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Global Weak Solutions for the Weakly Dissipative Camassa-Holm Equation

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Abstract. In this paper we prove the existence and uniqueness of global weak solutions to the weakly dissipative Camassa-Holm equation.

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1 Introduction

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad t > 0, \ \ x \in \mathbb{R},$$

is a model for wave motion on shallow water, where $u(t,x)$ represents the fluid's free surface above a flat bottom (or equivalently, the fluid velocity at time $t \geq 0$ in the spatial x direction).

Since the equation was derived physically by Camassa and Holm [1, 2], many researchers have paid extensive attention to it. The equation has a bi-Hamiltonian structure [3] and is completely integrable [2, 4–6]. It is a re-expression of geodesic flow on the diffeomorphism group of the circle [7] and geodesic exponential maps of the Virasoro group [8]. Its solitary waves are peaked [5, 9], and they are orbitally stable and interact like solitons [9–11]. These peaked waves are analogous to the exact traveling wave solutions of the governing equations for water waves representing waves of great height- see the recent discussions in [12, 13].

The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that this equation is locally well-posed [14–19] for initial data $u_0 \in$

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$H^s(\mathbb{R})$ with $s>3/2$. More interestingly, it has not only global strong solutions modelling permanent waves [14, 18–21] and but also blow-up solutions modelling wave breaking [14–21]. On the other hand, it has global weak solutions with initial data $u_0 \in H^1(\mathbb{R})$, cf. [22–24]. Morover, the initial boundary value problem for the Camassa-Holm equation on the half line and on a finite interval were studied recently in [25, 26]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [2, 14, 15].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. Ott and Sudan [27] investigated how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of the KdV equation, and Ghidaglia [28] investigated the long time behavior of solutions to the weakly dissipative KdV equation as a finite dimensional dynamical system.

Similarly, we would like to consider the dissipative Camassa-Holm equation:

$$u_t - u_{txx} + 3uu_x + L(u) = 2u_x u_{xx} + uu_{xxx}, \quad t>0, x\in\mathbb{R},$$

where $L(u)$ is a dissipative term, L can be a differential operator or a quasi-differential operator according to different physical situations. We are interested in the effect of the weakly dissipative term on the Camassa-Holm equation. In the paper, we would like to consider the Cauchy problem of the weakly dissipative Camassa-Holm equation:

$$\begin{cases} u_t - u_{txx} + 3uu_x + \lambda(u-u_{xx}) = 2u_x u_{xx} + uu_{xxx}, \quad t>0, x\in\mathbb{R}, \\ u(0,x) = u_0(x), \quad x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $L(u) = \lambda(I-\partial_x^2)u$ is the weakly dissipative term and $\lambda>0$ is a constant.

The local well-posedness, global existence and blow-up phenomena of the Cauchy problem of Eq. (1.1) on the line [29] and on the circle [30] were studied recently. We found that the behaviors of Eq. (1.1) are similar to the Camassa-Holm equation in a finite interval of time, such as, the local well-posedness and the blow-up phenomena, and that there are considerable differences between Eq. (1.1) and the Camassa-Holm equation in their long time behaviors. The global solutions of Eq. (1.1) decay to zero as time goes to infinite. This long time behavior is an important feature that the Camassa-Holm equation does not possess.

Eq. (1.1) has the same blow-up rate as the Camassa-Holm equation does when the blow-up occurs, cf. [29, 30]. This fact shows that the blow-up rate of the Camassa-Holm equation is not affected by the weakly dissipative term. But the occurrence of blow-up of Eq. (1.1) is affected by the dissipative parameter, cf. [29, 30].

