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A Generalized (G'/G)-Expansion Method to Find the Traveling Wave Solutions of Nonlinear Evolution Equations

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Abstract. In this article, we construct the exact traveling wave solutions for nonlinear evolution equations in the mathematical physics via the modified Kawahara equation, the nonlinear coupled KdV equations and the classical Boussinesq equations, by using a generalized (G'/G)-expansion method, where G satisfies the Jacobi elliptic equation. Many exact solutions in terms of Jacobi elliptic functions are obtained.

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1 Introduction

The investigation of the exact solutions for nonlinear evolution equations plays an important role in the study of soliton theory. In recent years, many powerful methods to construct exact solutions of nonlinear evolution equations have been established and developed such as the inverse scattering transform method [1], the Hirota method [2], the Backlund transform method [3], the exp- function method [4], truncated Painleve expansion method [5], the Weierstrass elliptic function method [6], the tanh- function method [7] and the Jacobi elliptic function expansion method [8,9]. There are other methods which can be found in [10, 11].

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Wang *et al.* [12] have introduced a simple method which is called, the $(G^{'}/G)$- expansion method to look for traveling wave solutions of nonlinear evolution equations, where $G=G(\xi)$ satisfies the second order linear ordinary differential equation $G^{''}(\xi)+\lambda G'(\xi)+\mu G(\xi)=0$ and λ μ are arbitrary constants. For further references, see the articles [13, 14]. Recently, Zayed [15] introduced an alternative approach, which is called a generalized $(\frac{G^{'}}{G})$- expansion method . The main idea of this alternative approach is that the traveling wave solutions of nonlinear differential equations can be expressed by a polynomial in $(G^{'}/G)$,where $G=G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2=e_2G^4(\xi)+e_1G^2(\xi)+e_0$, $\xi=x-Vt$ and e_2, e_1, e_0, V are arbitrary constants while $^{'}=\mathrm{d}/\mathrm{d}\xi$. The objective of this article is to apply the generalized $(G^{'}/G)$-expansion method to construct the traveling wave solutions for nonlinear evolution equations in the mathematical physics via the modified Kawahara equation, the coupled KdV equations and the classical Boussinesq equations, in terms of the Jacobi elliptic functions.

2 Description of a generalized $(G^{'}/G)$-expansion method

Suppose we have the following nonlinear partial differential equation

$$F(u,u_t,u_x,u_{tt},u_{xt},u_{xx},\cdots)=0, \tag{2.1}$$

where $u=u(x,t)$ is an unknown function, F is a polynomial in $u(x,t)$ and its various partial derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following we give the main steps of a generalized $(G^{'}/G)$-expansion method [15]:

Step 1. We start with, the traveling wave variable

$$u(x,t)=u(\xi), \qquad \xi=x-Vt, \tag{2.2}$$

where V is a constant which, permits us reducing Eq. (2.1) to an ODE for $u=u(\xi)$ in the form

$$P(u,u^{'},u^{''},u^{'''},\cdots)=0. \tag{2.3}$$

Step 2. Suppose the solution of Eq. (2.3) can be expressed by a polynomial in $(G^{'}/G)$ as follows

$$u(\xi)=\sum_{i=0}^{n}\alpha_i\left(\frac{G^{'}}{G}\right)^i, \tag{2.4}$$

where $G=G(\xi)$ satisfies the following Jacobi elliptic equation:

$$[G'(\xi)]^2=e_2G^4(\xi)+e_1G^2(\xi)+e_0, \tag{2.5}$$

where α_i ,e_2,e_1,e_0 and V are arbitrary constants to be determined later provided $\alpha_n\neq 0$. The positive integer "n" can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.1) or (2.3). Therefore, we can get the value of n in (2.4).

Step 3. Substituting (2.4) into (2.3) and using Eq. (2.5), we obtain polynomials in $G^j(\xi)$, $G'(\xi)G^j(\xi)$ $(j=0,\pm1,\pm2,\cdots)$. Equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for α_i ,e_2,e_1,e_0 and V.

Step 4. Since the general solutions of (2.5) have been well known for us (see Appendix A), then substituting α_i,V and the general solution of (2.5) into (2.4) we have many exact traveling wave solutions of the nonlinear partial differential equation (2.1).

3 Some applications

In this section, we apply the generalized (G'/G)-expansion method to construct a new traveling wave solutions for the modified Kawahara equation, the nonlinear coupled KdV equations and the classical Boussinesq system, which are very important nonlinear evolution equations in mathematical physics and have been paid attention by many researchers.

3.1 Example 1: the modified Kawahara equation

We start with the modified Kawahara equation [16] in the form:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\alpha\frac{\partial^5 u}{\partial x^5}=0, \tag{3.1}$$

where α and β are arbitrary constants. This equation has been derived by Kawahara [16] as a model for water waves in the long- wave regime for moderate values of surface tension. The Kawahara equation (3.1) gives an appropriate description of several phenomena observed in the dynamics of the water- wave problem.

