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Decay of Solutions to a 2D Schrödinger Equation

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Abstract. Let $u \in C(\mathbb{R}, H^1)$ be the solution to the initial value problem for a 2D semi-linear Schrödinger equation with exponential type nonlinearity, given in [1]. We prove that the L^r norms of u decay as $t \to \pm\infty$, provided that $r > 2$.

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1 Introduction

In this work, we study some asymptotic properties of solution to the following initial value Schrödinger equation

$$i\partial_t u + \Delta_x u = f(u), \qquad \text{in } \mathbb{R}_t \times \mathbb{R}_x^2, \tag{1.1}$$

with data

$$u_0 := u(0,.) \in H^1(\mathbb{R}^2), \tag{1.2}$$

where $u := u(t,x)$ is a complex-valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, and

$$f(u) := u\left(e^{4\pi|u|^2} - 1\right). \tag{1.3}$$

Two important conserved quantities of (1.1) are the mass and the Hamiltonian. The mass is defined by

$$M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^2)}, \tag{1.4}$$

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and the Hamiltonian is defined by

$$H(u(t)):=\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{4\pi}\|e^{4\pi|u(t)|^2}-1-4\pi|u(t)|^2\|_{L^1(\mathbb{R}^2)}. \tag{1.5}$$

We know [1] that the Cauchy problem (1.1)-(1.2) has a unique solution u in the space $C(\mathbb{R},H^1(\mathbb{R}^2))\cap L^4_{loc}(C^{1/2}(\mathbb{R}^2))$. Moreover, u satisfies conservation of the mass and the Hamiltonian. Our aim, in this paper, is to prove some asymptotic properties of such solution.

Before going further, let recall some historic facts about well-posedness of the monomial defocusing semilinear Schrödinger equation

$$i\partial_t u+\Delta_x u=|u|^{p-1}u, \quad p>1, \quad u:(-T^*,T^*)\times\mathbb{R}^d\to\mathbb{C}. \tag{1.6}$$

A solution u to (1.6) satisfies conservation of the mass and the Hamiltonian

$$H_p(u(t)):=\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)}+\frac{2}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}(t,x)\mathrm{d}x.$$

Moreover, for any $\lambda>0$,

$$\begin{aligned}&u_\lambda:(-T^*\lambda^2,T^*\lambda^2)\times\mathbb{R}^d\to\mathbb{C},\\&(t,x)\longmapsto\lambda^{\frac{2}{1-p}}u(\lambda^{-2}t,\lambda^{-1}x)\end{aligned}$$

is a solution to (1.6). Note also that for $s_c:=d/2-2/(p-1)$, the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is relevant in the well-posedness theory of (1.6) because it is invariant under the mapping

$$f(x)\longmapsto\lambda^{\frac{2}{1-p}}f(\lambda^{-1}x), \qquad \lambda>0.$$

We refer to Eq. (1.6) with the notation $NLS_p(\mathbb{R}^d)$ and we limit our discussion to the case $0\le s_c\le 1$. If $s_c>1$, (1.6) is locally well-posed in H^s, for $s>s_c$.

1. $NLS_p(\mathbb{R}^d)$ **local well-posedness in** $H^s(\mathbb{R}^d)$. It is known (see, e.g., [2–4]) that

(a) If $s>s_c$, then (1.6) is locally well-posed in H^s, with an existence interval depending only upon $\|u_0\|_{H^s}$.

(b) For $s=s_c$, (1.6) is locally well-posed in H^s, with an existence interval depending upon $e^{it\Delta}u_0$.

(c) If $s<s_c$, then (1.6) is ill-posed in H^s (see, e.g., [5–9]).

So, it is naturel to refer to H^{s_c} as the critical regularity for (1.6). 2. $NLS_p(\mathbb{R}^d)$ **global well-posedness** .

(a) *The energy subcritical case* $s_c < 1$. Using local well-posedness and conservation laws, we obtain global well-posedness of (1.6) in H^1. It is expected that the local H^{s_c} solutions of (1.6) extend to global solutions. For certain choice of p,d, there are results (see for instance [10–14]) which show that H^s initial data evolve into global solutions of (1.6) for $s \in (\tilde{s}_{p,d},1)$ with $s_c < \tilde{s}_{p,d} < 1$ such that $\tilde{s}_{p,d}$ is close to 1 and away from s_c. For all problems with $0 \le s_c < 1$, global well-posedness in the scale invariant space H^{s_c} is unknown but conjured to hold. Moreover, the solutions scatter when $p > p_* := 1 + 4/d$ [4,15].

(b) *The energy critical case* $s_c = 1$. Since the local existence interval does not depend only on $\|u_0\|_{H^1}$, an iteration of the local well-posedness theory fails to prove global well-posedness. But using new ideas of Bourgain in [11] (see also [16]) (which treated the radial case in dimension 3) and a new interaction Morawetz inequality [13], the energy critical case of (1.6) is now completely resolved [17–19]. Finite energy initial data u_0 evolve into global solution u with finite space-time size $\|u\|_{L_{t,x}^{[2(2+d)]/(d-2)}} < \infty$ and scatter.

(c) *The energy supercritical case* $s_c > 1$. Global well-posedness for the defocusing energy supercritical $NLS_p(\mathbb{R}^d)$ is an outstanding open problem (see [5,7,9] for some partial results).

3. **The two space dimensions case**. The initial value problem $NLS_p(\mathbb{R}^2)$ is energy subcritical for all $p > 1$. So it is natural to consider problems with exponential nonlinearities, which have several applications, as for example the self trapped beams in plasma [20]. Cazenave considered in [21] the Schrödinger equation with decreasing exponential nonlinearity and showed global well-posedness and scattering. With increasing exponentials the situation is more complicated because there's no a priori L^∞ control of the nonlinear term. Moreover, the two dimensional case is interesting because of its relation to the critical Moser-Trudinger inequalities (see [22, 23]). The two dimensional Schrödinger problems with exponential growth nonlinearities was studied, for small Cauchy data, by Nakamura and Ozawa in [24]. They proved global well-posedness and scattering. Later on, Colliander-Ibrahim-Majdoub-Masmoudi considered the Schrödinger Cau-chy problem (1.1)–(1.2).