With $y = u - u_{xx}$, Eq. (1.1) takes the form:

$$\begin{cases} y_t + uy_x + 2u_x y + \lambda y = 0, \quad t>0, x\in\mathbb{R}, \\ y(0,x) = u_0 - u_{0,xx}, \quad x\in\mathbb{R}. \end{cases} \tag{1.2}$$

Note that if $p(x)=\frac{1}{2}e^{-|x|},x\in\mathbb{R}$, then $(1-\partial_x^2)^{-1}f=p*f$ for all $f\in L^2(\mathbb{R})$ and $p*y=u$. Using this identity, we can rewrite (1.1) as follows:

$$\begin{cases} u_t+uu_x+\partial_x\left(p*(u^2+\frac{1}{2}u_x^2)\right)+\lambda u=0, & t>0,x\in\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.3}$$

In this paper we will prove the existence and uniqueness of global weak solutions for Eq. (1.1). We have the following theorem:

Theorem 1.1. *If $u_0\in H^1(\mathbb{R})$ is such that $y_0=u_0-u_{0,xx}\in\mathcal{M}(\mathbb{R})$, and assume that there is a $x_0\in\mathbb{R}$ such that* supp $y_0^-\subset(-\infty,x_0)$ *and* supp $y_0^+\subset(x_0,\infty)$*. Then equation* (1.3) *(or* (1.1)*) has a unique solution $u\in C^1(\mathbb{R}^+;L^2(\mathbb{R}))\cap C(\mathbb{R}^+;H^1(\mathbb{R}))$ with initial data $u(0)=u_0$ and such that*

$$y(t,\cdot)=u(t,\cdot)-u_{xx}(t,\cdot)\in L^\infty_{\rm loc}(\mathbb{R}^+;\mathcal{M}(\mathbb{R})).$$

We use the method of the proof in [23] for the Camassa-Holm equation. It relies on the approximation of the initial data $u_0\in H^1(\mathbb{R})$ by smooth functions producing a sequence of global solutions of (1.1) in $H^s(\mathbb{R}),s>\frac{3}{2}$. Suitable a priori estimates enable us to extract a subsequence of these solutions that converges weakly in $H^1(\mathbb{R})$. We use Helly's theorem [31] to pass to the limit in the nonlocal nonlinear term from (1.1).

In [23] it is prove that if $u_0\in H^1(\mathbb{R})$ is such that

$$y_0=u_0-u_{0,xx}\in\mathcal{M}^+(\mathbb{R}),$$

then there exists a global weak solution $u\in C^1(\mathbb{R}^+;L^2(\mathbb{R}))\cap C(\mathbb{R}^+;H^1(\mathbb{R}))$ of the Camassa-Holm equation, but in this paper we take a more general hypothesis of initial data u_0 to prove that there exists a global weak solution $u\in C^1(\mathbb{R}^+;L^2(\mathbb{R}))\cap C(\mathbb{R}^+;H^1(\mathbb{R}))$ of Eq. (1.1).

In addition,

$$E(u)=\int_{\mathbb{R}}(u^2+u_x^2)\mathrm{d}x$$

is a conservation law for the weak solutions $u(t,x)$ of the Camassa-Holm equation, which play an important role in the proof of the regularity $u\in C(\mathbb{R}^+;H^1(\mathbb{R}))$ of the weak solutions in time, cf. [23]. But $E(u)$ is no longer conservation law for the weak solutions $u(t,x)$ of Eq. (1.1), which brings about some difficulties in the proof of the regularity of the weak solutions in time.

Notation. $\mathcal{M}(\mathbb{R})$ is the space of Radon measures on $\mathbb{R}$ with bounded total variation, the norm in $\mathcal{M}(\mathbb{R})$ is written by $\|\cdot\|_{\mathcal{M}}$. For $1\le p\le\infty$, the norm in the Lebesgue space $L^p(\mathbb{R})$ will be written by $\|\cdot\|_{L^p}$, while $\|\cdot\|_{H^s},s>0$ will stand for the norm in the classical Sobolev spaces $H^s(\mathbb{R})$. Throughout this paper, we denote by $*$ the convolution.

2 Proof of the Theorem 1.1

In the section, we prove the Theorem 1.1. Let us first present some lemmas that will be of use in our approach.

