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Morrey Estimates for Nondivergence Parabolic Operators with Discontinuous Coefficients on Homogeneous Groups

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Abstract. In this paper, by establishing the boundedness of singular integral operators with variable kernels and their commutators with BMO functions on Morrey spaces of homogeneous groups, we prove a local a prior estimate in Sobolev-Morrey space for solutions to the nondivergence parabolic equation with discontinuous coefficients.

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1 Introduction

In recent years, there has been extensive study on a priori estimates for partial differential operators by using singular integral theory (see [1–3]). For example, authors in [4–7] proved Morrey estimates for nondivergence elliptic operators with discontinuous coefficients on Euclidean spaces. Bramanti and Brandolini proved Schauder estimates for parabolic operators of Hörmander type in [8] and L^p-estimates for hypoelliptic operators of Hörmander type in [9], respectively. The L^p-estimates and Morrey estimates for ultraparabolic operator of Kolmogorov-Fokker-Planck type

$$\mathcal{L}=\sum_{i,j=1}^{q} a_{ij}(x,t)\partial_{x_i}\partial_{x_j}+\sum_{i,j=1}^{N} x_i b_{ij}\partial_{x_j}-\partial_t, \quad (x,t)\in\mathbb{R}^{N+1} \tag{1.1}$$

were considered in [10] and [11], respectively, where the coefficients $a_{ij}\in VMO\cap L^\infty(\Omega)$, $\{b_{ij}\}$ is a constant real matrix with a suitable upper triangular structure. The class of

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operators (1.1) contains prototypes of the Fokker-Planck operators describing Brownian motions of a particle in fluid, as well as Kolmogorov operators depicting systems with $2n$ degrees of freedom (see [12]).

Let $a_{ij}=\delta_{ij}$, $X_i=\partial_{x_i}$, and $X_0=\sum_{i,j=1}^N x_i b_{ij}\partial_{x_j}-\partial_t$. Then (1.1) becomes a hypoelliptic operator

$$\mathcal{L}=\sum_{i,j=1}^{q} X_i^2+X_0,$$

where $X_0,X_1,\cdots,X_q$ satisfy the Hörmander condition, i.e., the Lie algebra generated at every point by the fields $X_0,X_1,\cdots,X_q$ is $\mathbb{R}^{N+1}$. In [12], the L^p-estimate for the following operator on homogeneous groups

$$\mathcal{L}=\sum_{i,j=1}^{q} a_{ij}X_iX_j+a_0X_0,$$

was proved, where a_0, $a_{ij}\in VMO\cap L^\infty(\Omega)$ and vector fields X_0, X_1, $\cdots,X_q$ satisfy the Hörmander condition.

The aim of this paper is to establish Morrey estimates for nondivergence parabolic operators with discontinuous coefficients and lower order terms on homogeneous groups

$$\mathbf{H}=\sum_{i,j=1}^{q} a_{ij}(z)X_iX_j+\sum_{i=1}^{q} b_i(z)X_i+c(z)-\partial_t,\quad z=(x,t)\in\mathbb{R}^{N+1}, \tag{1.2}$$

where $X_1,\cdots,X_q$ are the first layer of the basis of vector fields of homogeneous groups, the coefficients a_{ij},b_i,c satisfy the following several assumptions:

• (H1) uniform ellipticity condition: $a_{ij}(z)\in L^\infty(\Omega)$, $\Omega\subset\mathbb{R}^{N+1}$ and there exists $\mu>0$ such that

$$\frac{1}{\mu}\sum_{j}^{q}\zeta_j^2\le\sum_{i,j}^{q}a_{ij}(z)\zeta_i\zeta_j\le\mu\sum_{j}^{q}\zeta_j^2,\quad \forall(\zeta_1,\cdots,\zeta_q)\in\mathbb{R}^q.$$

• (H2) very weak regularity condition: $a_{ij}(z)\in VMO(\Omega)$ (the function space of "Vanishing Mean Oscillation" (see Definition 2.3 below)).

• (H3) the coefficients $b_i(z)$ and $c(z)$ are measurable functions in Ω,

$$b_i(z)\in\begin{cases}L^{Q+2}, & p+\lambda\le Q+2,\\ L^{p,\lambda}, & p+\lambda>Q+2,\end{cases}\qquad c(z)\in\begin{cases}L^{\frac{Q+2}{2}}, & 2p+\lambda\le Q+2,\\ L^{p,\lambda}, & 2p+\lambda>Q+2.\end{cases}$$

Let

$$\mathcal{H}=\sum_{i,j=1}^{q} a_{ij}(z)X_iX_j-\partial_t. \tag{1.3}$$

It is known that under condition (H1), for any fixed $z_0 \in \mathbb{R}^{N+1}$, the "frozen" operator

$$\mathcal{H}_0 = \sum_{i,j=1}^{q} a_{ij}(z_0) X_i X_j - \partial_t, \tag{1.4}$$

is hypoelliptic and homogeneous of degree 2 about the dilation in (2.4) below.

For an open set Ω in $\mathbb{R}^{N+1}$ and $p \in (1,\infty)$, the Sobolev-Morrey space and the Sobolev-Morrey norm are defined by

$$S^{p,\lambda}(\Omega,\mathbf{H}) = \Big\{ f \in L^{p,\lambda}(\Omega) : cf, b_i X_i f, X_i X_j f, \partial_t f \in L^{p,\lambda}(\Omega), i,j = 1,\cdots,q \Big\},$$

and

$$||f||_{S^{p,\lambda}(\Omega,\mathbf{H})} = \sum_{i,j=1}^{q} ||X_i X_j f||_{p,\lambda;\Omega} + ||\partial_t f||_{p,\lambda;\Omega}, \tag{1.5}$$

respectively, where we denote $L^{p,\lambda}(\Omega)$ and $||\cdot||_{p,\lambda;\Omega}$ by Morrey space and Morrey norm, respectively, see Section 2.