Definition 1.1. *The Cauchy problem* (1.1)–(1.2) *is said to be subcritical if*

$$H(u_0) < 1.$$

It is critical if $H(u_0) = 1$ *and supercritical if* $H(u_0) > 1$.

They obtained [1] global well-posedness in the energy space for both subcritical and critical cases. In the supercritical case, they obtained an instability result (similar results was proved for the wave equation [25, 26]). Recently, subtracting the cubic term of the nonlinearity (1.3), Ibrahim-Majdoub-Masmoudi-Nakanishi proved in [27] scattering for

$$i\partial_t u + \Delta_x u = u(e^{4\pi|u|^2} - 1 - 4\pi|u|^2), \qquad \text{on } \mathbb{R}_t \times \mathbb{R}_x^2 \tag{1.7}$$

in the subcritical case $(H(u_0)<1)$. They used a new interaction Morawetz estimate proved independently by Colliander et al. and Planchon-Vega [28, 29]. The critical case $(H(u_0)=1)$ is an open problem (similar results was proved for the wave equation [30, 31]).

In the light of [1, 27], we consider the Schrödinger equation (1.1), in both subcritical and critical cases $(H(u_0)\leq 1)$ and we show decay of solution in $L^r(\mathbb{R}^2)$ norm for $2<r<\infty$.

Remark 1.2. We mention that

1. In order to prove scattering, the authors in [27] have subtracted the cubic part from the nonlinearity to avoid the critical exponent p_*.
2. For $p_*=1+4/d$, a complete scattering theory is available in the conformal space of functions $f\in H^1(\mathbb{R}^d)$ such that $\int|x|^2|f(x)|^2\mathrm{d}x<\infty$ (see [32–34]).
3. The scattering result proved in [27] implies that, for any $r>2$, we have the following decay result
$$\lim_{t\to\infty}\|u(t)\|_{L^r(\mathbb{R}^2)}=0, \tag{1.8}$$
where $u\in C(\mathbb{R},H^1(\mathbb{R}^2))$ is the solution to (1.7)–(1.2).
4. In [27], scattering was established only in the subcritical case $(H(u_0)<1)$ and for Eq. (1.7).
5. Using the same estimates as in this paper, it is easier to prove the same decay result in the case of (1.7).
6. Recently, extending previous results obtained in [4,15], Viscigilia [35] proved a similar result of decay for the solution to the Cauchy problem associated to a Schrödinger equation with a monomial type nonlinearity.

1.1 Main result

Our main result can be stated as follows.

Theorem 1.3. *Let* $u_0\in H^1(\mathbb{R}^2)$ *such that* $H(u_0)\leq 1$ *and* $u\in C(\mathbb{R},H^1(\mathbb{R}^2))$ *be the solution to* (1.1)–(1.2). *Thus*

1. *If* $\limsup_{t\to\infty}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}<1$, *then*
$$\lim_{t\to\infty}\|u(t)\|_{L^r(\mathbb{R}^2)}=0, \qquad \text{for every } 2<r<\infty.$$
2. *If* $\limsup_{t\to\infty}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}=1$, *then*
$$\liminf_{t\to\infty}\|u(t)\|_{L^r(\mathbb{R}^2)}=0, \qquad \text{for every } 2<r<\infty.$$
Moreover for any sequence of positive real numbers (t_n) *tending to infinity, there exist a subsequence denoted* (s_n) *and a sequence of positive real numbers* (r_n), *such that*
$$\lim_{n\to\infty}r_n=0 \quad \text{and} \quad \lim_{n\to\infty}\|u(s_n+r_n)\|_{L^r(\mathbb{R}^2)}=0.$$

Remark 1.4. Consequently, if $H(u_0)<1$, then

$$\lim_{t\to\infty}\|u(t)\|_{L^r(\mathbb{R}^2)}=0, \qquad \text{for every } 2<r<\infty.$$

1.2 Tools

In what follows, we collect some estimates needed in the sequel. We say that a couple (q,r) is Schrödinger admissible (for short S-admissible), if

$$2\le q,r\le\infty, \quad (q,r)\neq(2,\infty) \quad \text{and} \quad \frac{1}{q}+\frac{1}{r}=\frac{1}{2}.$$

In order to control the solution of (1.1), we will use the following Strichartz estimate [36].

Proposition 1.5. (Strichartz estimate) *Let $I\subset\mathbb{R}$ be a time slab, $t_0\in I$ and (q,r), (α,β) two S-admissimble pairs. Then, a constant C exists such that, for any $u\in C(I,H^1(\mathbb{R}^2))$, we have*

$$\|u\|_{L^q(I,W^{1,r}(\mathbb{R}^2))}\le C\Big(\|u(t_0,.)\|_{H^1(\mathbb{R}^2)}+\|i\partial_t u+\Delta_x u\|_{L^{\alpha'}(I,W^{1,\beta'}(\mathbb{R}^2))}\Big). \tag{1.9}$$

In particular we have the following energy estimate.

Proposition 1.6. (Energy estimate) *With the same hypothesis we have*

$$\sup_{t\in I}\|u(t,.)\|_{H^1(\mathbb{R}^2)}\le C\Big(\|u(t_0,.)\|_{H^1(\mathbb{R}^2)}+\|i\partial_t u+\Delta_x u\|_{L^1(I,H^1(\mathbb{R}^2))}\Big). \tag{1.10}$$

In order to control the nonlinear part of the energy in $L^1_t(H^1_x)$, we will use the following Moser-Trudinder inequality [22,37,38].