Lemma 2.1. ([29]) *Assume* $u_0\in H^s(\mathbb{R}),s>3/2$ *is such that the associated potential* $y_0=u_0-u_{0,xx}$ *satisfies* $y_0(x)\le 0$ *on* $(-\infty,x_0]$ *and* $y_0(x)\ge 0$ *on* $[x_0,\infty)$ *for some point* $x_0\in\mathbb{R}$, *then the initial value problem* (1.1) *has a unique global solution* $u\in C(\mathbb{R}^+;H^s(\mathbb{R}))\cap C^1(\mathbb{R}^+;H^{s-1}(\mathbb{R}))$. *Moreover, we have*

$$\|u(t)\|^2_{H^1}=e^{-2\lambda t}\|u_0\|^2_{H^1},\quad t\in[0,\infty), \tag{2.1}$$

and

$$u_x(t,x)\ge-\frac{1}{\sqrt{2}}\|u_0\|_{H^1},\quad (t,x)\in[0,\infty)\times\mathbb{R}. \tag{2.2}$$

Consider the following differential equation associated with the solution u to Eq. (2.1)

$$\begin{cases} q_t=u(t,q), & t>0,\ x\in\mathbb{R},\\ q(0,x)=x, & x\in\mathbb{R}. \end{cases} \tag{2.3}$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following two results on $q(t,x)$.

Lemma 2.2. ([29]) *Let* $u_0\in H^s(\mathbb{R}),s\ge 3$, *and let* $T>0$ *be the maximal existence time of the corresponding solution* $u(t,x)$ *to Eq.* (1.1). *Then Eq.* (2.3) *has a unique solution* $q\in C^1([0,T)\times\mathbb{R},\mathbb{R})$. *Moreover, the map* $q(t,\cdot)$ *in an increasing diffeomorphism of* $\mathbb{R}$ *with* $q_x(t,x)>0$ *for all* $(t,x)\in[0,T)\times\mathbb{R}$, *and we have*

$$y(t,q(t,x))q_x^2(t,x)=y_0(x)e^{-\lambda t},\quad \forall(t,x)\in[0,T)\times\mathbb{R}.$$

Lemma 2.3. *Assume* $u_0\in H^s(\mathbb{R}),s\ge 3$, *and there exists* $x_0\in\mathbb{R}$ *such that* $y_0(x)\le 0$ *on* $(-\infty,x_0]$ *and* $y_0(x)\ge 0$ *on* $[x_0,\infty)$. *Let* u *be the corresponding solution of Eq.* (1.1) *with initial data* u_0, *then*

$$\begin{aligned}&\|u_x(t,\cdot)\|_{L^\infty}\le\tfrac{1}{2}\|y(t,\cdot)\|_{L^1},\\ &\|u(t,\cdot)\|_{L^1}\le\|y(t,\cdot)\|_{L^1},\\ &\|u_x(t,\cdot)\|_{L^1}\le\|y(t,\cdot)\|_{L^1}.\end{aligned}$$

Moreover, if $y_0\in L^1(\mathbb{R})$, *then we have*

$$\|y(t,\cdot)\|_{L^1}\le e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{L^1}.$$

Proof. Since $y(t,x)=u(t,x)-u_{xx}(t,x)$, it follows that $u=p*y$ and $u_x=p_x*y$. Note that $\|p_x\|_{L^\infty}=1/2,\|p\|_{L^1}=1,\|p_x\|_{L^1}=1$. Applying Young's inequality, one can easily obtain the three inequalities ahead.

We infer from the assumption and Lemma 2.2 that for $t\in[0,\infty)$

$$\begin{cases} y(t,x)\le 0, \ x\le q(t,x_0), \\ y(t,x)\ge 0, \ x\ge q(t,x_0), \end{cases}$$

and $y(t,q(t,x_0))=0,t\in[0,\infty)$. Using Eq. (1.2) and (2.2) in Lemma 2.1, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} y^+\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{q(t,x_0)}^{\infty} y\mathrm{d}x = -\int_{q(t,x_0)}^{\infty} uy_x\mathrm{d}x - 2\int_{q(t,x_0)}^{\infty} u_x y\mathrm{d}x - \lambda\int_{q(t,x_0)}^{\infty} y\mathrm{d}x \\ &= -\int_{q(t,x_0)}^{\infty} u_x y\mathrm{d}x - \lambda\int_{q(t,x_0)}^{\infty} y\mathrm{d}x \le \sup_{x\in\mathbb{R}}\{-u_x\}\int_{\mathbb{R}} y^+\mathrm{d}x - \lambda\int_{\mathbb{R}} y^+\mathrm{d}x \\ &\le \left(\frac{1}{\sqrt{2}}\|u_0\|_{H^1} - \lambda\right)\int_{\mathbb{R}} y^+\mathrm{d}x. \end{aligned}$$

