SHORT COMMUNICATION SECTION

Some Geometric Flows on Kähler Manifolds

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Received 28 October 2009; Accepted 6 January 2010

Abstract. We define a kind of KdV (Korteweg-de Vries) geometric flow for maps from a real line or a circle into a Kähler manifold $\mathbb{N}$ with complex structure $J$ and metric $h$ as the generalization of the vortex filament dynamics from a real line or a circle. By using the geometric analysis, the existence of the Cauchy problems of the KdV geometric flows will be investigated in this note.

AMS Subject Classifications: 35Q53, 35Q35, 53C44

Chinese Library Classifications: O175.29

Key Words: KdV geometric flow; conservation law; local and global existence.

1 The definition of some geometric flow

Let $(\mathbb{N}, J, h)$ be a Kähler manifold with complex structure $J$ and metric $h$. For any smooth map $u(x, t)$ from $S^1 \times \mathbb{R}$ into $(\mathbb{N}, J, h)$, let $\nabla_x$ denote the covariant derivative $\nabla \mid_{\pi}$ on the pull-back bundle $u^{-1}TN$ induced from the Levi-Civita connection $\nabla$ on $\mathbb{N}$. For the sake of convenience, we always denote $\nabla_x u$ and $\nabla_t u$ by $u_x$ and $u_t$ respectively. The energy of a smooth map $v : S^1 \to \mathbb{N}$ is defined as

$$E_1(v) \equiv \frac{1}{2} \int_{S^1} |v_x|^2 dx,$$

and the tension field of $v$ is written by $\tau(v) \equiv \nabla_x v_x$.

For the maps from a unit circle $S^1$ or a real line $\mathbb{R}$ into $\mathbb{N}$, we define a class of geometric flows, which we would like to call KdV geometric flow (Korteweg-de Vries), as follows:

$$\frac{\partial u}{\partial t} = \nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x,$$

(1.1)

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where $R$ is the curvature tensor on $N$ and $J_u \equiv J(u)$. If $(N,J,h)$ is a locally Hermitian symmetric space, it is easy to verify that the geometric flow is an energy conserved system. Moreover, the flow on any Kähler manifold $N$ always preserves the following moment density

$$E_2(u) \equiv \int \langle \nabla_x u_x, J u_x \rangle \, dx.$$

Let $(\Sigma,h)$ be a Riemann surface. It is easy to verify that on $\Sigma$ there holds true

$$R(u_x,J u u_x)J u_x = K(u) |u_x|^2 u_x,$$

where $K(\cdot)$ is the Gauss curvature (sectional curvature) function on $(\Sigma,h)$. So, the KdV geometric flow from $S^1$ or $\mathbb{R}$ into $(\Sigma,h)$ can be written by

$$\frac{\partial u}{\partial t} = \nabla^2_x u_x + \frac{1}{2} K(u) |u_x|^2 u_x. \tag{1.2}$$

It is not difficult to see that in the case $\Sigma$ is a 2-dimensional sphere $S^2$ the above KdV flow can be derived from the following curve flow $u$ from $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ into Euclidean space $\mathbb{R}^3$

$$u_t = u_s \times u_{ss} + \beta \left( u_{sss} + \frac{3}{2} u_s \times (u_s \times u_{ss}) \right), \tag{1.3}$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. For the mechanical background of the curve flow we refer to [2–4].

Now we recall another geometric flow, namely Schrödinger flow, from $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ into a Kähler manifold $(N,J,h)$ formulated by (see, e.g., [5–8])

$$\frac{\partial u}{\partial t} = J(u) \nabla_x u_x = J(u) \tau(u), \tag{1.4}$$

which is a Hamilton system with the energy functional.

Here we should mention that Terng and Uhlenbeck [9] constructed via gauge transformations an isomorphism from the phase space of Schrödinger flow from $\mathbb{R}$ into a Grassmann manifold to the phase space of the matrix valued Schrödinger equation so that the Schrödinger flow corresponds to the matrix valued Schrödinger flow. In fact, the theory of Terng and Uhlenbeck also implies that the above KdV geometric flow from $\mathbb{R}$ into a Grassmann manifold corresponds to the matrix valued KdV flow associated Grassmannian symmetric Lie algebra. Matrix valued Schrödinger equations and KdV flows (or vector valued mKdV equation) associated Hermitian symmetric spaces (Lie algebra) was introduced in [4] and [10].