We will prove a priori estimates in $S^{p,\lambda}(\Omega,\mathbf{H})$ for solutions to the equation $\mathbf{H}f = F$ when conditions (H1)-(H3) are fulfilled. The main result is the following.

Theorem 1.1. *Assume* (H1)-(H3) *hold. If* $\Omega' \subset\subset \Omega \subset \mathbb{R}^{N+1}$ $(\Omega,\Omega'$ *are bounded open sets*$)$, $F \in L^{p,\lambda}(\Omega), p \in (1,+\infty), \lambda \in (0,Q+2)$ $(Q$ *is the homogeneous dimension of the homogeneous group*$)$*, then*

$$||f||_{S^{p,\lambda}(\Omega',\mathbf{H})} \leq C\Big(||\mathbf{H}f||_{p,\lambda;\Omega} + ||f||_{p,\lambda;\Omega} \Big), \quad \forall f \in S^{p,\lambda}(\Omega), \tag{1.6}$$

where the constant C depends only on $p,\lambda,\mu,\Omega,\Omega'$ and the "VMO moduli" η_a of the coefficients a_{ij} (see Definition 2.3*).*

Remark 1.1. If f is independent of t, then from Theorem 1.1 we immediately obtain a local estimate in $S^{p,\lambda}(\Omega,\mathbf{H})$ for degenerate elliptic operators with lower order terms. If let $L^p(\Omega) = L^{p,0}(\Omega)$, then we easily get the L^p-estimate for operator $\mathbf{H}$ in (1.2) from the proof of Theorem 1.1.

The paper is organized as follows: In Section 2 we introduce the definitions of BMO, VMO and Morrey space on homogeneous groups. In Section 3, by the properties of the fundamental solutions on homogeneous groups, the solution of (1.1) can be reformulated as the singular integrals and their commutators with BMO functions. In Section 4 we prove the Morrey boundedness of the singular integrals and their commutators above using the content of Section 3. In Section 5 we establish the Morrey estimate of $X_i X_j f$ and $\partial_t f$, and then complete the proof of Theorem 1.1.

2 Notations and several lemmas

We start by introducing some notations on homogeneous groups, refer to [12] and [13] for more properties.

Let "$\cdot$" be an assigned Lie group law on $\mathbb{R}^N$, for which the identity is the origin. Suppose $\mathbb{R}^N$ is endowed with a homogeneous structure by the following dilations

$$\delta(\lambda)(x)=\delta(\lambda)(x_1,\cdots,x_N)=(\lambda^{\alpha_1}x_1,\cdots,\lambda^{\alpha_N}x_N), \tag{2.1}$$

where $\alpha_1\le\cdots\le\alpha_N$ are strictly positive integers, $\lambda>0$. We call $\mathbb{G}\equiv(\mathbb{R}^N,\cdot,\delta(\lambda))$ a homogeneous group. We denote by $Q=\alpha_1+\cdots+\alpha_N$ the homogeneous dimension of $\mathbb{G}$.

Definition 2.1. *For any $x\in\mathbb{G}\backslash\{0\}$, define the homogeneous norm $||x||_1$ of x as follows: $||x||_1=\rho_1$ if and only if $|\delta(\frac{1}{\rho_1})(x)|=1$, where $|\cdot|$ denotes the Euclidean norm, and $||0||_1=0$.*

Since (1.2) involves t, we introduce two geometric structures on $\mathbb{R}^{N+1}$, i.e., group law "$\circ$" and dilation transform $(D(\lambda))_{\lambda>0}$. For every $z=(x,t)$, $\eta=(y,s)\in\mathbb{R}^{N+1}$ with $x,y\in\mathbb{R}^N$ and $t,s\in\mathbb{R}$, let

$$z\circ\eta=(x,t)\circ(y,s)=(x\cdot y,t+s). \tag{2.2}$$

Then $(\mathbb{R}^{N+1},\circ)$ is a (noncommutative) group with neutral element $(0,0)$; the inverse of an element $z=(x,t)\in\mathbb{R}^{N+1}$ is

$$z^{-1}=(x,t)^{-1}=(x^{-1},t^{-1}). \tag{2.3}$$

The other geometric structure is a group of dilations on $\mathbb{R}^{N+1}$, denoted by $(D(\lambda))_{\lambda>0}$:

$$D(\lambda)(x,t)=(\delta(\lambda)x,\lambda^2t)=(\lambda^{\alpha_1}x_1,\cdots,\lambda^{\alpha_N}x_N,\lambda^2t). \tag{2.4}$$

Then $(\mathbb{R}^{N+1},\circ,D(\lambda))$ also constitute a homogeneous group $\mathbb{G}\times\mathbb{R}$. We denote by $Q+2$ the homogeneous dimension of $\mathbb{R}^{N+1}\equiv\mathbb{G}\times\mathbb{R}$, where Q is the homogeneous dimension of $\mathbb{G}$.