By Gronwall's inequality, we obtain

$$\int_{\mathbb{R}} y^+(t,x)\mathrm{d}x \le e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\int_{\mathbb{R}} y_0^+\mathrm{d}x.$$

Repeating the above proof, one can obtain a same estimate for $y^-(t,x)$. This complete the proof of the lemma. ☐

Next, we recall a partial integration result for Bochner spaces (below $\langle\cdot,\cdot\rangle$ is the $H^{-1}(\mathbb{R}),H^1(\mathbb{R})$ duality bracket).

Lemma 2.4. ([32]) *Let $T>0$. If*

$$f,g\in L^2((0,T);H^1(\mathbb{R})) \text{ and } \frac{\mathrm{d}f}{\mathrm{d}t},\frac{\mathrm{d}g}{\mathrm{d}t}\in L^2((0,T);H^{-1}(\mathbb{R})),$$

then f,g are a.e. equal to a function continuous from $[0,T]$ into $L^2(\mathbb{R})$ and

$$\langle f(t),g(t)\rangle - \langle f(s),g(s)\rangle = \int_s^t \left\langle \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau},g(\tau)\right\rangle \mathrm{d}\tau + \int_s^t \left\langle \frac{\mathrm{d}g(\tau)}{\mathrm{d}\tau},f(\tau)\right\rangle \mathrm{d}\tau$$

for all $s,t\in[0,T]$.

We now prove the existence of the weak solution of Eq. (1.1).

2.1 Proof of existence

We denote by $\{\rho_n\}_{n\ge 1}$ the mollifiers

$$\rho_n(x) := \left(\int_{\mathbb{R}}\rho(\xi)\mathrm{d}\xi\right)^{-1} n\rho(nx), \qquad x\in\mathbb{R}, \ n\ge 1,$$

where $\rho\in C_c^\infty(\mathbb{R})$ is defined by

$$\rho(x):=\begin{cases} e^{1/(x^2-1)}, & \text{for } |x|<1,\\ 0, & \text{for } |x|\geq 1.\end{cases}$$

Set

$$y_0^n=l\left(-\frac{1}{n}\right)(\rho_n*y_0^+)-l\left(\frac{1}{n}\right)(\rho_n*y_0^-),$$

where $l(r)$ denotes right translation by $r\in\mathbb{R}$, $l(r)f(x)=f(x+r)$. By the definition of ρ_n and the assumptions of the theorem, we have

$$\operatorname{supp}(\rho_n*y_0^-)\subset\left(-\infty,x_0+\frac{1}{n}\right] \text{ and } \operatorname{supp}(\rho_n*y_0^+)\subset\left[x_0-\frac{1}{n},\infty\right).$$

It follows that

$$\begin{cases} y_0^n\leq 0, & \text{if } x\leq x_0,\\ y_0^n\geq 0, & \text{if } x\geq x_0.\end{cases}$$

Let us define $u_0^n:=(1-\partial_x^2)^{-1}y_0^n=p*y_0^n\in H^\infty(\mathbb{R})$ for $n\geq 1$. By Lemma 2.1, we obtain a global strong solution of Eq. (1.1) with initial data u_0^n

$$u^n\in C(\mathbb{R}^+;H^s(\mathbb{R}))\cap C^1(\mathbb{R}^+;H^{s-1}(\mathbb{R})),$$

for every $s>3/2$. Note that

$$p*(y_0^{n\pm})\in H^1(\mathbb{R}) \qquad \text{and} \qquad \left\|l\left(\mp\frac{1}{n}\right)\rho_n\right\|_{L^1}=1.$$