Proposition 1.7. (Moser-Trudinger inequality) *Let $\alpha\in(0,4\pi)$, a constant $\mathcal{C}_\alpha$ exists such that for all $u\in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_{L^2(\mathbb{R}^2)}\le 1$, we have*

$$\int_{\mathbb{R}^2}\Big(e^{\alpha|u(x)|^2}-1\Big)\mathrm{d}x\le\mathcal{C}_\alpha\|u\|^2_{L^2(\mathbb{R}^2)}. \tag{1.11}$$

Moreover, (1.11) *is false if $\alpha\ge 4\pi$.*

Remark 1.8. $\alpha=4\pi$ becomes admissible if we take $\|u\|_{H^1(\mathbb{R}^2)}\le 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)}\le 1$. In this case

$$\mathcal{K}:=\sup_{\|u\|_{H^1(\mathbb{R}^2)}\le 1}\int_{\mathbb{R}^2}\Big(e^{4\pi|u(x)|^2}-1\Big)\mathrm{d}x<\infty, \tag{1.12}$$

and this is false for $\alpha>4\pi$. See [23] for more details.

Thanks to the following L^∞ logarithmic estimate, coupled with the previous inequalities, we will be able to control $\|e^{4\pi|u|^2}-1\|_{L^1_T L^2}$.

Proposition 1.9. (Log estimate) *Let* $\beta\in]0,1[$. *For any* $\lambda>\frac{1}{2\pi\beta}$ *and any* $0<\mu\le 1$, *a constant* C_λ *exists such that, for any function* $u\in H^1(\mathbb{R}^2)\cap C^\beta(\mathbb{R}^2)$, *we have*

$$\|u\|^2_{L^\infty(\mathbb{R}^2)}\le\lambda\|u\|^2_\mu\log\left(C_\lambda+\frac{8^\beta\|u\|_{C^\beta(\mathbb{R}^2)}}{\mu^\beta\|u\|_\mu}\right), \tag{1.13}$$

where

$$\|u\|^2_\mu:=\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+\mu^2\|u\|^2_{L^2(\mathbb{R}^2)}. \tag{1.14}$$

Recall that $C^\beta(\mathbb{R}^2)$ denotes the space of β-Hölder continuous functions endowed with the norm

$$\|u\|_{C^\beta(\mathbb{R}^2)}:=\|u\|_{L^\infty(\mathbb{R}^2)}+\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\beta}.$$

We refer to [40] for the proof of this Proposition and for more details. We just point out that the condition $\lambda>1/(2\pi\beta)$ in (1.13) is optimal.

Finally, we recall the following abstract result.

Lemma 1.10. (Bootstrap Lemma) *Let* $T>0$ *and* $X\in C([0,T],\mathbb{R}_+)$ *such that*

$$X\le a+bX^\theta,\qquad \text{on } [0,T],$$

where a, $b>0$, $\theta>1$, $a<(1-1/\theta)(\theta b)^{-1/\theta}$ *and* $X(0)\le(\theta b)^{-1/(\theta-1)}$. *Then*

$$X\le\frac{\theta}{\theta-1}a,\qquad \text{on } [0,T].$$

We mention that C denotes an absolute positive constant which may vary from line to line. If A and B are nonnegative real numbers, $A\lesssim B$ means that $A\le CB$. Moreover, we denote for $1\le r\le\infty$ and $1\le s,T<\infty$,

$$\|u\|_{L^s_TL^r}:=\Big(\int_0^T\|u(t)\|^s_{L^r(\mathbb{R}^2)}\mathrm{d}t\Big)^{\frac{1}{s}},\quad \|u\|_{L^sL^r}:=\Big(\int_0^{+\infty}\|u(t)\|^s_{L^r(\mathbb{R}^2)}\mathrm{d}t\Big)^{\frac{1}{s}}.$$

This paper is organized as follows. The next section is devoted to give some technical results. In the last section we prove our main result.

2 Preliminary results

In this section, we give some technical Lemmas needed to prove our main result about decay of solution to the Shrödinger equation (1.1).
For any time slab $I\subset\mathbb{R}$ and any $\varphi\in H^1(\mathbb{R}^2)$, we denote

$$\|u\|_{S^1(I)}:=\|u\|_{L^\infty(I,H^1(\mathbb{R}^2))}+\|u\|_{L^4(I,W^{1,4}(\mathbb{R}^2))},$$

and the Hamiltonian

$$H(\varphi):=\|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{4\pi}\|e^{4\pi|\varphi|^2}-1-4\pi|\varphi|^2\|_{L^1(\mathbb{R}^2)}.$$

For small time, we have the following uniforme estimate.

Lemma 2.1. *Let* $0<\eta<1$, (φ_n) *a sequence of* $H^1(\mathbb{R}^2)$ *satisfying* $\sup_n\|\varphi_n\|_{H^1(\mathbb{R}^2)}<\infty$ *and* $H(\varphi_n)\le 1$. *We denote by* u_n *the solution in* $C(\mathbb{R},H^1(\mathbb{R}^2))$ *to* (1.1) *with data* φ_n. *Assume that for some* $T_1>0$,

$$\sup_n\|\nabla u_n(t)\|_{L^2(\mathbb{R}^2)}\le\eta,\qquad \forall t\in[0,T_1].$$

Then there exist $T>0$ *and a constant* $C(\eta)$ *such that*

$$\sup_n\Big(\|u_n\|_{S^1(0,T)}\Big)\le C(\eta).$$

Proof. Using Strichartz estimate (1.9) we have

$$\begin{aligned}\|u_n\|_{S^1(0,T)}&\le C\Big(\|\varphi_n\|_{H^1(\mathbb{R}^2)}+\|f(u_n)\|_{L^1_T(H^1(\mathbb{R}^2))}\Big)\\&\lesssim 1+\|f(u_n)\|_{L^1_T(H^1(\mathbb{R}^2))}\\&\lesssim 1+\|f(u_n)\|_{L^1_T(L^2(\mathbb{R}^2))}+\|\nabla f(u_n)\|_{L^1_T(L^2(\mathbb{R}^2))}.\end{aligned}\tag{2.1}$$