Definition 2.2. (*see* [14]) *For any $z\in\mathbb{R}^{N+1}\backslash\{0\}$, define the homogeneous norm of z as $||z||$: $||z||=\rho$ if and only if $|D(\frac{1}{\rho})(x)|=1$ and $||0||=0$.*

Proposition 2.1. *The norm $||\cdot||$ has the following properties:*

(i) $||D(\lambda)z||=\lambda||z||$ *for every* $z\in\mathbb{R}^{N+1}$, $\lambda>0$;

(ii) *The set* $\{z\in\mathbb{R}^{N+1}:||z||=1\}$ *is the Euclidean unit sphere* Σ_{N+1};

(iii) *Let us define*

$$|||(x,t)|||=||x||_2+|t|^{1/2},$$

where

$$||x||_2=\sum_{j=1}^N|x_j|^{1/\alpha_i},\quad (x,t)\in\mathbb{R}^{n+1}.$$

Then $|||\cdot|||$ *satisfies* (i) *above, and*

$$\frac{1}{N+1}||z|| \leq |||z||| \leq (N+1)||z||.$$

Proof. (i) and (ii) follow immediately from (2.4) and Definition 2.2; (iii) can be proved by using Definition 2.1 and an argument similar to that in [10, Proposition 1.3]. □

In view of Proposition 2.1, it is natural to define the "quasidistance" d on $\mathbb{R}^{N+1}$:

$$d(z,\eta) = ||\eta^{-1} \circ z||.$$

For every $z \in \mathbb{R}^{N+1}$ and $r > 0$, we define the balls with respect to d as

$$B(z,r) := B_r(z) = \left\{ \eta \in \mathbb{R}^{N+1} : d(z,\eta) < r \right\}.$$

Note that $|B(z,r)| = |B(0,r)| = |B(0,1)| r^{Q+2}$. It means that the Lebesgue measure dz is a doubling measure with respect to d, that is,

$$|B(z,2r)| \leq C|B(z,r)|, \quad \text{for every } z \in \mathbb{R}^{N+1} \text{ and } r > 0.$$

Then the space $(\mathbb{R}^{N+1}, dz, d)$ is a space of homogeneous type.

Next, we will give the definitions of BMO, VMO and Morrey space on homogeneous spaces.

Definition 2.3. (*see* [10]) *For a measurable function* $f \in L^1_{loc}(\mathbb{R}^{N+1})$, *define*

$$\eta_f(R) = \sup_{r \leq R} \frac{1}{|B_r|} \int_{B_r} |f(z) - f_{B_r}| \mathrm{d}z,$$

where

$$f_{B_r} = \frac{1}{|B_r|} \int_{B_r} f(z) \mathrm{d}z.$$

Then $f \in BMO(\mathbb{R}^{N+1})$ (*bounded mean oscillation*) *if*

$$||f||_* := \sup_R \eta_f(R) < +\infty,$$

while $f \in VMO(\mathbb{R}^{N+1})$ (*vanishing mean oscillation*) *if*

$$\lim_{R \to 0} \eta_f(R) = 0.$$

For a given domain $\Omega \subset \mathbb{R}^{N+1}$, the spaces $BMO(\Omega)$ and $VMO(\Omega)$ are defined as Definition 2.3, just taking $B_r \cap \Omega$ instead of B_r.

Definition 2.4. *We say that a measurable function* $f\in L^1_{loc}(\mathbb{R}^{N+1})$ *belongs to the Morrey space* $L^{p,\lambda}(\mathbb{R}^{N+1})$, $p\in(1,+\infty),\lambda\in(0,Q+2)$, *if the following norm:*

$$||f||_{p,\lambda}:=\left(\sup_{r>0}\frac{1}{r^\lambda}\int_{B_r}|f(z)|^p\mathrm{d}z\right)^{\frac{1}{p}} \tag{2.5}$$

is finite. Similarly, the space $L^{p,\lambda}(\Omega)$ *and the norm* $||f||_{p,\lambda;\Omega}$ *are defined by taking* $B_r\cap\Omega$ *instead of* B_r *in* (2.5). *When* $\lambda=0$, $L^{p,\lambda}$ *coincides with the Lebesgue space* L^p *and* (2.5) *gives rise to the norm* $||f||_{p,0}:=||f||_p$.

3 Fundamental solutions for the frozen operator and representation formulas

For any given $z_0\in\mathbb{R}^{N+1}$, the frozen operator in (1.4) has a fundamental solution (see [13]). Let us denote it by $\Gamma(z_0;\cdot)$ which depends on the frozen coefficients $a_{ij}(z_0)$. For any $i,j=1,\cdots,q$, let

$$\Gamma_{ij}(z_0;z)=X_iX_j[\Gamma(z_0;\cdot)](z). \tag{3.1}$$

Lemma 3.1. ([13, 15]) *For every fixed* $z_0\in\mathbb{R}^{N+1}$, *we have*

(i) $\Gamma(z_0;\cdot)\in C^\infty(\mathbb{R}^{N+1}\backslash\{0\})$ *and* $\Gamma(z_0;\cdot)$ *is* $D(\lambda)$*-homogeneous of degree* $-Q$;

(ii) $\Gamma_{ij}(z_0;\cdot)\in C^\infty(\mathbb{R}^{N+1}\backslash\{0\})$ *and* $\Gamma_{ij}(z_0;\cdot)$ *is* $D(\lambda)$*-homogeneous of degree* $-(Q+2)$;

(iii) *For* $i,j=1,\cdots,q$ *and every test function* $f\in C_0^\infty(\mathbb{R}^{N+1})$, *there exist constants* $\alpha_{ij}(z_0)$ *such that*

$$X_iX_jf(z)=-P.V.\int_{\mathbb{R}^{N+1}}\Gamma_{ij}(z_0;\eta^{-1}\circ z)\mathcal{H}_0f(\eta)\mathrm{d}\eta+\alpha_{ij}(z_0)\mathcal{H}_0f(z). \tag{3.2}$$

Lemma 3.2. (Cutoff functions) *For every* $0<\gamma<r$, $z=(x,t)\in\mathbb{R}^{N+1}$, *there exists* $\varphi\in C_0^\infty(\mathbb{R}^{N+1})$ *with the following properties:*

(i) $0\le\varphi\le1$, $\varphi\equiv1$ *on* $B_\gamma(z)$, *and* $supp\varphi\subset B_r(z)$;

(ii) *There exists a constant* C *such that*

$$|D^2\varphi|=\sum_{1\le i,j\le q}|X_iX_j\varphi|\le\frac{C}{(r-\gamma)^2},\quad |\partial_t\varphi|\le\frac{C}{(r-\gamma)^2}.$$

We will write $B_\gamma(z)\prec\varphi\prec B_r(z)$ as φ satisfies the above two properties.