Since $\operatorname{supp}(l(\mp 1/n)\rho_n)\to 0$, as $n\to\infty$, it follows that

$$\begin{aligned} u_0^n&=p*\left[l\left(-\frac{1}{n}\right)(\rho_n*y_0^+)-l\left(\frac{1}{n}\right)(\rho_n*y_0^-)\right]\\ &=l\left(-\frac{1}{n}\right)(\rho_n*(p*y_0^+))-l\left(\frac{1}{n}\right)(\rho_n*(p*y_0^-))\\ &\to p*y_0^+-p*y_0^-=u_0, \qquad \text{in } H^1(\mathbb{R}),\ \text{as } n\to\infty.\end{aligned}$$

Note that (see (2.2) in [23, pp.48])

$$\|\rho_n*y_0\|_{L^1}\leq\|y_0\|_{\mathcal{M}}. \tag{2.4}$$

We use Young's inequality to get for $n\geq 1$

$$\begin{aligned} \|u_0^n\|_{L^2}&=\left\|p*\left[l\left(-\frac{1}{n}\right)(\rho_n*y_0^+)-l\left(\frac{1}{n}\right)(\rho_n*y_0^-)\right]\right\|_{L^2}\\ &\leq\|p\|_{L^2}\left\|l\left(-\frac{1}{n}\right)(\rho_n*y_0^+)-l\left(\frac{1}{n}\right)(\rho_n*y_0^-)\right\|_{L^1}\\ &\leq\|p\|_{L^2}(\|\rho_n*y_0^+\|_{L^1}+\|\rho_n*y_0^-\|_{L^1})\\ &\leq\|p\|_{L^2}(\|y_0^+\|_{\mathcal{M}}+\|y_0^-\|_{\mathcal{M}})=\|p\|_{L^2}\|y_0\|_{\mathcal{M}},\end{aligned}$$

and

$$\begin{aligned}\|u^n_{0,x}\|_{L^2} &= \left\| p_x * \left[l\left(-\frac{1}{n}\right)(\rho_n * y_0^+) - l\left(\frac{1}{n}\right)(\rho_n * y_0^-)\right]\right\|_{L^2} \\ &\le \|p_x\|_{L^2} \left\| l\left(-\frac{1}{n}\right)(\rho_n * y_0^+) - l\left(\frac{1}{n}\right)(\rho_n * y_0^-)\right\|_{L^1} \\ &\le \|p_x\|_{L^2}\|y_0\|_{\mathcal{M}}.\end{aligned}$$

Then

$$\|u_0^n\|_{H^1} \le \|p\|_{H^1}\|y_0\|_{\mathcal{M}}. \tag{2.5}$$

Further, we use Young's inequality, (2.1) and (2.5) to obtain that

$$\begin{aligned}&\left\| \partial_x \left(p * \left([u^n(t)]^2 + \frac{1}{2}[u^n_x(t)]^2 \right)\right)\right\|_{L^2} \le \|p_x\|_{L^2} \|[u^n(t)]^2 + \frac{1}{2}[u^n_x(t)]^2\|_{L^1} \\ &\le \|p\|_{H^1}\|u^n(t)\|^2_{H^1} \le \|p\|_{H^1}\|u^n_0\|^2_{H^1} \le \|p\|^3_{H^1}\|y_0\|^2_{\mathcal{M}},\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\|u^n(t)u^n_x(t)\|_{L^2} &\le \|u^n(t)\|_{L^\infty}\|u^n_x(t)\|_{L^2} \le \|u^n(t)\|^2_{H^1} \\ &\le \|u^n_0\|^2_{H^1} \le \|p\|^2_{H^1}\|y_0\|^2_{\mathcal{M}}.\end{aligned} \tag{2.7}$$

From (2.6), (2.7) and (1.3), we get

$$\|u^n_t(t,\cdot)\|_{L^2} \le \|p\|^2_{H^1}\|y_0\|^2_{\mathcal{M}} + \|p\|^3_{H^1}\|y_0\|^2_{\mathcal{M}} + \lambda\|p\|_{H^1}\|y_0\|_{\mathcal{M}}, \tag{2.8}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} ([u^n(t,x)]^2 + [u^n_x(t,x)]^2 + [u^n_t(t,x)]^2)\mathrm{d}x \\ &\le \|p\|^2_{H^1}\|y_0\|^2_{\mathcal{M}} + \left(\|p\|^2_{H^1}\|y_0\|^2_{\mathcal{M}} + \|p\|^3_{H^1}\|y_0\|^2_{\mathcal{M}} + \lambda\|p\|_{H^1}\|y_0\|_{\mathcal{M}}\right)^2.\end{aligned} \tag{2.9}$$