Let us now solve Eq. (3.1) by the generalized (G'/G)-expansion method. To this end, we see that the following traveling wave variable:

$$u(x,t)=u(\xi), \qquad \xi=x-Vt, \tag{3.2}$$

where V is a constant, permits us converting Eq.(3.1) into the following ODE:

$$3(1-V)u+u^3+3\beta u''+3\alpha u^{(4)}+3C_1=0, \tag{3.3}$$

where C_1 is the integration constant.

We suppose that the solution of Eq. (3.3) can be expressed by a polynomial in (G'/G) as the following form:

$$u(\xi)=\sum_{i=0}^{n}\alpha_i\left(\frac{G'}{G}\right)^i, \tag{3.4}$$

where α_i $(i=0,1,2,\cdots,n)$ are arbitrary constants, while $G(\xi)$ satisfies the Jacobi elliptic equation (2.5).

Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.3), we deduce that $n=2$. Thus, we get

$$u(\xi)=\alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0,\qquad \alpha_2\neq 0. \tag{3.5}$$

From (2.5) and (3.5) we have the following derivatives:

$$\begin{aligned}
&u'=2\alpha_2G'[e_2G-e_0G^{-3}]+\alpha_1[e_2G^2-e_0G^{-2}],\\
&u''=2\alpha_2[2e_1e_2G^2+2e_1e_0G^{-2}+3e_2^2G^4+3e_0^2G^{-4}+2e_0e_2]+2\alpha_1G'[e_2G+e_0G^{-3}],\\
&u'''=2\alpha_1[2e_1e_2G^2-2e_1e_0G^{-2}+3e_2^2G^4-3e_0^2G^{-4}]+8\alpha_2G'[e_1e_2G+3e_2^2G^3-e_1e_0G^{-3}-3e_0^2G^{-5}],
\end{aligned}$$

and

$$\begin{aligned}
u^{(4)}=&8\alpha_1G'[16e_0e_1e_2+16e_2e_1^2G^2+120e_1e_2^2G^4+72e_2^2e_0G^2\\
&+120e_2^3G^6+120e_0^2e_1G^{-4}+16e_0e_1^2G^{-2}+120e_0^3G^{-6}+72e_0^2e_2G^{-2}]\\
&+8\alpha_1G'[e_1e_2G+3e_2^2G^3+e_1e_0G^{-3}+3e_0^2G^{-5}].
\end{aligned}\tag{3.6}$$

Substituting (3.6) and (3.5) into (3.3) we get the following polynomial:

$$\begin{aligned}
&G^6[360\,\alpha\alpha_2e_2^3+\alpha_2^3e_2^3]+G^4[18\beta\alpha_2e_2^2+3\alpha_1^2\alpha_2e_2^2+360\,\alpha\alpha_2e_2^2e_1+3\alpha_2^3e_1e_2^2+3\alpha_2^2\alpha_0e_2^2]\\
&\quad+3\alpha_2\alpha_0^2e_2+3\alpha_2^3e_1^2e_2+48\alpha\alpha_2e_2e_1^2+12\beta\alpha_2e_2e_1+6\alpha_1^2\alpha_2e_1e_2+3\alpha_2^3e_2^2e_0+6\alpha_2^2\alpha_0e_1e_2]\\
&\quad+G'G\,[6\beta\alpha_1e_2+\alpha_1^3e_2+24\alpha\alpha_1e_2e_1+6\alpha_1\alpha_2\alpha_0e_2+6\alpha_1\alpha_2^2e_1e_2]+G'G^{-1}[-3V\alpha_1\\
&\quad+6\alpha_1\alpha_2^2e_2e_0+6\alpha_1\alpha_2\alpha_0e_1+\alpha_1^3e_1+3\alpha_1+3\alpha_1\alpha_2^2e_1^2+3\alpha_1\alpha_0^2]+G^{-2}[48\alpha\alpha_2e_0e_1^2\\
&\quad+6\alpha_1^2\alpha_2e_1e_0+3\alpha_1^2\alpha_0e_0-3V\alpha_2e_0+6\alpha_2^2\alpha_0e_1e_0+3\alpha_2\alpha_0^2e_0+3\alpha_2e_0+216\alpha\alpha_2e_0^2e_2\\
&\quad+3\alpha_2^3e_2e_0^2+3\alpha_2^3e_1^2e_0+12\beta\alpha_2e_0e_1]+G^{-4}[6\beta\alpha_1e_0+6\alpha_1\alpha_2^2e_1e_0+\alpha_1^3e_0+24\alpha\alpha_1e_0e_1]\\
&\quad+G'G^{-3}[360\alpha\alpha_2e_0^2e_1+6\alpha_1\alpha_2\alpha_0e_0+3\alpha_2^2\alpha_0e_0^2+18\beta\alpha_2e_0^2+3\alpha_1^2\alpha_2e_0^2+3\alpha_2^3e_1e_0^2]\\
&\quad+G'G\,[3\alpha_1\alpha_2^2e_0^2+72\alpha\alpha_1e_0^2]+G^{-6}[\alpha_2^3e_0^3+360\alpha\alpha_2e_0^3]+3\alpha_1^2\alpha_2e_1^2+6\alpha_2^2\alpha_0e_2e_0-3V\alpha_2e_1\\
&\quad-3V\alpha_0+\alpha_0^3+3\alpha_1^2\alpha_0e_1+12\beta\alpha_2e_0e_2+3\alpha_2\alpha_0^2e_1+3\alpha_2^2\alpha_0e_1^2+\alpha_2^3e_1^3+3\alpha_0+3C_1\\
&\quad+6\alpha_1^2\alpha_2e_2e_0+3\alpha_2e_1+6\alpha_2^3e_1e_2e_0+48\alpha\alpha_2e_0e_1e_2=0.
\end{aligned}\tag{3.7}$$