Let $\varepsilon>0$, there exists a positive real number C_ε such that

$$\begin{aligned}\|f(u_n)\|_{L^1_TL^2}&\le C_\varepsilon\left\|u_n\Big(e^{4\pi(1+\varepsilon)|u_n|^2}-1\Big)\right\|_{L^1_TL^2}\\&\le C_\varepsilon\|u_n\|_{L^4_TL^4}\left\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\right\|_{L^{\frac43}_TL^4}\\&\le C_\varepsilon\|u_n\|_{S^1(0,T)}\left\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\right\|_{L^{\frac43}_TL^4}.\end{aligned}\tag{2.2}$$

For ε small enough and $T\le T_1$,

$$\begin{aligned}\left\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\right\|_{L^{\frac43}_TL^4}&\le\left\|\left\|e^{16\pi(1+\varepsilon)|u_n|^2}-1\right\|^{\frac14}_{L^1}\right\|_{L^{\frac43}_T}\\&\le\left\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\right\|^{\frac14}_{L^\infty_TL^1}\left\|e^{3\pi(1+\varepsilon)\|u_n\|^2_{L^\infty}}\right\|_{L^{\frac43}_T}\\&\lesssim\left\|e^{3\pi(1+\varepsilon)\|u_n\|^2_{L^\infty}}\right\|_{L^{\frac43}_T}.\end{aligned}\tag{2.3}$$

In fact, using Moser-Trudinger inequality (1.11) for $(1+\varepsilon)\eta^2<1$ and $t\le T$, we obtain

$$\begin{aligned}\|e^{4\pi(1+\varepsilon)|u_n(t)|^2}-1\|_{L^1(\mathbb{R}^2)}&=\left\|e^{4\pi(1+\varepsilon)\eta^2\left(\frac{|u_n(t)|}{\eta}\right)^2}-1\right\|_{L^1(\mathbb{R}^2)}\\&\lesssim\|u_n(t)\|^2_{L^2(\mathbb{R}^2)}\lesssim\|\varphi_n\|^2_{L^2(\mathbb{R}^2)}\lesssim 1.\end{aligned}$$

For any $\lambda>\frac{1}{\pi}$ and $\mu\in]0,1]$, by the logarithmic inequality (1.13), we have

$$\begin{aligned} e^{3(1+\varepsilon)\pi\|u_n\|^2_{L^\infty_x}} &\leq \Big(C+2\sqrt{\frac{2}{\mu}}\frac{\|u_n\|_{C^{\frac{1}{2}}}}{\|u_n\|_\mu}\Big)^{3(1+\varepsilon)\lambda\pi\|u_n\|^2_\mu} \\ &\leq \left(C+2\sqrt{\frac{2}{\mu(\eta^2+M\mu^2)}}\|u_n\|_{C^{\frac{1}{2}}}\right)^{3(1+\varepsilon)(\eta^2+M\mu^2)\lambda\pi} \\ &\lesssim \Big(1+\|u_n\|_{C^{\frac{1}{2}}}\Big)^{3(1+\varepsilon)(\eta^2+M\mu^2)\lambda\pi}, \end{aligned} \tag{2.4}$$

where $M:=\sup_n\|\varphi_n\|^2_{L^2}$. Taking ε,μ close to zero, λ close to $1/\pi$ and choosing suitably η, there exists a nonnegative real r such that

$$4(1+\varepsilon)(\eta^2+M\mu^2)\lambda\pi\leq r<4. \tag{2.5}$$

Then, using (2.4), for some real number a satisfying $1/r=1/4+1/a$, we have

$$\begin{aligned} \|e^{3(1+\varepsilon)\pi\|u_n\|^2_{L^\infty_x}}\|_{L^{\frac{4}{3}}_T} &\lesssim \|1+\|u_n\|_{C^{\frac{1}{2}}}\|^{\frac{3r}{4}}_{L^r_T} \\ &\lesssim \Big(T^{\frac{1}{r}}+T^{\frac{1}{a}}\|u_n\|_{L^4_TC^{\frac{1}{2}}}\Big)^{\frac{3r}{4}} \\ &\lesssim T^{\frac{3}{4}}+T^{\frac{3r}{4a}}\|u_n\|^{\frac{3r}{4}}_{S^1(0,T)}. \end{aligned} \tag{2.6}$$

Plugging the estimates (2.2)-(2.3)-(2.6) together, we obtain for small T,

$$\|f(u_n)\|_{L^1_TL^2}\lesssim\Big(T^{\frac{3}{4}}+T^{\frac{3r}{4a}}\|u_n\|^{\frac{3r}{4}}_{S^1(0,T)}\Big)\|u_n\|_{S^1(0,T)}. \tag{2.7}$$

In what follows, we control $\|f(u_n)\|_{L^1_T\dot{H}^1}$. For any $\varepsilon>0$, we have

$$\begin{aligned} \|\nabla f(u_n)\|_{L^1_TL^2} &\lesssim \Big\|\nabla u_n\Big(e^{4\pi(1+\varepsilon)|u_n|^2}-1\Big)\Big\|_{L^1_TL^2} \\ &\lesssim \|\nabla u_n\|_{L^4_TL^4}\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\|_{L^{\frac{4}{3}}_TL^4} \\ &\lesssim \|u_n\|_{S^1(0,T)}\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\|_{L^{\frac{4}{3}}_TL^4}. \end{aligned}$$