For fixed $T>0$, the inequality (2.9) shows that the sequence $\{u^n\}_{n\ge 1}$ is uniformly bounded in the space $H^1((0,T)\times\mathbb{R})$. Therefore, it has a subsequence such that

$$u^{n_k} \rightharpoonup u \text{ weakly in } H^1((0,T)\times\mathbb{R}), \quad n_k\to\infty,$$

and

$$u^{n_k} \to u, \text{ a.e. on } (0,T)\times\mathbb{R}, \quad n_k\to\infty, \tag{2.10}$$

for some $u\in H^1((0,T)\times\mathbb{R})$. Note that for fixed $t\in(0,T)$, we have by Lemma 2.3 and (2.4) that the sequence $u^{n_k}_x(t,\cdot)\in BV(\mathbb{R})$ with

$$\begin{aligned}\mathbb{V}[u^{n_k}_x(t,\cdot)] &= \|u^{n_k}_{xx}(t,\cdot)\|_{L^1} \le \|u^{n_k}(t,\cdot)\|_{L^1} + \|y^{n_k}(t,\cdot)\|_{L^1} \\ &\le 2\|y^{n_k}(t,\cdot)\|_{L^1} \le 2e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0^{n_k}\|_{L^1} \\ &\le 2e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}},\end{aligned}$$

and

$$\|u_x^{n_k}(t,\cdot)\|_{L^\infty} \le \frac{1}{2}\|y^{n_k}(t,\cdot)\|_{L^1} \le \frac{1}{2}e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}}.$$

Here $BV(\mathbb{R})$ is the space of functions with bounded variation and $\mathbb{V}(f)$ is the total variation of $f\in BV(\mathbb{R})$, cf. [31]. By Helly's theorem (see [31]), there exists a subsequence, denoted again $\{u_x^{n_k}(t,\cdot)\}$, which converges at every point to some function $v(t,\cdot)$ of finite variation with

$$\mathbb{V}(v(t,\cdot)) \le 2e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}}.$$

The limit (2.10) implies that $u_x^{n_k}(t,\cdot)\to u_x(t,\cdot)$ in $D'(\mathbb{R})$ for almost all $t\in(0,T)$. This enables us to identify $v(t,\cdot)$ with $u_x(t,\cdot)$ for a.e. $t\in(0,T)$. Therefor

$$u_x^{n_k}\to u_x,\ \text{a.e. on } (0,T)\times\mathbb{R}, \quad n_k\to\infty, \tag{2.11}$$

and for a.e. $t\in(0,T)$,

$$\mathbb{V}[u_x(t,\cdot)] = \|u_{xx}(t,\cdot)\|_{\mathcal{M}} \le 2e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}}. \tag{2.12}$$

Let us again fix $t\in(0,T)$. From Lemma 2.1, Lemma 2.3, (2.4) and (2.5) we have that

$$\begin{aligned}\|[u^n(t)]^2\|_{L^2} &\le \|u^n(t)\|_{L^\infty}\|u^n(t)\|_{L^2} \le \|u^n(t)\|_{H^1}^2 \le \|u_0^n\|_{H^1}^2\\ &\le \|p\|_{H^1}^2\|y_0\|_{\mathcal{M}}^2,\end{aligned}$$

and

$$\begin{aligned}&\|[u_x^n(t)]^2\|_{L^2} \le \|u_x^n(t)\|_{L^\infty}\|u_x^n(t)\|_{L^2}\\ \le&\frac{1}{2}e^{(\frac{1}{\sqrt{2}}\|u_0^n\|_{H^1}-\lambda)t}\|y_0^n\|_{L^1}\|u_0^n\|_{H^1} \le \frac{1}{2}e^{(\frac{1}{\sqrt{2}}\|p\|_{H^1}\|y_0\|_{\mathcal{M}}-\lambda)t}\|p\|_{H^1}\|y_0\|_{\mathcal{M}}^2.\end{aligned}$$