By equating the coefficients of the polynomial (3.7) to zero, we have a system of algebraic equations which can be solved by the Maple or Mathematica to obtain the following results:

$$\begin{aligned}
&\alpha_2=6\sqrt{-10\alpha},\qquad \alpha_1=0,\qquad \alpha_0=\frac{40\alpha e_1-\beta}{\sqrt{-10\alpha}},\\
&V=\frac{-1}{10\alpha}[240\alpha^2e_1^2+\beta^2-10\alpha+2880\,\alpha^2e_0e_2],\\
&C_1=\frac{\sqrt{-10\alpha}}{150\alpha^2}\{240\,\alpha^2e_1^2\beta-3200\alpha^3e_1^3-\beta^3+115200\,\alpha^3e_0e_1e_2+2880\,\alpha^2e_0e_2\beta\}.
\end{aligned}\tag{3.8}$$

Substituting (3.8) into (3.5) yields

$$u(\xi)=6\sqrt{-10\alpha}\left(\frac{G'}{G}\right)^2+\frac{40\alpha e_1-\beta}{\sqrt{-10\alpha}}, \tag{3.9}$$

where

$$\xi=x+\frac{t}{10\alpha}[240\alpha^2 e_1^2+\beta^2-10\alpha+2880\,\alpha^2 e_0 e_2]. \tag{3.10}$$

According to appendix A, we have the following families of exact solutions:

Family 1. If $e_0=1$, $e_1=-(m^2+1)$, $e_2=m^2$, then we get

$$u(\xi)=6\sqrt{-10\alpha}cs^2(\xi)dn^2(\xi)-\frac{40\alpha(1+m^2)+\beta}{\sqrt{-10\alpha}}, \tag{3.11}$$

or

$$u(\xi)=6\sqrt{-10\alpha}(1-m^2)^2sc^2(\xi)nd^2(\xi)-\frac{40\alpha(1+m^2)+\beta}{\sqrt{-10\alpha}}, \tag{3.12}$$

where $\xi=x+t[240\alpha^2(m^2+1)^2+\beta^2-10\alpha+2880\alpha^2m^2]/(10\alpha)$.

Family 2. If $e_0=1-m^2$, $e_1=2m^2-1$, $e_2=-m^2$, then we get

$$u(\xi)=6\sqrt{-10\alpha}sc^2(\xi)dn^2(\xi)+\frac{40\alpha(2m^2-1)-\beta}{\sqrt{-10\alpha}}, \tag{3.13}$$

where$\xi=x+t[240\alpha^2(2m^2-1)^2+\beta^2-10\alpha-2880\alpha^2(1-m^2)m^2]/(10\alpha)$.

Family 3. If $e_0=m^2-1$, $e_1=2-m^2$, $e_2=-1$, then we get

$$u(\xi)=6\sqrt{-10\alpha}m^4sd^2(\xi)cn^2(\xi)+\frac{40\alpha(2-m^2)-\beta}{\sqrt{-10\alpha}}, \tag{3.14}$$

where $\xi=x+t[240\alpha^2(2-m^2)^2+\beta^2-10\alpha-2880\alpha^2(m^2-1)]/(10\alpha)$.

Family 4. If $e_0=m^2$, $e_1=-(m^2+1)$, $e_2=1$, then we get

$$u(\xi)=6\sqrt{-10\alpha}ds^2(\xi)cn^2(\xi)-\frac{40\alpha(m^2+1)+\beta}{\sqrt{-10\alpha}}, \tag{3.15}$$

or

$$u(\xi)=6\sqrt{-10\alpha}(1-m^2)^2sd^2(\xi)nc^2(\xi)-\frac{40\alpha(m^2+1)+\beta}{\sqrt{-10\alpha}}, \tag{3.16}$$

where $\xi=x+t[240\alpha^2(m^2+1)^2+\beta^2-10\alpha+2880\alpha^2m^2]/(10\alpha)$.

Family 5. If $e_0=-m^2$, $e_1=2m^2-1$, $e_2=1-m^2$, then we get

$$u(\xi)=6\sqrt{-10\alpha}sn^2(\xi)dc^2(\xi)+\frac{40\alpha(2m^2-1)-\beta}{\sqrt{-10\alpha}}, \tag{3.17}$$

where $\xi=x+t[240\alpha^2(2m^2-1)^2+\beta^2-10\alpha-2880\alpha^2(1-m^2)m^2]/(10\alpha)$.

Family 6. If $e_0=-1$, $e_1=2-m^2$, $e_2=m^2-1$, then we get

$$u(\xi)=6\sqrt{-10\alpha}m^4sn^2(\xi)cd^2(\xi)+\frac{40\alpha(2-m^2)-\beta}{\sqrt{-10\alpha}}, \tag{3.18}$$

where $\xi=x+t[240\alpha^2(2-m^2)^2+\beta^2-10\alpha-2880\alpha^2(m^2-1)]/(10\alpha)$.

