi-\lambda\eta)}v_k(\xi-\lambda\eta)\ \mathrm{d}\eta\ e^{\delta\xi}v_i(\xi)\mathrm{d}\xi\\&=\int_{-\infty}^{\xi_0}\int_0^{\infty}e^{\delta_0\eta}\boldsymbol{B}(\eta,\mathbf{0})\boldsymbol{w}(\xi-\lambda\eta)\mathrm{d}\eta\cdot\boldsymbol{w}(\xi)\mathrm{d}\xi\ge 0.\end{aligned} \tag{4.9}$$

Taking ξ_0 sufficiently negative, we have

$$\sup_{\substack{\theta\in[0,1]\\ \xi\in(-\infty,\xi_0]}}\sum_{j,l=1}^{3}\int_0^{\infty}|\omega_j||\omega_l|\big|A_{ijkl}\big|e^{\delta_0\eta}\mathrm{d}\eta\cdot\int_0^{\infty}e^{\delta_0\eta}c(\eta)\mathrm{d}\eta<\frac{\varepsilon^2}{288}. \tag{4.10}$$

Then, by Cauchy inequality, we get

$$\begin{aligned}|I_3|\le&\sum_{i,j,k,l=1}^{3}\int_{-\infty}^{\xi_0}\int_0^1\int_0^{\infty}|\omega_j\omega_l|\ \big|A_{ijkl}\big|\ e^{\delta_0\eta}\ |v_k(\xi-\lambda\eta)|\ e^{\delta(\xi-\lambda\eta)}\ \mathrm{d}\eta\ \mathrm{d}\theta\ e^{\delta\xi}\ |v_i(\xi)|\mathrm{d}\xi\\ \le&\sum_{i,j,k,l=1}^{3}\int_{-\infty}^{\xi_0}\int_0^1\Big(\int_0^{\infty}|\omega_j\omega_l|\ \big|A_{ijkl}\big|\ e^{\delta_0\eta}\mathrm{d}\eta\Big)^{\frac12}\\&\cdot\Big(\int_0^{\infty}|\omega_j\omega_l|\ \big|A_{ijkl}\big|\ e^{\delta_0\eta}\ |v_k(\xi-\lambda\eta)|^2\ e^{2\delta(\xi-\lambda\eta)}\mathrm{d}\eta\Big)^{\frac12}\ \mathrm{d}\theta\ e^{\delta\xi}\ |v_i(\xi)|\ \mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&\le \frac{B_\varepsilon}{\sqrt{2}} \sum_{i,j,k,l=1}^{3} \int_{-\infty}^{\xi_0} \int_0^1 \Big(\int_0^\infty |\omega_j \omega_l| \, \big| A_{ijkl} \big| \, e^{\delta_0 \eta} \, |v_k(\xi-\lambda\eta)|^2 \, e^{2\delta(\xi-\lambda\eta)} \mathrm{d}\eta \Big)^{\frac{1}{2}} \, e^{\delta\xi} \, |v_i(\xi)| \mathrm{d}\theta \, \mathrm{d}\xi \\
&\le B_\varepsilon \sum_{i,k=1}^{3} \int_{-\infty}^{\xi_0} \Big(\int_0^\infty e^{\delta_0 \eta} c(\eta) |v_k(\xi-\lambda\eta)|^2 \, e^{2\delta(\xi-\lambda\eta)} \mathrm{d}\eta \Big)^{\frac{1}{2}} e^{\delta\xi} \, |v_i(\xi)| \, \mathrm{d}\xi \\
&\le B_\varepsilon \sum_{i,k=1}^{3} \Big(\int_0^\infty e^{\delta_0 \eta} c(\eta) \int_{-\infty}^{\xi_0} |v_k(\xi-\lambda\eta)|^2 \, e^{2\delta(\xi-\lambda\eta)} \mathrm{d}\xi \mathrm{d}\eta \int_{-\infty}^{\xi_0} e^{2\delta\xi} \, |v_i(\xi)|^2 \mathrm{d}\xi \Big)^{\frac{1}{2}} \\
&\le \frac{\varepsilon}{4} \int_{-\infty}^{\xi_0} e^{2\delta\xi} |\boldsymbol{v}(\xi)|^2 \mathrm{d}\xi.
\end{aligned} \tag{4.11}
$$

where A_{ijkl} is defined by (4.4), and

$$
B_\varepsilon = \frac{\varepsilon}{12} \Big(\int_0^\infty e^{\delta_0 \eta} c(\eta) \mathrm{d}\eta \Big)^{-\frac{1}{2}}.
$$

Finally, combining (4.8), (4.9) with (4.11) yields

$$
\lambda^2 \int_{-\infty}^{\xi_0} e^{2\delta\xi} |\boldsymbol{v}(\xi)|^2 \mathrm{d}\xi \le \Big(\rho(A(\mathbf{0})) + \frac{\varepsilon}{2} \Big) \int_{-\infty}^{\xi_0} e^{2\delta\xi} |\boldsymbol{v}(\xi)|^2 \mathrm{d}\xi. \tag{4.12}
$$

Then, we obtain $\boldsymbol{v}(\xi) \equiv \mathbf{0}$ on $(-\infty, \xi_0]$ immediately. Let

$$
\xi^* = \sup\{ \xi \mid \boldsymbol{v}(\zeta) \equiv \mathbf{0}, -\infty < \zeta < \xi \}. \tag{4.13}
$$

If $\xi^* = +\infty$, it means that $\boldsymbol{v}(\xi) \equiv \mathbf{0}$ on $\mathbb{R}$. Now we suppose $\xi^* < \infty$. Then the system (2.3) can be written as

$$
\begin{aligned}
\lambda^2 v_i(\xi) = & \sum_{j,k,l=1}^{3} \omega_j \omega_l \int_0^1 a_{ijkl}(\theta \boldsymbol{v}(\xi) \otimes \boldsymbol{\omega}) \mathrm{d}\theta \, v_k(\xi) \\
& - \int_0^{\frac{1}{\lambda}(\xi-\xi^*)} \sum_{j,k,l=1}^{3} \omega_j \omega_l b_{ijkl}(\eta, \mathbf{0}) v_k(\xi-\lambda\eta) \mathrm{d}\eta \\
& - \int_0^{\frac{1}{\lambda}(\xi-\xi^*)} \sum_{j,k,l=1}^{3} \omega_j \omega_l \int_0^1 A_{ijkl} \mathrm{d}\theta \, v_k(\xi-\lambda\eta) \mathrm{d}\eta.
\end{aligned} \tag{4.14}
$$

By Lemma 4.2, using the same proof as above, we can get $\boldsymbol{v}(\xi) \equiv \mathbf{0}$ on $[\xi^*, \xi^*+h]$ for sufficiently small h. This is a contradiction. The proof of Theorem 2.3 is complete. □

The proof for the case of

$$
\lambda^2 < r(\boldsymbol{A}(\mathbf{0})) - \rho\Big(\int_0^\infty \boldsymbol{B}(\eta, \mathbf{0}) \mathrm{d}\eta \Big)
$$

is similar and is omitted here.

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