The above two inequalities show that the sequence $\{[u^n(t,\cdot)]^2+[u_x^n(t,\cdot)]^2/2\}_{n\ge1}$ is uniformly bounded in $L^2(\mathbb{R})$ for fixed $t\in(0,T)$. Therefor, it has a subsequence, denoted again $\{[u^{n_k}(t,\cdot)]^2+[u_x^{n_k}(t,\cdot)]^2/2\}$, converging weakly in $L^2(\mathbb{R})$. For a.e. $t\in(0,T)$, we deduce from (2.10)–(2.11) that the weak $L^2(\mathbb{R})$-limit is $[u(t,\cdot)]^2+\frac{1}{2}[u_x(t,\cdot)]^2$. As $p_x\in L^2(\mathbb{R})$, a.e. on $(0,T)\times\mathbb{R}$ we have

$$\partial_x p*\left[(u^{n_k})^2+\frac{1}{2}(u_x^{n_k})^2\right]\to\partial_x p*\left(u^2+\frac{1}{2}u_x^2\right), \qquad n_k\to\infty. \tag{2.13}$$

From the relations (2.10), (2.11) and (2.13) we obtain that u satisfies Eq. (1.3) in $D'((0,T)\times\mathbb{R})$.

From (2.8) we know that the sequence $u_t^{n_k}(t,\cdot)$ is uniformly bounded in $L^2(\mathbb{R})$ as $t\in\mathbb{R}^+$. We also infer from (2.1) and (2.5) that $\|u^{n_k}(t,\cdot)\|_{H^1}$ is uniformly bounded for all $t\in\mathbb{R}^+$ and all $n_k's$. Hence the family $t\mapsto u^{n_k}(t,\cdot)\in H^1(\mathbb{R})$ is weakly equicontinuous on $[0,T]$ for any $T>0$. It follows from the Arzela-Ascoli theorem that $\{u^{n_k}\}$ contains a subsequence, which we denote again by $\{u^{n_k}\}$, which converges u weakly in $H^1(\mathbb{R})$, uniformly in t, and u is weakly continuous from $\mathbb{R}^+$ into $H^1(\mathbb{R})$. Then, we have by (2.1) and (2.5)

$$\begin{aligned}\|u(t,\cdot)\|_{H^1} &\leq \liminf_{n_k\to\infty}\|u^{n_k}(t,\cdot)\|_{H^1} \leq \liminf_{n_k\to\infty}\|u_0^{n_k}\|_{H^1}\\ &\leq \|p\|_{H^1}\|y_0\|_{\mathcal{M}}, \qquad \text{for a.e. } t\in\mathbb{R}^+.\end{aligned}$$

This implies that $u\in L^\infty(\mathbb{R}^+\times\mathbb{R})$.

By Lemma 2.3 and (2.4), for all $t\in\mathbb{R}^+$, we have

$$\|u_x^n(t,\cdot)\|_{L^\infty} \leq \frac{1}{2}\|y^n(t,\cdot)\|_{L^1} \leq \frac{1}{2}e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}}.$$

Combining this with (2.11), we deduce that $u_x\in L^\infty_{\rm loc}(\mathbb{R}^+\times\mathbb{R})$.

In order to prove that $u\in C(\mathbb{R}^+;H^1(\mathbb{R}))$, it is enough to show that the equality (2.1) is still valid for u here. Indeed, as $u\in C_w(\mathbb{R}^+,H^1(\mathbb{R}))$, according to (2.1) we have

$$\begin{aligned}\|u(t)-u(s)\|^2_{H^1} &= \|u(t)\|^2_{H^1} - 2(u(t),u(s))_{H^1} + \|u(t)\|^2_{H^1}\\ &= e^{-2\lambda t}\|u_0\|^2_{H^1} - 2(u(t),u(s))_{H^1} + e^{-2\lambda s}\|u_0\|^2_{H^1}\\ &\to 0, \qquad \text{as } s\to t.\end{aligned}$$

Now, we prove that (2.1) is still valid for weak solution u here.