Family 7. If $e_0=1-m^2$, $e_1=2-m^2$, $e_2=1$, then we get

$$u(\xi)=6\sqrt{-10\alpha}nc^2(\xi)ds^2(\xi)+\frac{40\alpha(2-m^2)-\beta}{\sqrt{-10\alpha}}, \tag{3.19}$$

where $\xi=x+t[240\alpha^2(2-m^2)^2+\beta^2-10\alpha+2880\alpha^2(1-m^2)]/(10\alpha)$.

Family 8. If $e_0=1$, $e_1=2-m^2$, $e_2=1-m^2$, then we get

$$u(\xi)=6\sqrt{-10\alpha}ns^2(\xi)dc^2(\xi)+\frac{40\alpha(2-m^2)-\beta}{\sqrt{-10\alpha}}, \tag{3.20}$$

where $\xi=x+t[240\alpha^2(2-m^2)^2+\beta^2-10\alpha+2880\alpha^2(1-m^2)]/(10\alpha)$.

Family 9. If $e_0=1$, $e_1=2m^2-1$, $e_2=m^2(m^2-1)$, then we get

$$u(\xi)=6\sqrt{-10\alpha}\ ns^2(\xi)cd^2(\xi)+\frac{40\alpha(2m^2-1)-\beta}{\sqrt{-10\alpha}}, \tag{3.21}$$

where $\xi=x+t[240\alpha^2(2m^2-1)^2+\beta^2-10\alpha+2880\alpha^2m^2(m^2-1)]/(10\alpha)$.

Family 10. If $e_0=m^2(m^2-1)$, $e_1=2m^2-1$, $e_2=1$, then we get

$$u(\xi)=6\sqrt{-10\alpha}nd^2(\xi)cs^2(\xi)+\frac{40\alpha(2m^2-1)-\beta}{\sqrt{-10\alpha}}, \tag{3.22}$$

where $\xi=x+t[240\alpha^2(2m^2-1)^2+\beta^2-10\alpha+2880\alpha^2m^2(m^2-1)]/(10\alpha)$.

Family 11. If $e_0=1/4$, $e_1=(1-2m^2)/2$, $e_2=1/4$, then we get

$$u(\xi)=6\sqrt{-10\alpha}ds^2(\xi)+\frac{20\alpha(1-2m^2)-\beta}{\sqrt{-10\alpha}}, \tag{3.23}$$

where $\xi=x+t[60\alpha^2(1-2m^2)^2+\beta^2-10\alpha+180\alpha^2]/(10\alpha)$.

Family 12. If $e_0=(1-m^2)/4$, $e_1=(1+m^2)/2$, $e_2=(1-m^2)/4$, then we get

$$u(\xi)=6\sqrt{-10\alpha}dc^2(\xi)+\frac{20\alpha(1+m^2)-\beta}{\sqrt{-10\alpha}}, \tag{3.24}$$

where $\xi=x+t[60\ \alpha^2(1+m^2)^2+\beta^2-10\alpha+180\alpha^2(1-m^2)^2]/10\alpha$.

Family 13. If $e_0=m^2/4$, $e_1=(m^2-2)/2$, $e_2=1/4$, then we get

$$u(\xi)=6\sqrt{-10\alpha}cs^2(\xi)+\frac{20\alpha(m^2-2)-\beta}{\sqrt{-10\alpha}}, \tag{3.25}$$

where $\xi=x+t[60\alpha^2(m^2-2)^2+\beta^2-10\alpha+180\alpha^2m^2]/10\alpha$.

Family 14. If $e_0=m^2/4$, $e_1=(m^2-2)/2$, $e_2=m^2/4$, then we get

$$u(\xi)=6\sqrt{-10\alpha}dn^2(\xi)+\frac{20\alpha(m^2-2)-\beta}{\sqrt{-10\alpha}}, \tag{3.26}$$

where $\xi=x+t[60\alpha^2(m^2-2)^2+\beta^2-10\alpha+180\alpha^2m^4]/10\alpha$.

3.2 Example 2: the nonlinear coupled KdV equations

In this subsection, we consider the following nonlinear coupled KdV equations [17] in the forms:

$$\begin{aligned}&u_t+L_1u_x+L_2u\,u_x+L_3u_{xxx}+L_4v_x=0,\\&v_t+L_5v_x+L_6v\,v_x+L_7v_{xxx}+L_8u_x=0,\end{aligned} \tag{3.27}$$

where L_1-L_5 is the detaining parameter which measure the difference in the linear long-wave speed of uncoupled system, L_4, L_8 are the coupling parameter, while L_2,L_6 and L_3,L_7 are nonlinear and linear dispersive coefficients, respectively .