Arguing as previously, we obtain

$$\|\nabla f(u_n)\|_{L^1_TL^2}\lesssim\Big(T^{\frac{3}{4}}+T^{\frac{3r}{4a}}\|u_n\|^{\frac{3r}{4}}_{S^1(0,T)}\Big)\|u_n\|_{S^1(0,T)}.$$

Thus, by (2.1),

$$\|u_n\|_{S^1(0,T)}\lesssim 1+\Big(T^{\frac{3}{4}}+T^{\frac{3r}{4a}}\|u_n\|^{\frac{3r}{4}}_{S^1(0,T)}\Big)\|u_n\|_{S^1(0,T)}.$$

Let $X_n(T) := \|u_n\|_{S^1(0,T)} + 1$. For small T, we have

$$\begin{aligned} X_n(T) &\lesssim 1 + \left(T^{\frac{3}{4}} + T^{\frac{3r}{4a}} \|u_n\|_{S^1(0,T)}^{\frac{3r}{4}}\right) X_n(T) \\ &\lesssim 1 + T^{\frac{3r}{4a}} \left(1 + \|u_n\|_{S^1(0,T)}\right)^{\frac{3r}{4}} X_n(T) \\ &\lesssim 1 + T^{\frac{3r}{4a}} X_n(T)^{1+\frac{3r}{4}}. \end{aligned}$$

Taking account of Lemma 1.10, the previous inequality and (2.5), we obtain, for small time T,

$$\sup_n \left(\|u_n\|_{S^1(0,T)}\right) \lesssim C(\eta).$$

The proof of Lemma 2.1 is achieved. □

Our next preliminary result is the following

Lemma 2.2. *Let* (φ_n) *a sequence of* $H^1(\mathbb{R}^2)$ *such that* $\sup_n \|\varphi_n\|_{H^1(\mathbb{R}^2)} < \infty$, $H(\varphi_n) \le 1$ *and* φ_n *converging weakly to* φ *in* $H^1(\mathbb{R}^2)$*. Then,*

$$H(\varphi) \le 1.$$

Proof. We denote by

$$\begin{aligned} &F(x) := e^{4\pi x^2} - 4\pi x^2 - 1, \\ &b_n := \frac{1}{4\pi} \int_{\mathbb{R}^2} F(\varphi_n(x))\mathrm{d}x, \qquad b := \frac{1}{4\pi} \int_{\mathbb{R}^2} F(\varphi(x))\mathrm{d}x, \\ &a_n := \|\nabla \varphi_n\|_{L^2(\mathbb{R}^2)}^2, \qquad a := \|\nabla \varphi\|_{L^2(\mathbb{R}^2)}^2. \end{aligned}$$

It follows that

$$H(\varphi_n) = a_n + b_n \quad \text{and} \quad H(\varphi) = a + b.$$

Since φ_n converges weakly to φ in H^1, then, up to subsequence extraction, φ_n converges to φ in L^2_{loc}. Hence, φ_n converges almost everywhere to φ. Then, with Fatou Lemma, we have

$$b \le \liminf_{n\to\infty} b_n.$$

Thanks to the previous inequality with the fact that $a_n + b_n = H(\varphi_n) \le 1$, we have

$$\limsup_{n\to\infty} a_n \le 1 - \liminf_{n\to\infty} b_n \le 1 - b.$$

Let $\psi \in L^2(\mathbb{R}^2)$ such that $\|\psi\|_{L^2(\mathbb{R}^2)} = 1$. By duality argument

$$|\langle \nabla \varphi_n, \psi \rangle_{L^2(\mathbb{R}^2)}| \le \sqrt{a_n}.$$

Taking the limit as n tends to infinity, we obtain

$$|\langle\nabla\varphi,\psi\rangle_{L^2(\mathbb{R}^2)}|^2\le\limsup_{n\to\infty} a_n,$$

which implies that

$$a=\sup_{\|\phi\|_{L^2(\mathbb{R}^2)}=1}\Big|\langle\nabla\varphi,\phi\rangle_{L^2(\mathbb{R}^2)}\Big|^2\le\limsup_{n\to\infty} a_n.$$

Thus $H(\varphi)\le 1$. The proof of Lemma 2.2 is achieved. ☐

Using Lemmas 2.1–2.2, we obtain the following result.

Lemma 2.3. *Let $\chi\in C_0^\infty(\mathbb{R}^2)$ to be a cut-off function, $0<\eta<1$ and (φ_n) a sequence in $H^1(\mathbb{R}^2)$ satisfying $\sup_n\|\varphi_n\|_{H^1(\mathbb{R}^2)}<\infty$, $H(\varphi_n)\le 1$ and $\varphi_n\rightharpoonup\varphi$ in $H^1(\mathbb{R}^2)$. Let u_n (respectively u) to be the solution in $C(\mathbb{R},H^1)$ to (1.1) with initial data φ_n (respectively φ). Assume that for some $T>0$, $\sup_n\|\nabla u_n(t)\|_{L^2(\mathbb{R}^2)}\le\eta$, $\quad\forall t\in[0,T]$. Then, for every $\varepsilon>0$, there exist $T_\varepsilon>0$ and $n_\varepsilon\in\mathbb{N}$ such that*

$$\|\chi(u_n-u)\|_{L^\infty_{T_\varepsilon}L^2}<\varepsilon,\quad \forall n>n_\varepsilon.$$

Remark 2.4. Note that the existence of $u\in C(\mathbb{R},H^1)$ in Lemma 2.3 is guaranteed by Lemma 2.2.