We define another geometric flow for maps from $S^1$ or $\mathbb{R}$ into a Kähler manifold $(N,J,h)$ as follows

$$u_t = \alpha J_u \tau(u) + \beta \left( \nabla^2_x u_x + \frac{1}{2} R(u_x,J u u_x)J u_x \right), \tag{1.5}$$
where $\alpha$ and $\beta$ are two real constants. The flow is a direct extension to a Kähler manifold of the curve flow characterized the motion of vortex filament (see [11]). The flows can also be regarded as the geometric generalization of Hirota equation [12]

$$i\gamma_t + i\gamma_1 q_x + \gamma_2 (q_{xx} + 2|q|^2 q) + i\gamma (q_{xxx} + 6|q|^2 q_x) = 0,$$

(1.6) which describes the pulse in optical fibres. Here $\gamma, \gamma_1$ and $\gamma_2$ are real parameters here.

2 Main results

First we consider the local existence for the Cauchy problem of KdV geometric flow on a Kähler manifold $N$ defined by

$$\begin{cases}
    u_t = \nabla^2_x u_x + \frac{1}{2} R(u_x, J u_x) J u_x, & x \in S^1 \text{ or } \mathbb{R}; \\
    u(x, 0) = u_0(x).
\end{cases}$$

(2.1)

Furthermore, some results on global existence are established when $N$ is some kind of special locally Hermitian symmetric spaces.

In order to state our main theorems we need to introduce several definitions on Sobolev spaces with vector bundle value. Let $(E, M, \pi)$ be a vector bundle with base manifold $M$. If $(E, M, \pi)$ is equipped with a metric, then we may define so-called vector bundle value Sobolev spaces as follows:

**Definition 2.1.** $H^m(M, E)$ is the completeness of the set of smooth sections with compact supports denoted by $\{s | s \in C^\infty_0(M, E)\}$ with respect to the norm

$$\|s\|_{H^m(M, E)} = \sum_{i=0}^{m} \int_M |\nabla^i s|^2 dM.$$  

Here $\nabla$ is the connection on $E$ which is compatible with the metric on $E$.

**Definition 2.2.** The Sobolev space of maps from $\mathbb{R}$ into a Riemannian manifold $(N, h)$ is defined by

$$H^{m+1}(\mathbb{R}; N) = \left\{ u \in C(\mathbb{R}; N) | u_x \in H^m(\mathbb{R}; TN) \right\},$$

where $u_x \in H^m(\mathbb{R}; TN)$ means that $u_x$ satisfies

$$\|u_x\|_{H^m(\mathbb{R}; TN)}^2 = \sum_{j=0}^{m} \int_{\mathbb{R}} h(u(x))(\nabla^j_x u_x(x), \nabla^j_x u_x(x))dx < +\infty;$$

and

$$H^{m+1}_Q(\mathbb{R}; N) = \left\{ u \in C(\mathbb{R}; N) | d_h(u(x), Q) \in L^2(\mathbb{R}), u_x \in H^m(\mathbb{R}; TN) \right\},$$

where $d_h(u(x), Q)$ denotes the distance between $u(x)$ and $Q$. 

Theorem 2.1. Let \((N,J,h)\) be a complete Kähler manifold. Then the Cauchy problem (2.1) with the initial value map \(u_0 \in H^k(S^1,N)\) for any integer \(k \geq 3\) admits a local solution \(u \in L^\infty((0,T),H^k(S^1,N))\), where \(T = T(N,||u_0||_{H^k})\).

Theorem 2.2. Let \((N,J,h)\) be a compact Kähler manifold. Then the Cauchy problem (2.1) with the initial map \(u_0 \in H^k_Q(R,N)\) for any integer \(k \geq 3\), admits a local solution \(u\) such that \(u \in L^\infty((0,T),H^k_Q(R,N))\), where \(T = T(N,||\nabla_x u_0||_{H^k})\).