Let us now solve Eqs. (3.27) by the proposed method. To this end, we see that the traveling wave variables $u=u(\xi)$,$v=v(\xi)$ and $\xi=x-Vt$, permit us converting (3.27) into the following ODEs:

$$\begin{aligned}&C_1+(L_1-V)u+\frac{1}{2}L_2u^2+L_3u''+L_4v=0,\\&C_2+(L_5-V)v+\frac{1}{2}L_6v^2+L_7v''+L_8u=0,\end{aligned} \tag{3.28}$$

where C_1 and C_2 are the integration constants. Suppose that the solutions of Eqs. (3.28) can be expressed by a polynomials in (G'/G) as follows:

$$u(\xi)=\sum_{i=0}^{n}\alpha_i\left(\frac{G'}{G}\right)^i, \tag{3.29}$$

$$v(\xi)=\sum_{i=0}^{m}\beta_i\left(\frac{G'}{G}\right)^i, \tag{3.30}$$

where V, α_i $(i=0,1,\cdots,n)$ and β_i $(i=0,1,\cdots,n)$ are arbitrary constants to be determined provided $\alpha_n,\beta_n\neq 0$, while $G(\xi)$ satisfies the Jacobi elliptic equation (2.5). Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in (3.28), we get $n=m=2$. Thus, the solutions of Eqs. (3.28) have the following forms:

$$u(\xi)=\alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \tag{3.31}$$

and

$$v(\xi)=\beta_2\left(\frac{G'}{G}\right)^2+\beta_1\left(\frac{G'}{G}\right)+\beta_0. \tag{3.32}$$

Substituting (3.31) and (3.32) into system (3.28) and collecting all terms with the same power of $G^j(\xi)$, $G'(\xi)G^j(\xi)$ $(j=0,\pm1,\pm2,\cdots)$. By equating the coefficients of the polynomials to zero, yields a set of simultaneous algebraic equations and for the sake of brevity we omit them. Solving these algebraic equations by Maple or Mathematica, we have the formulae of the solutions of system (3.28) as follows:

$$u(\xi)=-\frac{12L_3}{L_2}\left(\frac{G'}{G}\right)^2+\alpha_0, \tag{3.33}$$

and

$$\begin{aligned}v(\xi)=&-\frac{12L_7}{L_6}\left(\frac{G'}{G}\right)^2+\frac{1}{L_3L_6^2L_7L_2}\Big(L_7L_2^2L_3\alpha_0L_6+L_7L_2L_1L_3L_6\\&-8L_7L_2L_3^2e_1L_6+L_7^2L_2^2L_4-L_3L_6L_5L_7L_2-L_3^2L_6^2L_8+8L_3L_6L_7^2e_1L_2\Big),\end{aligned} \tag{3.34}$$

where

$$\xi=x-\frac{t}{L_3L_6}\Big(L_3\alpha_0L_6L_2+L_1L_3L_6-8L_3^2e_1L_6+L_4L_7L_2\Big), \tag{3.35}$$

$$\begin{aligned}C_1=&\frac{1}{2L_3L_6^2L_7L_2}\Big(L_2^2\alpha_0^2L_3L_6^2L_7-16L_3^2\alpha_0e_1L_6^2L_7L_2-2L_4L_7L_2^2L_3\alpha_0L_6-2L_4L_7L_2L_1L_3L_6\\&+16L_4L_7L_2L_3^2e_1L_6-2L_7^2L_2^2L_4^2+2L_4L_3L_6L_5L_7L_2+2L_4L_3^2L_6^2L_8-16L_4L_7^2e_1L_3L_6L_2\\&+2\alpha_0L_6L_7^2L_2^2L_4-192L_3^3e_0e_2L_6^2L_7+48L_3^3e_1^2L_6^2L_7\Big),\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}C_2=&\frac{1}{2L_6^3L_3^2L_7^2L_2^2}\Big(-2L_7^3L_2^3L_4L_3L_6L_5+L_3^2L_6^2L_5^2L_7^2L_2^2-L_3^4L_6^4L_8^2-2L_8\alpha_0L_6^3L_3^2L_7^2L_2\\&-192L_7^4e_0e_2L_6^2L_3^2L_2^2+16L_8L_3^3e_1L_6^3L_7^2L_2-16L_7^4e_1^2L_6^2L_3^2L_2^2+L_7^4L_2^4L_4^2+L_7^2L_2^4L_3^2\alpha_0^2L_6^2\\&+2L_7^2L_2^3L_3^2\alpha_0L_6^2L_1-16L_7^2L_2^3L_3^3\alpha_0L_6^2e_1+2L_7^3L_2^4L_3\alpha_0L_6L_4-2L_7^2L_2^3L_3^2\alpha_0L_6^2L_5\\&+L_7^2L_2^2L_1^2L_3^2L_6^2-16L_7^2L_2^2L_1L_3^3L_6^2e_1+2L_7^3L_2^3L_1L_3L_6L_4-2L_7^2L_2^2L_1L_3^2L_6^2L_5\\&+64L_7^2L_2^2L_3^4e_1^2L_6^2-16L_7^3L_2^3L_3^2e_1L_6L_4+16L_7^2L_2^2L_3^3e_1L_6^2L_5\Big).\end{aligned} \tag{3.37}$$

According to appendix A, we have the following families of exact solutions:

Family 1. If $e_0=1$, $e_1=-(m^2+1)$, $e_2=m^2$, then we get

$$u(\xi)=-\frac{12L_3}{L_2}cs^2(\xi)\ dn^2(\xi)+\alpha_0, \tag{3.38}$$

and

$$\begin{aligned}v(\xi)=&-\frac{12L_7}{L_6}cs^2(\xi)\ dn^2(\xi)+\frac{1}{L_3L_6^2L_7L_2}\Big(L_7L_2^2L_3\alpha_0L_6+L_7L_2L_1L_3L_6\\&+8L_7L_2L_3^2(m^2+1)L_6+L_7^2L_2^2L_4-L_3L_6L_5L_7L_2-L_3^2L_6^2L_8\\&-8L_3L_6L_7^2(m^2+1)L_2\Big),\end{aligned} \tag{3.39}$$

or

$$u(\xi)=-\frac{12L_3(1-m^2)^2}{L_2}sd^2(\xi)\ nc^2(\xi)+\alpha_0, \tag{3.40}$$

and

$$\begin{aligned}v(\xi)=&-\frac{12L_7(1-m^2)^2}{L_6}\ sd^2(\xi)\ nc^2(\xi)+\frac{1}{L_3L_6^2L_7L_2}\Big(L_7L_2^2L_3\alpha_0L_6+L_7L_2L_1L_3L_6\\&+8L_7L_2L_3^2(m^2+1)L_6+L_7^2L_2^2L_4-L_3L_6L_5L_7L_2-L_3^2L_6^2L_8\\&-8L_3L_6L_7^2(m^2+1)L_2\Big),\end{aligned} \tag{3.41}$$

where $\xi=x-t[L_3\alpha_0L_6L_2+L_1L_3L_6+8L_3^2(m^2+1)L_6+L_4L_7L_2]/(L_3L_6)$.

Family 2. If $e_0=1-m^2$, $e_1=2m^2-1$, $e_2=-m^2$, then we get

$$u(\xi)=-\frac{12L_3}{L_2}sc^2(\xi)\ dn^2(\xi)+\alpha_0, \tag{3.42}$$

and

$$\begin{aligned}v(\xi)=&-\frac{12L_7}{L_6}sc^2(\xi)\ dn^2(\xi)+\frac{1}{L_3L_6^2L_7L_2}\Big(L_7L_2^2L_3\alpha_0L_6+L_7L_2L_1L_3L_6\\&-8L_7L_2L_3^2(2m^2-1)L_6+L_7^2L_2^2L_4-L_3L_6L_5L_7L_2-L_3^2L_6^2L_8\\&+8L_3L_6L_7^2(2m^2-1)L_2\Big),\end{aligned} \tag{3.43}$$

where $\xi=x-t[L_3\alpha_0L_6L_2+L_1L_3L_6-8L_3^2(2m^2-1)L_6+L_4L_7L_2]/(L_3L_6)$.

Family 3. If $e_0=m^2-1$, $e_1=2-m^2$, $e_2=-1$, then we get

$$u(\xi)=-\frac{12L_3}{L_2}m^4sn^2(\xi)\ cd^2(\xi)+\alpha_0, \tag{3.44}$$

and

$$v(\xi)=-\frac{12L_7}{L_6}m^4sn^2(\xi)\ cd^2(\xi)+\frac{1}{L_3L_6^2L_7L_2}\Big(L_7L_2^2L_3\alpha_0L_6+L_7L_2L_1L_3L_6 -8L_7L_2L_3^2(2-m^2)L_6+L_7^2L_2^2L_4-L_3L_6L_5L_7L_2-L_3^2L_6^2L_8 +8L_3L_6L_7^2(2-m^2)L_2\Big), \tag{3.45}$$

where $\xi=x-t(L_3\alpha_0L_6L_2+L_1L_3L_6-8L_3^2(2-m^2)L_6+L_4L_7L_2)/(L_3L_6)$.

Family 4. If $e_0=m^2$, $e_1=-(m^2+1)$, $e_2=1$, then we get

$$u(\xi)=-\frac{12L_3}{L_2}ds^2(\xi)\ cn^2(\xi)+\alpha_0, \tag{3.46}$$

and

$$v(\xi)=-\frac{12L_7}{L_6}ds^2(\xi)\ cn^2(\xi)+\frac{1}{L_3L_6^2L_7L_2}\Big(L_7L_2^2L_3\alpha_0L_6+L_7L_2L_1L_3L_6 +8L_7L_2L_3^2(m^2+1)L_6+L_7^2L_2^2L_4-L_3L_6L_5L_7L_2-L_3^2L_6^2L_8 -8L_3L_6L_7^2(m^2+1)L_2\Big), \tag{3.47}$$

where $\xi=x-t(L_3\alpha_0L_6L_2+L_1L_3L_6+8L_3^2(m^2+1)L_6+L_4L_7L_2)/(L_3L_6)$.

Similarly, we can write down the other families of exact solutions of Eq. (3.28) which are omitted for convenience.

3.3 Example 3: the classical Boussinesq equations

Lastly, we consider the classical Boussinesq equations [18,19] in the form:

$$v_t+[(1+v)u]_x+\frac{1}{3}u_{xxx}=0, \tag{3.48a}$$

$$u_t+u\ u_x+v_x=0, \tag{3.48b}$$

The system (3.48) is integrable and has three Hamiltonian structures [18]. Wu and Zhang [19] derive three sets of classical Boussinesq model equations for modeling nonlinear and dispersive long gravity wave traveling in two horizontal directions on shallow water of uniform depth.