Proof. $B_\gamma\prec\varphi\prec B_r$ implies $B_{\gamma'}\prec\varphi\prec B_r$ for every $0<\gamma'<\gamma$. Without loss of generality, we can assume $\gamma\ge r/2$. Choose a function $f:[0,\infty)\to[0,1]$ satisfying:

$$\begin{aligned}&f\equiv1\ \text{ in }[0,\gamma];\quad f\equiv1\ \text{ in }[r,\infty];\ f\in C^\infty(0,\infty),\\ &|f^{(k)}|\le\frac{c_k}{(r-\gamma)^k}\ \text{ for }\ k\ge1.\end{aligned}$$

Let $\varphi(y,s)=f(d((x,t),(y,s)))$. Then

$$X_i\varphi(y,s)=f'(d((x,t),(y,s)))X_i(d((x,t),(\cdot,s)))(y);$$
$$X_iX_j\varphi(y,s)=f''(d((x,t),(y,s)))X_i(d((x,t),(\cdot,s)))(y)X_j(d((x,t),(\cdot,s)))(y) + f'(d((x,t),(y,s)))X_iX_j(d((x,t),(\cdot,s)))(y).$$

From Definitions 2.1 and 2.2, we get the relation between the quasidistance d on $\mathbb{R}^{N+1}$ and the quasidistance d_1 on $\mathbb{R}^N$:

$$d((x,t),(y,s))=\sqrt{\frac{d_1(x,y)^2+\sqrt{d_1(x,y)^4+4|t-s|^2}}{2}}.$$

Therefore,

$$\begin{aligned} X_i(d((x,t),(\cdot,s)))(y) &= \frac{d_1(x,y)X_i(d_1(x,\cdot))(y)+\dfrac{d_1(x,y)^3X_i(d_1(x,y))}{\sqrt{d_1(x,y)^4+4|t-s|^2}}}{2d((x,t),(y,s))} \\ &= \frac{d_1(x,y)X_i(\rho_1(\cdot))(y^{-1}\cdot x)+\dfrac{d_1(x,y)^3X_i(\rho_1(\cdot))(y^{-1}\cdot x)}{\sqrt{d_1(x,y)^4+4|t-s|^2}}}{2d((x,t),(y,s))}. \end{aligned}$$

By homogeneity of the norm ρ_1, $X_i(\rho_1(\cdot))(y^{-1}\cdot x)$ is bounded. Consequently,

$$|X_i(d((x,t),(\cdot,s)))(y)|\le C.$$

Analogously,

$$|X_iX_j(d((x,t),(\cdot,s)))(y)|\le \frac{C}{d((x,t),(y,s))},$$

for $d((x,t),(y,s))$ small enough. Then,

$$|X_i\varphi(y,s)|\le \frac{C}{r-\gamma}.$$

Since $f'(d((x,t),(y,s)))\neq 0$ for $d((x,t),(y,s))>\gamma$, we have

$$\begin{aligned} |X_iX_j\varphi(y,s)| &\le \frac{C}{(r-\gamma)^2}+\frac{C}{\gamma(r-\gamma)} \\ &\le \frac{Cr}{\gamma(r-\gamma)^2}\le \frac{C}{(r-\gamma)^2}. \end{aligned}$$

Also, note

$$\partial_s\varphi(y,s)=f'(d((x,t),(y,s)))\frac{|t-s|}{d((x,t),(y,s))\sqrt{d_1(x,y)^4+4|t-s|^2}},$$

which yields

$$|\partial_s\varphi(y,s)|\le\frac{C}{\gamma(r-\gamma)}\le\frac{Cr}{\gamma(r-\gamma)^2}\le\frac{C}{(r-\gamma)^2}.$$

This completes the proof of Lemma 3.2. □

Theorem 3.1. *For any fixed $z_0\in\mathbb{R}^{N+1}$, let*

$$\Gamma^0_{ij}(z,\eta)=X_iX_j\Gamma(z_0;\cdot)(\eta^{-1}\circ z).$$

Then we have

(i) *the growth condition:*

$$|\Gamma^0_{ij}(z,\eta)|\le\frac{C}{|B(z,d(z,\eta)|},$$

for every $z,\eta\in\mathbb{R}^{N+1}$ and some constant C;

(ii) *the Hörmander inequality: there exists a constant C such that for every $z_1\in\mathbb{R}^{N+1},r>0,z\in B_r(z_1),\eta\in\partial B_{2r}(z_1)$ (spherical surface of $B_{2r}(z_1)$),*

$$\big|\Gamma^0_{ij}(z,\eta)-\Gamma^0_{ij}(z_1,\eta)\big|+\big|\Gamma^0_{ij}(\eta,z)-\Gamma^0_{ij}(\eta,z_1)\big|\le C\frac{d(z_1,z)}{d(z_1,\eta)^{Q+3}}.$$