Proof of Lemma 2.3. Let $v_n:=\chi u_n$ and $v:=\chi u$. We compute

$$i\partial_t v_n+\Delta v_n=\Delta\chi u_n+2\nabla\chi\nabla u_n+\chi f(u_n),\quad v_n(0)=\chi\varphi_n,$$

and

$$i\partial_t v+\Delta v=\Delta\chi u+2\nabla\chi\nabla u+\chi f(u),\quad v(0)=\chi\varphi.$$

With the integral formula, we obtain

$$v_n(t,x)=e^{it\Delta}\chi\varphi_n+i\int_0^t e^{i(t-s)\Delta}\Big(\Delta\chi u_n+2\nabla\chi\nabla u_n+\chi f(u_n)\Big)\mathrm{d}s,$$

and

$$v(t,x)=e^{it\Delta}\chi\varphi+i\int_0^t e^{i(t-s)\Delta}\Big(\Delta\chi u+2\nabla\chi\nabla u+\chi f(u)\Big)\mathrm{d}s.$$

We denote $w_n:=v_n-v$, $z_n:=u_n-u$. By Strichartz estimate

$$\begin{aligned}\|w_n\|_{L^\infty_T L^2}\lesssim&\|\chi(\varphi_n-\varphi)\|_{L^2(\mathbb{R}^2)}+\|\Delta\chi z_n\|_{L^1_T L^2}\\&+2\|\nabla\chi\nabla z_n\|_{L^1_T L^2}+\|\chi(f(u_n)-f(u))\|_{L^1_T L^2}.\end{aligned}\tag{2.8}$$

Thanks to Rellich Theorem, up to subsequence extraction, we have

$$\lim_{n\to\infty}\|\chi(\varphi_n-\varphi)\|_{L^2(\mathbb{R}^2)}=0.\tag{2.9}$$

Moreover, by Hölder inequality

$$\begin{aligned}\|\Delta\chi z_n\|_{L^1_TL^2}+2\|\nabla\chi\nabla z_n\|_{L^1_TL^2}&\leq N\Big(\|\Delta\chi\|_{L^1_TL^4}+2\|\nabla\chi\|_{L^1_TL^4}\Big)\\&\leq NT\Big(\|\Delta\chi\|_{L^4(\mathbb{R}^2)}+2\|\nabla\chi\|_{L^4(\mathbb{R}^2)}\Big)\lesssim T,\end{aligned}\tag{2.10}$$

where $N:=\|u\|_{L^\infty H^1}+\sup_n\|u_n\|_{L^\infty H^1}$.

Using a convexity argument, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|f(z_1)-f(z_2)|\leq C_\varepsilon|z_1-z_2|\sum_{i=1,2}\Big(e^{4\pi(1+\varepsilon)|z_i|^2}-1\Big).$$

Since $\|w_n\|_{L^4_TL^4}\leq NT^{1/4}$, we have for any $\varepsilon>0$,

$$\begin{aligned}\|\chi(f(u_n)-f(u))\|_{L^1_TL^2}&\lesssim\|w_n\|_{L^4_TL^4}\Big(\|e^{4\pi(1+\varepsilon)|u|^2}-1\|_{L^{\frac{4}{3}}_TL^4}+\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\|_{L^{\frac{4}{3}}_TL^4}\Big)\\&\lesssim T^{\frac14}\Big(\|e^{4\pi(1+\varepsilon)|u|^2}-1\|_{L^{\frac{4}{3}}_TL^4}+\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\|_{L^{\frac{4}{3}}_TL^4}\Big).\end{aligned}\tag{2.11}$$

Arguing as previously and using (2.6) with Lemma 2.1, there exist some positive real numbers $a,r,\alpha>0$ satisfying $1/r=1/4+1/a$ and

$$\|e^{4\pi(1+\varepsilon)|u_n|^2}-1\|_{L^{\frac{4}{3}}_TL^4}\lesssim T^{\frac34}+T^{\frac{3r}{4a}}\|u_n\|^{\frac{3r}{4}}_{S^1(0,T)}\lesssim T^\alpha.\tag{2.12}$$

Moreover, using a continuity argument with the fact that

$$\|\nabla u(0)\|_{L^2(\mathbb{R}^2)}\leq\liminf_{n\to\infty}\|\nabla u_n(0)\|_{L^2(\mathbb{R}^2)}\leq\eta,$$

there exist a positive time denoted also $T>0$ and a real number $0<\eta_1<1$ such that $\sup_{[0,T]}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}\leq\eta_1$. So, arguing as previously, there exists a real number, denoted also $\alpha>0$, such that

$$\|e^{4\pi(1+\varepsilon)|u|^2}-1\|_{L^{\frac{4}{3}}_TL^4}\lesssim T^{\frac34}+T^{\frac{3r}{4a}}\|u\|^{\frac{3r}{4}}_{S^1(0,T)}\lesssim T^\alpha.\tag{2.13}$$

As a consequence of (2.11)-(2.12)-(2.13),

$$\|\chi(f(u_n)-f(u))\|_{L^1_TL^2}\lesssim T^\alpha,\quad\alpha>0.\tag{2.14}$$

The proof of Lemma 2.3 is achieved thanks to (2.8)-(2.9)-(2.10)-(2.14). □

We conclude this section with the following result.

Lemma 2.5. *Let $u_0 \in H^1$ such that $H(u_0) \le 1$ and $u \in C(\mathbb{R}, H^1)$ be the solution to* (1.1) *with initial data u_0. Take (t_n) a sequence of positive real numbers tending to infinity. Then, there are two possible cases:*

1. *There exist two real numbers $T > 0$ and $0 < \eta < 1$, such that*

$$\sup_n \|\nabla u(t_n + t)\|_{L^2(\mathbb{R}^2)} \le \eta, \quad \forall t \in [0, T]. \tag{2.15}$$

2. *There exist a subsequence denoted by (s_n) and sequence of positive real numbers (r_n) such that*

$$\lim_{n\to\infty} r_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\nabla u(s_n + r_n)\|_{L^2(\mathbb{R}^2)} = 1. \tag{2.16}$$