Now let us solve system (3.48) by the proposed method. To this end, we see that the traveling wave variables $u=u(\xi)$, $v=v(\xi)$ and $\xi=x-Vt$, permit us converting (3.48) into the following ODEs:

$$C_1-Vv+(1+v)u+\frac{1}{3}u''=0,$$

$$C_2-Vu+\frac{1}{2}u^2+v=0, \tag{3.49}$$

where C_1 and C_2 are the integration constants. By considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.49), we get $n=1$ and $m=2$. Then, the solutions of Eqs. (3.49) have the following forms:

$$u(\xi)=\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \qquad \alpha_1\neq 0, \tag{3.50}$$

and

$$v(\xi)=\beta_2\left(\frac{G'}{G}\right)^2+\beta_1\left(\frac{G'}{G}\right)+\beta_0, \qquad \beta_2\neq 0. \tag{3.51}$$

Substituting (3.50) and (3.51) into system (3.49) , collecting all terms with the same power of $G^j(\xi)$, $G'(\xi)G^j(\xi)$ $(j=0,\pm1,\pm2,\cdots)$ and equating the coefficients of the polynomials to zero, yield a set of simultaneous algebraic equations. For the sake of brevity we omit them. Solving these algebraic equations by Maple or Mathematica, we have the formulae of the solutions of Eqs. (3.49) as follows:

$$u(\xi)=\pm\frac{2}{\sqrt{3}}\left(\frac{G'}{G}\right)+V, \tag{3.52}$$

$$v(\xi)=-\frac{2}{3}\left(\frac{G'}{G}\right)^2+\frac{2}{3}e_1-1, \tag{3.53}$$

where $C_1=-V$, $C_2=\frac{V^2}{2}+1-\frac{2}{3}e_1$ and $\xi=x-V\,t$.

According to the appendix A, we have the following families of exact solutions

Family 1. If $e_0=1$, $e_1=-(m^2+1)$, $e_2=m^2$, then we get

$$u(\xi)=\pm\frac{2}{\sqrt{3}}cs(\xi)\ dn(\xi)+V, \tag{3.54}$$

and

$$v(\xi)=-\frac{2}{3}cs^2(\xi)\ dn^2(\xi)-\frac{2}{3}(m^2+1)-1, \tag{3.55}$$

or

$$u(\xi)=\mp\frac{2}{\sqrt{3}}(1-m^2)\ sd(\xi)\ nc\ (\xi)+V, \tag{3.56}$$

and

$$v(\xi)=-\frac{2}{3}(1-m^2)^2\ sd^2(\xi)\ nc^2\ (\xi)-\frac{2}{3}(m^2+1)-1. \tag{3.57}$$

Family 2. If $e_0=1-m^2$, $e_1=2m^2-1$, $e_2=-m^2$, then we get

$$u(\xi)=\mp\frac{2}{\sqrt{3}}sc(\xi)\ dn(\xi)+V, \tag{3.58}$$

and

$$v(\xi)=-\frac{2}{3}sc^2(\xi)\ dn^2(\xi)+\frac{2}{3}(2m^2-1)-1. \tag{3.59}$$

Family 3. If $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, $e_2 = -1$, then we get

$$u(\xi) = \mp \frac{2}{\sqrt{3}} m^2 sn(\xi)\ cd(\xi) + V, \tag{3.60}$$

and

$$v(\xi) = -\frac{2}{3} m^4 sn^2(\xi) cd^2(\xi) + \frac{2}{3}(2 - m^2) - 1. \tag{3.61}$$

Family 4. If $e_0 = m^2$, $e_1 = -(m^2 + 1)$, $e_2 = 1$, then we get

$$u(\xi) = \mp \frac{2}{\sqrt{3}} ds(\xi) cn(\xi) + V, \tag{3.62}$$

and

$$v(\xi) = -\frac{2}{3} ds^2(\xi) cn^2(\xi) - \frac{2}{3}(m^2 + 1) - 1, \tag{3.63}$$

or

$$u(\xi) = \pm \frac{2}{\sqrt{3}} (1 - m^2) nc(\xi) sd(\xi) + V, \tag{3.64}$$

and

$$v(\xi) = -\frac{2}{3} nc^2(\xi) sd^2(\xi) - \frac{2}{3}(m^2 + 1) - 1. \tag{3.65}$$

Similarly, we can write down the other families of exact solutions of Eqs. (3.48) which are omitted for convenience.

Remark 3.1. Some of these solutions presented in this latter have been checked with Maple by putting them back into the original equations.

Remark 3.2. The generalized (G'/G)-expansion method is simple but its results are very cumbersome. The results of this method contain many arbitrary constants compare to the results of other method. The performance of generalized (G'/G)-expansion method is reliable, simple, direct, concise and gives more new exact solutions compared to the other method. This method allowed us to solve more complicated PDEs in the mathematical physics.