Proof. (i) By the uniform Gaussian estimates proved in [15] for the fundamental solution of $\mathcal{H}_0$, we know that

$$\Big|\partial_t^kX_{i1}\cdots X_{ir}\Gamma((x_0,t_0);(x,t))\Big|\le C_1\frac{e^{-C_2(||x||_1)^2/t}}{t^{Q/2+k+r/2}},\quad z=(x,t)\in\mathbb{R}^{N+1},\tag{3.3}$$

with C_1,C_2 independent of $z_0=(x_0,t_0)$ and $||\cdot||_1$ defined in Definition 2.1. Then by (iii) of Proposition 2.1 and (3.3),

$$\begin{aligned}|\Gamma^0_{ij}(x,t)|&\le C_1\Big(\frac{||x||_1+|t|^{1/2}}{t^{1/2}}\Big)^{Q+2}\frac{e^{-C_2(||x||_1)^2/t}}{(||x||_1+|t|^{1/2})^{Q+2}}\\&\le\frac{C_3}{(||x||_1+|t|^{1/2})^{Q+2}}\le\frac{C}{||z||^{Q+2}},\end{aligned}\tag{3.4}$$

where $z=(x,t)$ and the constant C_3 is independent of $z_0=(x_0,t_0)$. This implies that

$$|\Gamma^0_{ij}(z,\eta)|\le\frac{C}{||\eta^{-1}\circ z||^{Q+2}}=\frac{C}{d(z,\eta)^{Q+2}}\le\frac{C}{|B(z,d(z,\eta))|}.$$

(ii) Fix $z_1,\eta\in\mathbb{R}^{N+1}$ and let $\varphi(z)$ be a cutoff function satisfying $B_r(z_1)\prec\varphi(z)\prec B_{3r/2}(z_1)$ and

$$|\partial_t^kD^h\varphi|=\sum_{1\le i,j\le q}|\partial_t^kX_{i1}\cdots X_{ih}\varphi|\le\frac{C_{h,k}}{r^{h+2k}}.\tag{3.5}$$

Set $f(z)=\Gamma_{ij}^0(z,\eta)\varphi(z)$. Then $f\in C_0^1(B_{3r/2}(z_1))$, and for $d(z_1,z)<r$, we have

$$\begin{aligned}&\left|\Gamma_{ij}^0(z,\eta)-\Gamma_{ij}^0(z_1,\eta)\right|=|f(z)-f(z_1)|\\&\leq d(z_1,z)\left\{\sup_{\zeta\in B_{3/2r}(z_1)}|Xf(\zeta)|+\frac{3}{2}r\sup_{\zeta\in B_{3/2r}(z_1)}|f_t(\zeta)|\right\}.\end{aligned}\tag{3.6}$$

It follows from (3.3)-(3.5) that

$$\begin{aligned}|Xf(\zeta)|&=|X\Gamma_{ij}^0(\zeta,\eta)\varphi(\zeta)|+|\Gamma_{ij}^0(\zeta,\eta)X\varphi(\zeta)|\\&\leq\frac{C}{d(\zeta,\eta)^{Q+3}}\leq\frac{C}{r^{Q+3}}\leq\frac{C}{d(z_1,\eta)^{Q+3}}.\end{aligned}$$

Similarly,

$$f_t(\zeta)=\partial_t\Gamma_{ij}^0(\zeta,\eta)\varphi(\zeta)+\Gamma_{ij}^0(\zeta,\eta)\varphi_t(\zeta)\leq\frac{C}{r^{Q+4}}.$$

Therefore, by (3.6), we get

$$|\Gamma_{ij}^0(z,\eta)-\Gamma_{ij}^0(z_1,\eta)|\leq C\frac{d(z_1,z)}{d(z_1,\eta)^{Q+3}},\quad\text{for }\ d(z_1,\eta)\geq 2d(z_1,z).$$

Similarly we can obtain

$$|\Gamma_{ij}^0(\eta,z)-\Gamma_{ij}^0(\eta,z_1)|\leq C\frac{d(z_1,z)}{d(z_1,\eta)^{Q+3}},\quad\text{for }\ d(z_1,\eta)\geq 2d(z_1,z).$$

This completes the proof of this theorem. □

Theorem 3.2. *For every multi-index β, there exists a constant $C=C(\beta,N,\mu)$ such that*

$$\sup_{z_0\in\mathbb{R}^{N+1},|\eta||=1}\left|\left(\frac{\partial}{\partial\eta}\right)^{\beta}\Gamma_{ij}(z_0;\eta)\right|\leq C.$$

Proof. This result can be obtained from Theorem 3.1 and [12, Theorem 12]. □

Now consider (3.2). Writing $\mathcal{H}_0$ as $\mathcal{H}_0=\mathcal{H}+(\mathcal{H}_0-\mathcal{H})$, and then substituting z for z_0, we have a representation formula for the second order derivatives X_iX_jf.

Theorem 3.3. *Let $f\in C_0^\infty(\mathbb{R}^{N+1})$, $f=0$ for $t\leq 0$ and $z=(x,t)\in supp f$. Then for $i,j=1,\cdots,q$,*

$$\begin{aligned}X_iX_jf(z)=\ &P.V.\int_{\mathbb{R}^{N+1}}\Gamma_{ij}(z;\eta^{-1}\circ z)\left\{\sum_{h,k=1}^{q}[a_{hk}(\eta)-a_{hk}(z)]X_hX_kf(\eta)\right.\\&\left.-\mathcal{H}f(\eta)\right\}\mathrm{d}\eta+\alpha_{ij}(z)\mathcal{H}f(z),\end{aligned}\tag{3.7}$$

where

$$\alpha_{ij}(z)=\int_{\Sigma_{N+1}}\Gamma_j(z;\eta)\nu_i \mathrm{d}\sigma(\eta),$$

ν_i is the i-th component of the unit outward normal to the surface Σ_{N+1}.