Proof. We proceed by contradiction. Assume that (2.15) is false. Then, there exists a sequence (r_n) of positive real numbers such that

$$0 < r_p \le \frac{1}{p} \quad \text{and} \quad 1 - \frac{1}{2p} < \sup_n \|\nabla u(t_n + r_p)\|_{L^2(\mathbb{R}^2)} \le 1, \quad \forall p \ge 1.$$

If there exist infinitely many p such that

$$\sup_n \|\nabla u(t_n + r_p)\|_{L^2(\mathbb{R}^2)} = \|\nabla u(t_{n(p)} + r_p)\|_{L^2(\mathbb{R}^2)},$$

then

$$\lim_p \|\nabla u(t_{n(p)} + r_p)\|_{L^2(\mathbb{R}^2)} = 1. \tag{2.17}$$

Now, if $|\{n(p), p \ge 1\}| < \infty$, we have

$$\sup_p \|\nabla u(t_{n(p)} + r_p)\|_{L^2(\mathbb{R}^2)} \le \sup_{[0, 1+\max t_{n(p)}]} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} < 1.$$

This contradicts (2.17). So, up to subsequence extraction, we have

$$1 - \frac{1}{2p} < \|\nabla u(s_p + r_p)\|_{L^2(\mathbb{R}^2)} \le 1, \qquad \forall p \ge 1.$$

In particular, we have (2.16).

Now, assume that there exist infinitely many p such that $\sup_n \|\nabla u(t_n + r_p)\|_{L^2(\mathbb{R}^2)}$ is not atteigned. So, up to extraction of (r_p), for any p there exist infinitely many n such that

$$\left| \sup_m \|\nabla u(t_m + r_p)\|_{L^2(\mathbb{R}^2)} - \|\nabla u(t_n + r_p)\|_{L^2(\mathbb{R}^2)} \right| \le \frac{1}{2p}.$$

Thus, for any p there exist infinity many n such that

$$1 - \frac{1}{p} \le \|\nabla u(t_n + r_p)\|_{L^2(\mathbb{R}^2)} \le 1.$$

So, there exists an increasing integer function φ_p such that

$$1-\frac{1}{p}\leq\|\nabla u(t_{\varphi_p(n)}+r_p)\|_{L^2(\mathbb{R}^2)}\leq 1,\quad \forall p\geq 1,\forall n\in\mathbb{N}.$$

Then

$$1-\frac{1}{p}\leq\|\nabla u(t_{\varphi_p(p)}+r_p)\|_{L^2(\mathbb{R}^2)}\leq 1,\quad \forall p\geq 1.$$

Finally, for some subsequence of (t_n) denoted by (s_n), we have

$$\lim_{n\to\infty}\|\nabla u(s_n+r_n)\|_{L^2(\mathbb{R}^2)}=1.$$

The proof of Lemma 2.5 is finished. □

Now, we are ready to prove of the main result of this paper.

3 Proof of Theorem 1.3

By an interpolation argument it is sufficiant to prove Theorem 1.3 for $r=3$. We recall the following Gagliardo-Nirenberg inequality

$$\|u(t)\|^3_{L^3(\mathbb{R}^2)}\leq C\|u(t)\|^2_{H^1(\mathbb{R}^2)}\Big(\sup_x\|u(t)\|_{L^2(Q_1(x))}\Big), \tag{3.1}$$

where $Q_a(x)$ denotes the square centered at x whose edge has length a.

First case: $\limsup_{t\to\infty}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}<1.$

We proceed by contradiction. Assume that there exist a sequence (t_n) of positive real numbers and $\varepsilon>0$ such that $\lim_{n\to\infty}t_n=\infty$ and

$$\|u(t_n)\|_{L^3(\mathbb{R}^2)}>\varepsilon,\qquad \forall n\in\mathbb{N}. \tag{3.2}$$

By (3.2) and (3.1), there exist a sequence (x_n) in $\mathbb{R}^2$ and a positive real number denoted also by $\varepsilon>0$ such that

$$\|u(t_n)\|_{L^2(Q_1(x_n))}\geq\varepsilon,\qquad \forall n\in\mathbb{N}. \tag{3.3}$$

Let $\varphi_n(x):=u(t_n,x+x_n)$. Using the conservation laws, we obtain $\sup_n\|\varphi_n\|_{H^1}<\infty$. Then, up to a subsequence extraction, there exists $\varphi\in H^1$ such that φ_n converges weakly to φ in H^1. By Rellich Theorem, up to a subsequence extraction, we have

$$\lim_{n\to\infty}\|\varphi_n-\varphi\|_{L^2(Q_1(0))}=0. \tag{3.4}$$

Now, (3.3) implies that, $\|\varphi_n\|_{L^2(Q_1(0))}\geq\varepsilon$. So, using (3.4), there exists a positive real number denoted also $\varepsilon>0$ such that

$$\|\varphi\|_{L^2(Q_1(0))}\geq\varepsilon. \tag{3.5}$$

We denote by $\bar{u}\in C(\mathbb{R},H^1)$ the solution of (1.1) with data φ. Take a cut-off functin $\chi\in C_0^\infty(\mathbb{R}^2)$ satisfying $0\le\chi\le 1$, $\chi=1$ on $Q_1(0)$ and $\text{supp}(\chi)\subset Q_2(0)$. Using (3.5) with a continuity argument, there exists $T>0$ such that

$$\inf_{t\in[0,T]}\|\chi\bar{u}(t)\|_{L^2(\mathbb{R}^2)}\ge\frac{\varepsilon}{2}. \tag{3.6}$$

Since $H(\varphi_n)=H(u)\le 1$, there exists a unique $u_n\in C(\mathbb{R},H^1)$, solution to (1.1) with data φ_n. Moreover,

$$u_n(t,x)=u(t+t_n,x+x_n).$$

Using Lemma 2.5, there exist two real numbers $0<\eta<1$ and $T>0$, such that

$$\sup_n\|\nabla u_n(t)\|_{L^2(\mathbb{R}^2)}\le\eta,\qquad\forall t\in[0,T]. \tag{3.7}$$