Appendix A

The general solutions to the Jacobi elliptic equation (2.5) and their derivatives (see for example [8, 9, 15]) are listed as follows:

e_0	e_1	e_2	$G(\xi)$	$G'(\xi)$
1	$-(1+m^2)$	m^2	$sn(\xi)$ or $cd(\xi)$	$cn(\xi)\,dn(\xi)$ $-(1-m^2)sd(\xi)nd(\xi)$
$1-m^2$	$2m^2-1$	$-m^2$	$cn(\xi)$	$-sn(\xi)dn(\xi)$
m^2-1	$2-m^2$	-1	$dn(\xi)$	$-m^2sn(\xi)cn(\xi)$
m^2	$-(m^2+1)$	1	$ns(\xi)$ or $dc(\xi)$	$-ds(\xi)cs(\xi)$ $(1-m^2)nc(\xi)sc(\xi)$
$-m^2$	$2m^2-1$	$1-m^2$	$nc(\xi)$	$sc(\xi)dc(\xi)$
-1	$2-m^2$	m^2-1	$nd(\xi)$	$m^2sd(\xi)cd(\xi)$
$1-m^2$	$2-m^2$	1	$cs(\xi)$	$-ns(\xi)ds(\xi)$
1	$2-m^2$	$1-m^2$	$sc(\xi)$	$nc(\xi)dc(\xi)$
1	$2m^2-1$	$m^2(m^2-1)$	$sd(\xi)$	$nd(\xi)cd(\xi)$
$m^2(m^2-1)$	$2m^2-1$	1	$ds(\xi)$	$-cs(\xi)ns(\xi)$
$\frac{1}{4}$	$\frac{1}{2}(1-2m^2)$	$\frac{1}{4}$	$ns(\xi)\pm cs(\xi)$	$-ds(\xi)cs(\xi)\mp ns(\xi)ds(\xi)$
$\frac{1}{4}(1-m^2)$	$\frac{1}{2}(1+m^2)$	$\frac{1}{4}(1-m^2)$	$nc(\xi)\pm sc(\xi)$	$sc(\xi)dc(\xi)\pm nc(\xi)dc(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{1}{4}$	$ns(\xi)\pm ds(\xi)$	$-ds(\xi)cs(\xi)\mp cs(\xi)ns(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{m^2}{4}$	$sn(\xi)\pm icn(\xi)$	$cn(\xi)dn(\xi)\mp i\,sn(\xi)dn(\xi)$

where $0<m<1$ is the modulus of the Jacobi elliptic functions and $i=\sqrt{-1}$.

Appendix B

The Jacobi elliptic functions $sn(\xi),cn(\xi),dn(\xi),ns(\xi),cs(\xi),ds(\xi),sc(\xi),sd(\xi)$ generate into hyperbolic functions when $m\to 1$ as follows:

$sn(\xi)\longrightarrow \tanh(\xi)$,	$cn(\xi)\longrightarrow \text{sech}(\xi)$,	$dn(\xi)\longrightarrow \text{sech}(\xi)$,	$ns(\xi)\longrightarrow \coth(\xi)$,
$cs(\xi)\longrightarrow \text{cosech}(\xi)$,	$ds\xi)\longrightarrow \text{cosech}(\xi)$,	$sc(\xi)\longrightarrow \sinh(\xi)$,	$sd(\xi)\longrightarrow \sinh(\xi)$,

and into trigonometric functions when $m\to 0$ as follows:

$sn(\xi)\longrightarrow \sin(\xi)$,	$cn(\xi)\longrightarrow \cos(\xi)$,	$dn(\xi)\longrightarrow 1$,	$ns(\xi)\longrightarrow \text{cosec}(\xi)$,
$cs(\xi)\longrightarrow \cot(\xi)$,	$ds\xi)\longrightarrow \text{cosec}(\xi)$,	$sc(\xi)\longrightarrow \tan(\xi)$,	$sd(\xi)\longrightarrow \sin(\xi)$.

Appendix C

$cd(\xi)=\frac{cn(\xi)}{dn(\xi)}$,	$dc(\xi)=\frac{dn(\xi)}{cn(\xi)}$,	$nc(\xi)=\frac{1}{cn(\xi)}$,	$nd(\xi)=\frac{1}{dn(\xi)}$,
$cs(\xi)=\frac{cn(\xi)}{sn(\xi)}$,	$\text{sc}(\xi)=\frac{sn(\xi)}{cn(\xi)}$,	$sd(\xi)=\frac{sn(\xi)}{dn(\xi)}$,	$ds(\xi)=\frac{dn(\xi)}{sn(\xi)}$.

4 Conclusions

The main idea of the generalized $(G^{'}/G)$-expansion method is that the traveling wave solutions of nonlinear partial differential equations can be expressed as a polynomial in $(G^{'}/G)$, where $G(\xi)$ satisfies the Jacobi elliptic equation (2.5) to some nonlinear PDEs in mathematical physics via the modified Kawahara equation, the nonlinear coupled KdV equations and the classical Boussinesq system. We have obtained families of exact solutions of these equations in terms of Jacobi elliptic functions. Finally, we conclude according to the Appendix B that our results in terms of Jacobi elliptic functions generate into hyperbolic functions when $m\rightarrow 1$ and generate into trigonometric functions when $m\rightarrow 0$.

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