For convenience, we will introduce some notations. Set

$$\mathcal{K}_{ij}f(z)=P.V.\int_{\mathbb{R}^{N+1}}\Gamma_{ij}(z;\eta^{-1}\circ z)f(\eta)\mathrm{d}\eta.$$

For a singular integral operator $\mathcal{K}$ and a function $a\in BMO\cap L^\infty(\mathbb{R}^{N+1})$, define the commutator

$$\mathcal{C}[\mathcal{K},a]f=\mathcal{K}(af)-a\mathcal{K}(f).$$

Then (3.7) becomes

$$X_iX_jf=-\mathcal{K}_{ij}(\mathcal{H}f)+\sum_{h,k=1}^{q}\mathcal{C}\left[\mathcal{K}_{ij},a_{hk}\right]X_hX_kf+\alpha_{ij}\mathcal{H}f,\quad \forall i,j=1,\cdots,q. \tag{3.8}$$

4 Estimates of singular integrals

The L^p-estimates and $L^{p,\lambda}$-estimates for singular integrals with constant kernel and their commutators with BMO functions have been given in [3] and [16], respectively, in the context of general homogeneous spaces. Note that singular integrals here are with variable kernel. The main result in this section is the boundedness of $\mathcal{K}_{ij}$ and $\mathcal{C}\left[\mathcal{K}_{ij},a\right]$ in Morrey space $L^{p,\lambda}(\mathbb{R}^{N+1})$.

Theorem 4.1. *Let $p\in(1,+\infty),\lambda\in(0,Q+2)$ and $a\in BMO(\mathbb{R}^{N+1})$. For any $f\in L^{p,\lambda}(\mathbb{R}^{N+1})$, the singular integrals $\mathcal{K}_{ij}f$ and their commutators $\mathcal{C}\left[\mathcal{K}_{ij},a\right]f$ are bounded from $L^{p,\lambda}(\mathbb{R}^{N+1})$ into itself, that is,*

$$||\mathcal{K}_{ij}(f)||_{p,\lambda}\le C||f||_{p,\lambda} \tag{4.1}$$

and

$$||\mathcal{C}\left[\mathcal{K}_{ij},a\right](f)||_{p,\lambda}\le C||a||_*||f||_{p,\lambda}, \tag{4.2}$$

with $C=C(p,\lambda,Q,N)$.

To prove Theorem 4.1, we need to apply Calderón-Zygmund's technique of expansion in spherical harmonics (see [1,2]). Let $\{Y_{km}\}$ $(m\ge 0, 1\le k\le g_m)$ be a complete orthogonal system of spherical harmonics in $L^2(\Sigma_{N+1})$. Denote by m the degree of the polynomial and by g_m the dimension of the space of spherical harmonics of degree m in $\mathbb{R}^{N+1}$. It is known that

$$g_m\le C(N)m^{N-1},\quad \forall m=1,2,\cdots. \tag{4.3}$$

For any fixed $z\in\mathbb{R}^{N+1}$, $\eta\in\Sigma_{N+1}$, there is an expansion:

$$\Gamma_{ij}(z;\eta)=\sum_{m=0}^{\infty}\sum_{k=1}^{g_m}C_{ij}^{km}(z)Y_{km}(\eta),\quad \text{for } i,j=1,\cdots,q. \tag{4.4}$$

For any given $\eta\in\mathbb{R}^{N+1}$, let $\eta'=D(||\eta||^{-1})\eta$. Then $\eta'\in\Sigma_{N+1}$ by Proposition 2.3 (ii). By (4.4) and homogeneity of Γ_{ij}, we have

$$\Gamma_{ij}(z;\eta)=\sum_{m=0}^{\infty}\sum_{k=1}^{g_m}C_{ij}^{km}(z)\frac{Y_{km}(\eta')}{||\eta||^{Q+2}}. \tag{4.5}$$

The coefficients C_{ij}^{km} in the above expansion have the following bound: for every positive integer l, there exists a constant $C=C(l,\mu,N)$ such that

$$\sup_{z\in\mathbb{R}^{N+1}}|C_{ij}^{km}(z)|\le Cm^{-2l},\quad \forall m=1,2,\cdots,\quad k=1,\cdots,g_m. \tag{4.6}$$

Now, for $z\in\mathbb{R}^{N+1}$, $z'\in\Sigma_{N+1}$, let $H_{km}(z)=Y_{km}(z')/||z||^{Q+2}$. For the following singular integral operators

$$\mathcal{K}_{km}f(z)=:\int_{\mathbb{R}^{N+1}}H_{km}(\eta^{-1}\circ z)f(\eta)\mathrm{d}\eta,$$

and their commutators

$$\mathcal{C}\,[\mathcal{K}_{km},a]f(z):=\int_{\mathbb{R}^{N+1}}H_{km}(\eta^{-1}\circ z)\,[a(\eta)-a(z)]f(\eta)\mathrm{d}\eta,$$

the results in [3] and [16] show that they are bounded from $L^{p,\lambda}(\mathbb{R}^{N+1})$ into itself, namely,

$$||\mathcal{K}_{km}(f)||_{p,\lambda}\le Cm^{(N+1)/2}||f||_{p,\lambda}, \tag{4.7}$$

$$||\mathcal{C}\,[\mathcal{K}_{km},a]\,(f)||_{p,\lambda}\le Cm^{(N+1)/2}||a||_{*}||f||_{p,\lambda}, \tag{4.8}$$

with $C=C(p,\lambda,Q,N)$.

To prove Theorem 4.1 in our situation $(\mathbb{R}^{N+1},dz,d)$, it suffices to repeat the argument of [7, Theorem 2.1] making use of spherical harmonic expansion, (4.7) and (4.8).

5 Estimates of solutions in Sobolev-Morrey space

To complete the proof of Theorem 1.1, we verify the $L^{p,\lambda}$-estimate of two order derivation X_iX_jf of the solution to the equation $\mathcal{H}f=F$, and then deduce the estimate for lower order term of **H** by using interpolation inequalities. We first consider the localization of estimate (4.2): If $a\in VMO(\mathbb{R}^{N+1})$, then for any $\varepsilon>0$, there exists $r_0>0$ depending on ε and the VMO modulus η_a of a, such that for every $r\in(0,r_0)$,supp$f\subseteq B_r$, (4.2) can be localized as:

$$||\mathcal{C}\left[\mathcal{K}_{ij},a\right](f)||_{p,\lambda;B_r}\le C(p,\lambda,Q,N)\cdot\varepsilon||f||_{p,\lambda;B_r},\quad \text{for } f\in L^{p,\lambda}(B_r). \tag{5.1}$$

From (3.8), (4.1) and (5.1), we derive immediately the Morrey estimate of X_iX_jf on sufficiently small balls.