Now, by Lemma 2.3 and (3.7), there is a positive time denoted also T and $n_\varepsilon\in\mathbb{N}$ such that

$$\|\chi(u_n-\bar{u})\|_{L_T^\infty L_x^2}\le\frac{\varepsilon}{4},\qquad\forall n\ge n_\varepsilon. \tag{3.8}$$

Hence, for all $t\in[0,T]$ and $n\ge n_\varepsilon$,

$$\|\chi u_n(t)\|_{L^2(\mathbb{R}^2)}\ge\|\chi\bar{u}(t)\|_{L^2(\mathbb{R}^2)}-\|\chi(u_n-\bar{u})(t)\|_{L^2(\mathbb{R}^2)}\ge\frac{\varepsilon}{4}. \tag{3.9}$$

By the proprieties of χ and the last inequality, for all $t\in[0,T]$ and $n\ge n_\varepsilon$,

$$\|u(t+t_n)\|_{L^2(Q_2(x_n))}=\|u_n(t)\|_{L^2(Q_2(0))}\ge\frac{\varepsilon}{4}. \tag{3.10}$$

This implies that

$$\|u(t)\|_{L^2(Q_2(x_n))}\ge\frac{\varepsilon}{4},\quad\forall t\in[t_n,t_n+T],\quad\forall n\ge n_\varepsilon. \tag{3.11}$$

Since, by Hölder inequality, we have

$$\|u(t)\|_{L^2(Q_2(x_n))}\lesssim\|u(t)\|_{L^8(Q_2(x_n))},$$

then, there exists a real number $\alpha>0$ such that

$$\|u(t)\|_{L^8(Q_2(x_n))}\ge\alpha,\quad\forall t\in[t_n,t_n+T],\quad\forall n\ge n_\varepsilon. \tag{3.12}$$

Moreover, as $\lim_{n\to\infty}t_n=\infty$, we can suppose that $t_{n+1}-t_n>T$ for $n\ge n_\varepsilon$. Therefore

$$\begin{aligned}\|u\|_{L^4L^8}^4&=\int_0^\infty\|u(t)\|_{L^8}^4\mathrm{d}t\ge\sum_{n\ge n_\varepsilon}\int_{t_n}^{t_n+T}\|u(t)\|_{L^8}^4\mathrm{d}t\\&\ge\sum_{n\ge n_\varepsilon}\int_{t_n}^{t_n+T}\|u(t)\|_{L^8(Q_2(x_n))}^4\mathrm{d}t\ge\sum_{n\ge n_\varepsilon}\alpha^4T=\infty.\end{aligned} \tag{3.13}$$

This obviously contradicts the fact that u belongs to $L_t^4 L_x^8$. Hence

$$\lim_{n\to\infty}\|u(t_n)\|_{L^3(\mathbb{R}^2)}=0. \tag{3.14}$$

Second case: $\limsup_{t\to\infty}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}=1.$

Let (t_n) be a sequence of real numbers such that $\lim_{n\to\infty}t_n=\infty$. If we are in the case (2.15), the same arguments can be applied.

Assume that we are in the case (2.16). Recall that by Lemma 2.5 there exist (s_n) a susequence of (t_n) and a sequence of positive real numbers (r_n) such that

$$\lim_{n\to\infty} r_n=0 \quad \text{and} \quad \lim_{n\to\infty}\|\nabla u(s_n+r_n)\|_{L^2(\mathbb{R}^2)}=1.$$

We denote $y_n:=s_n+r_n$. We shall prove, by contradiction, that

$$\lim_{n\to\infty}\|u(y_n)\|_{L^3(\mathbb{R}^2)}=0.$$

Assume that there exists a positive real number $\varepsilon>0$ and a subsequence such that

$$\|u(y_n)\|_{L^3(\mathbb{R}^2)}>\varepsilon, \qquad \forall n\in\mathbb{N}. \tag{3.15}$$

By (3.1), there exist a sequence (x_n) in $\mathbb{R}^2$ and a positive real number denoted also by $\varepsilon>0$, such that

$$\|u(y_n)\|_{L^2(Q_1(x_n))}\geq\varepsilon, \qquad \forall n\in\mathbb{N}.$$

Take $\varphi_n(x):=u(y_n,x+x_n)$. Then

$$\|\varphi_n\|_{L^2(Q_1(0))}\geq\varepsilon, \qquad \forall n\in\mathbb{N}. \tag{3.16}$$

A staight forward computation leads to

$$H(\varphi_n)=H(u)=1, \qquad \lim_{n\to\infty}\|\nabla\varphi_n\|_{L^2(\mathbb{R}^2)}=1, \tag{3.17}$$

where

$$\begin{aligned} H(\varphi_n)&=\|\nabla\varphi_n\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{4\pi}\|F(\varphi_n)\|_{L^1(\mathbb{R}^2)}\\ &=\|\nabla\varphi_n\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{4\pi}\int_{\mathbb{R}^2}\left(e^{4\pi|\varphi_n|^2}-1-4\pi|\varphi_n|^2\right)\mathrm{d}x. \end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|F(\varphi_n)\|_{L^1(\mathbb{R}^2)}=0.$$

Using the inequality $x^4\lesssim F(x)$, we obtain

$$\lim_{n\to\infty}\|\varphi_n\|_{L^4}=0. \tag{3.18}$$

This implies that

$$\lim_{n\to\infty}\|\varphi_n\|_{L^2(Q_1(0))}=0,$$

which contradicts (3.16). Finally

$$\lim_{n\to\infty}\|u(s_n)\|_{L^3(\mathbb{R}^2)}=0.$$

This completes the proof of Theorem 1.3. □

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