Theorem 2.3. Let \((N,J,h)\) be a complete Kähler manifold. Then for any integer \(k \geq 4\), the local solutions of the Cauchy problem (2.1) with the initial map \(u_0 \in H^k(S^1,N)\) (or \(u_0 \in H^k_Q(R,N)\)) are unique in \(H^k(S^1,N)\) (or in \(H^k_Q(R,N)\)).

Theorem 2.4. Assume that \(N = M_1 \times M_2 \times \cdots \times M_n\) is a product manifold where \((M_i,h^i)\) \((i = 1,2,\cdots,n)\) is a complete locally Hermitian symmetric space satisfying

\[
h^i(R^i(Y,X)X,R^i(X,JX)JX) \equiv 0,
\]

where \(R^i\) is the Riemannian curvature on \(M_i\). Then the Cauchy problem (2.1) from \(S^1\) into \(N\) with the initial map \(u_0 \in H^k(S^1,N)\) for any integer \(k \geq 3\) admits a global solution \(u \in L^\infty([0,\infty), H^k(S^1,N))\). Moreover, when \(k \geq 4\), the solutions are unique.

Theorem 2.5. Assume that \(N = M_1 \times M_2 \times \cdots \times M_n\) is a product manifold where \((M_i,h^i)\) \((i = 1,2,\cdots,n)\) is a compact locally Hermitian symmetric space satisfying

\[
h^i(R^i(Y,X)X,R^i(X,JX)JX) \equiv 0,
\]

where \(R^i\) is the Riemannian curvature on \(M_i\). Then the Cauchy problem (2.1) from \(R\) into \(N\) with the initial map \(u_0 \in H^k(R,N)\) for any integer \(k \geq 3\) admits a global solution \(u \in L^\infty([0,\infty), H^k_Q(R,N))\). Moreover, when \(k \geq 4\), the solution is unique.

Remark 2.1. We will show that the identity on Riemannian curvature in Theorem 2.4 holds on Kähler manifolds with constant holomorphic sectional curvature, complex Grassmannians, the first class of bounded symmetric domains. The examples of Kähler manifolds with constant holomorphic sectional curvature are \(C^k\), the flat complex torus \(CT^l\), the complex projective spaces \(CP^n\), complex hyperbolic spaces \(CH^n\) and the compact quotient spaces of complex hyperbolic space modulo by a torsion free discrete subgroup of automorphism group of \(CH^n\) etc.

We will adopt the parabolic approximation and employ the geometric energy method developed in [6,7] to show these local existence problems. To prove the global existence we need to exploit some conservation laws and semi-conservation laws. We define

\[
E_3(u) \equiv \int |\nabla_x u_x|^2 dx - \frac{1}{4} \int \langle u_x, R(u_x, u_x) J u_x \rangle dx,
\]

(2.2)
and
\[
E_4(u) \equiv 2 \int |\nabla_x^2 u_x|^2 \, dx - 3 \int \langle \nabla_x u_x, R(\nabla_x u_x, u_x) \rangle \, dx \\
- 5 \int \langle \nabla_x u_x, R(\nabla_x u_x, J u_x) \rangle \, dx.
\] (2.3)

If $N$ is a locally Hermitian symmetric space satisfying some geometric conditions, for the smooth solution $u$ to the Cauchy problem (2.1) we establish some conservation laws and semi-conservation laws, e.g.,
\[
\frac{d}{dt} E_1(u) = 0, \quad \frac{d}{dt} E_3(u) = 0, \\
\frac{d}{dt} E_4(u) \leq C(N, E_1(u_0), E_3(u_0)) (1 + E_4(u)).
\] (2.4) (2.5)

We can make use of the above conservation laws with respect to $E_1(u)$ and $E_3(u)$ to obtain a uniform a priori bound of $|| \nabla_x u_x ||_{L^2}$ independent of $T$. By virtue of (2.5), we can obtain some results on global existence.

For the Cauchy problem of the following flow
\[
u_t = \alpha J_0 \nabla_x u_x + \beta \left( \nabla_x^2 u_x + \frac{1}{2} R(\nabla_x J_0 u_x) \right),
\] (2.6)
we can also obtain some similar results with the KdV flows.

**Acknowledgments**

This research is partially supported by 973 project of China (Grant No. 2006CB805902) and NSFC (Grant No. 10990013).

**References**