Lemma 5.1. *For every* $p\in(1,+\infty)$ *and* $\lambda\in(0,Q+2)$*, there exist* $C=C(p,\lambda,\mu)$ *and* $r_0=r_0(p,\lambda,\mu,\eta_a)$ *such that for* $f\in S_0^p\cap S^{p,\lambda}(B_r,\mathcal{H})$*,* $f=0$ *for* $t\le 0$*,* $\mathcal{H}f\in L^{p,\lambda}(B_r), 0<r<r_0$*,* $i,j=1,\cdots,q$*, the following estimate holds:*

$$||X_iX_jf||_{p,\lambda;B_r}\le C||\mathcal{H}f||_{p,\lambda;B_r}. \tag{5.2}$$

With techniques of cutoff function and interpolation inequalities, from Lemma 5.1, we obtain the following local a priori estimate.

Lemma 5.2. ($L^{p,\lambda}$-estimate without lower order terms) *For every* $p\in(1,+\infty),\lambda\in(0,Q+2)$ *and every open set* $\Omega'\subset\subset\Omega$*, there exists* $C=C(p,\lambda,\mu,\eta_a,|\Omega|,dist(\Omega',\partial\Omega))$ *such that for every* $f\in S^{p,\lambda}(\Omega,\mathcal{H})$*,* $f=0$ *for* $t\le 0$*,* $\mathcal{H}f\in L^{p,\lambda}(\Omega)$*,* $i,j=1,\cdots,q$*,*

$$||X_iX_jf||_{p,\lambda;\Omega'}\le C\Big\{||\mathcal{H}f||_{p,\lambda;\Omega}+||f||_{p,\lambda;\Omega}\Big\}. \tag{5.3}$$

Proof. For every $r_0=r_0(p,\lambda,\mu,\eta_a)>0$, let $r<r_0$. For $\sigma\in(0,1)$, let $\sigma'=(1+\sigma)/2$ and choose a cutoff function $\varphi\in C_0^\infty(B_r)$ and $B_{\sigma r}\prec\varphi\prec B_{\sigma' r}$ satisfying

$$|X_i\varphi|^2+|X_iX_j\varphi|\le\frac{C}{(1-\sigma)^2r^2},\quad\forall i,j=1,\cdots,q. \tag{5.4}$$

Set $g:=f\varphi$. Then $g\in S_0^p\cap S^{p,\lambda}(B_r)$. It follows from (5.2) and (5.4) that

$$\begin{aligned}&||X_iX_jf||_{p,\lambda;B_{\sigma r}}=||X_iX_jg||_{p,\lambda;B_{\sigma r}}\le C||\mathcal{H}f||_{p,\lambda;B_{\sigma' r}}\\ \le& C\Big(||\varphi\mathcal{H}f||_{p,\lambda;B_{\sigma' r}}+\sum_{i,j=1}^q||X_if\,X_j\varphi||_{p,\lambda;B_{\sigma' r}}+\sum_{i,j=1}^q||f\,X_iX_j\varphi||_{p,\lambda;B_{\sigma' r}}\Big)\\ \le& C\Big(||\mathcal{H}f||_{p,\lambda;B_r}+\frac{1}{(1-\sigma)r}||Xf||_{p,\lambda;B_{\sigma' r}}+\frac{1}{(1-\sigma)^2r^2}||f||_{p,\lambda;B_{\sigma' r}}\Big).\end{aligned} \tag{5.5}$$

Define the weighted Morrey seminorms:

$$\Phi_k=\sup_{\sigma\in(0,1)}(1-\sigma)^kr^k||X^kf||_{p,\lambda;B_{\sigma r}},\quad k=0,1,2, \tag{5.6}$$

where $||X^kf||_{p,\lambda;B_{\sigma r}}=\sum||X_{j1}\cdots X_{jk}f||_{p,\lambda;B_{\sigma r}}$, with $X_{j1}\cdots X_{jk}$ being the homogeneous of degree k. Consequently, (5.5) becomes

$$\Phi_2\le C\Big(r^2||\mathcal{H}f||_{p,\lambda;B_r}+\Phi_1+\Phi_0\Big). \tag{5.7}$$

For Φ_k, we claim the following version of the interpolation inequality

$$\Phi_1\le\varepsilon\Phi_2+\frac{C}{\varepsilon}\Phi_0,\quad\forall\varepsilon>0. \tag{5.8}$$

In fact, from (5.6), for every $\theta>0$, there exists $\sigma(\theta)\in(0,1)$ such that

$$\Phi_1 \le \big(1-\sigma(\theta)\big)r||Xf||_{p,\lambda;B_{\sigma(\theta)r}}+\theta.$$

Using [12, Proposition 2] (or [9, Theorem 3.6]) gives

$$||Xf||_{p,\lambda;B_{\sigma(\theta)r}} \le \delta||X^2f||_{p,\lambda;B_{\sigma(\theta)r}}+\frac{C}{\delta}||f||_{p,\lambda;B_{\sigma(\theta)r}}, \quad \forall\delta>0. \tag{5.9}$$

Hence, taking $\delta=\varepsilon(1-\sigma(\theta))r$ in (5.9) yields

$$\begin{aligned}\Phi_1 &\le \varepsilon(1-\sigma(\theta))^2r^2||X^2f||_{p,\lambda;B_{\sigma(\theta)r}}+\frac{C}{\varepsilon}||f||_{p,\lambda;B_{\sigma(\theta)r}}+\theta\\ &\le \varepsilon\Phi_2+\frac{C}{\varepsilon}\Phi_0+\theta.\end{aligned}$$

We obtain (5.8) by letting $\theta\to 0$ in the above inequality. Using (5.8) in (5.7), we then get

$$||X_iX_jf||_{p,\lambda;B_{\sigma r}} \le \frac{C}{(1-\sigma)^2r^2}\Big(r^2||\mathcal{H}f||_{p,\lambda;B_r}+||f||_{p,\lambda;B_r}\Big).$$

Finally, (5.3) follows by taking $\sigma=\frac{1}{2}$ and covering Ω' with a finite number of balls of radius $r/2$ for $r\le\min\{\mathrm{dist}(\Omega',\partial\Omega),r_0\}$. □

5.1 Proof of Theorem 1.1

For every $f\in S_0^p\cap S^{p,\lambda}(B_r,\mathbf{H})$ and r small enough, we claim

$$||b_iX_if||_{p,\lambda;B_r} \le \begin{cases} C||b_i||_{Q+2;B_r}||X^2f||_{p,\lambda;B_r}, & p+\lambda\le Q+2,\\ C||b_i||_{p,\lambda;B_r}||X^2f||_{p,\lambda;B_r}, & p+\lambda> Q+2.\end{cases} \tag{5.10}$$

In fact, for $p+\lambda\le Q+2$, by using the Hölder inequality and the Sobolev inequality [13], it derives

$$\begin{aligned}\left(\rho^{-\lambda}\int_{B_\rho\cap B_r}|b_iX_if|^p\right)^{1/p} &\le C\rho^{-\lambda/p}||b_i||_{Q+2;B_r}||X_if||_{\frac{(Q+2)p}{Q+2-p};B_r}\\ &\le C||b_i||_{Q+2;B_r}\rho^{-\lambda/p}||X_if||_{W^{1,p}(B_r)}\\ &\le C||b_i||_{Q+2;B_r}||X^2f||_{p,\lambda;B_r}.\end{aligned} \tag{5.11}$$

For $p+\lambda>Q+2$, since $W^{1,p}(B_r)\hookrightarrow C^0(B_r)$ (see [13]) and $L^{p,\lambda}(B_r)\hookrightarrow L^p(B_r)$, we obtain

$$\begin{aligned}\left(\rho^{-\lambda}\int_{B_\rho\cap B_r}|b_iX_if|^p\right)^{1/p} &\le \left(\rho^{-\lambda}\int_{B_\rho\cap B_r}|b_i|^p\right)^{1/p}\sup_{B_r}|X_if|\\ &\le C||b_i||_{p,\lambda;B_r}||X_if||_{W^{1,p}(B_r)}\\ &\le C||b_i||_{p,\lambda;B_r}||X^2f||_{p,\lambda;B_r}.\end{aligned} \tag{5.12}$$

The estimate (5.10) can be verified from (5.11) and (5.12). Similarly, we can obtain

$$||cf||_{p,\lambda;B_r} \le \begin{cases} C||c||_{\frac{Q+2}{2};B_r}||X^2f||_{p,\lambda;B_r}, & 2p+\lambda \le Q+2, \\ C||c||_{p,\lambda;B_r}||X^2f||_{p,\lambda;B_r}, & 2p+\lambda > Q+2. \end{cases} \tag{5.13}$$

Hence, it follows from (5.2), (5.10), (5.13) and (1.2) that

$$\begin{aligned} \sum_{i=1}^{q}|| X_iX_jf||_{p,\lambda;B_r} &\le C\Big(||\mathbf{H}f-\sum_{i=1}^{q}b_iX_if-cf||_{p,\lambda;B_r}\Big) \\ &\le C\Big(||\mathbf{H}f||_{p,\lambda;B_r}+\sum_{i,j=1}^{q}||X_iX_jf||_{p,\lambda;B_r}+||f||_{p,\lambda;B_r}\Big), \end{aligned}$$

which yields

$$||X_iX_jf||_{p,\lambda;B_r} \le C\Big(||\mathbf{H}f||_{p,\lambda;B_r}+||f||_{p,\lambda;B_r}\Big).$$

Now, we consider $f\in S^{p,\lambda}(\Omega,\mathbf{H})$. For $B_r\subset\Omega$ with r small enough, and $\sigma'=(1+\sigma)/2$ with $\sigma\in(0,1)$, let $\varphi\in C_0^\infty(B_r)$ be a cutoff function such that $B_{\sigma r}\prec\varphi\prec B_{\sigma' r}$. Then

$$||X_iX_jf||_{p,\lambda;B_{\sigma r}} \le C\Big(||\mathbf{H}(f\varphi)||_{p,\lambda;B_{\sigma' r}}+||f\varphi||_{p,\lambda;B_{\sigma' r}}\Big).$$

A similar argument as in Lemma 5.2 gives

$$||X_iX_jf||_{p,\lambda;B_{\sigma r}} \le \frac{C}{(1-\sigma)^2r^2}\Big(r^2||\mathbf{H}f||_{p,\lambda;B_r}+||f||_{p,\lambda;B_r}\Big). \tag{5.14}$$

From (1.2), (5.9) and (5.14), we have

$$\begin{aligned} ||\partial_tf||_{p,\lambda;B_{\sigma r}} &\le C\Big(||\mathbf{H}f||_{p,\lambda;B_r}+||X^2f||_{p,\lambda;B_r}+||Xf||_{p,\lambda;B_r}+||f||_{p,\lambda;B_r}\Big) \\ &\le \frac{C}{(1-\sigma)^2r^2}\Big(r^2||\mathbf{H}f||_{p,\lambda;B_r}+||f||_{p,\lambda;B_r}\Big). \end{aligned} \tag{5.15}$$

Combining (5.14) and (5.15) and using Lemma 5.2 conclude Theorem 1.1. □

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