As u solves (1.3) in distribution sense, we see that for a.e. $t\in\mathbb{R}^+$,

$$\rho_n*u_t+\rho_n*(uu_x)+\rho_n*p_x*\left(u^2+\frac{1}{2}u_x^2\right)+\lambda\rho_n*u=0, \qquad n\geq 1. \tag{2.14}$$

Multiplying with ρ_n*u, we obtain by integration and in view of Lemma 2.4 that for a.e. $t\in\mathbb{R}^+$ and all $n\geq 1$,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\rho_n*u)^2\mathrm{d}x + \int_{\mathbb{R}}(\rho_n*u)[\rho_n*(uu_x)]\mathrm{d}x\\ &\quad + \int_{\mathbb{R}}(\rho_n*u)\left[\rho_n*p_x*\left(u^2+\frac{1}{2}u_x^2\right)\right]\mathrm{d}x + \lambda\int_{\mathbb{R}}(\rho_n*u)^2\mathrm{d}x = 0.\end{aligned} \tag{2.15}$$

By differentiation of (2.14) we obtain a relation which multiplied by ρ_n*u_x yields after integration and in view of Lemma 2.4 that for a.e. $t\in\mathbb{R}$ and all $n\geq 1$,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\rho_n*u_x)^2\mathrm{d}x + \int_{\mathbb{R}}(\rho_n*u_x)[\rho_{n,x}*(uu_x)]\mathrm{d}x\\ &\quad + \int_{\mathbb{R}}(\rho_n*u_x)\left[\partial_x^2\rho_n*p*\left(u^2+\frac{1}{2}u_x^2\right)\right]\mathrm{d}x + \lambda\int_{\mathbb{R}}(\rho_n*u_x)^2\mathrm{d}x = 0.\end{aligned} \tag{2.16}$$

Since $u^2(t,\cdot)+u_x^2(t,\cdot)/2\in L^2(\mathbb{R})$, we have

$$\partial_x^2\rho_n*p*\left(u^2+\frac{1}{2}u_x^2\right) = \rho_n*p*\left(u^2+\frac{1}{2}u_x^2\right) - \rho_n*\left(u^2+\frac{1}{2}u_x^2\right).$$

Then we can rewrite relation (2.16) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\rho_n*u_x)^2\mathrm{d}x+\int_{\mathbb{R}}(\rho_n*u_x)[\rho_{n,x}*(uu_x)]\mathrm{d}x+\int_{\mathbb{R}}(\rho_n*u_x)\left[\rho_n*p*\left(u^2+\frac{1}{2}u_x^2\right)\right]\mathrm{d}x$$
$$-\int_{\mathbb{R}}(\rho_n*u_x)\left[\rho_n*\left(u^2+\frac{1}{2}u_x^2\right)\right]\mathrm{d}x+\lambda\int_{\mathbb{R}}(\rho_n*u_x)^2\mathrm{d}x=0. \tag{2.17}$$

Adding (2.15) and (2.17), integration by parts yields that for a.e. $t\in\mathbb{R}^+$ and all $n\geq 1$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[(\rho_n*u)^2+(\rho_n*u_x)^2\right]\mathrm{d}x-\frac{3}{2}\int_{\mathbb{R}}(\rho_n*u_x)(\rho_n*u^2)\mathrm{d}x+\int_{\mathbb{R}}(\rho_n*u_x)\left[\rho_{n,x}*(uu_x)\right]\mathrm{d}x$$
$$-\frac{1}{2}\int_{\mathbb{R}}(\rho_n*u_x)(\rho_n*u_x^2)\mathrm{d}x+\lambda\int_{\mathbb{R}}\left[(\rho_n*u)^2+(\rho_n*u_x)^2\right]\mathrm{d}x=0. \tag{2.18}$$

Let us now denote

$$E_n(t):=\int_{\mathbb{R}}\left[(\rho_n*u)^2+(\rho_n*u_x)^2\right]\mathrm{d}x,\qquad t\in\mathbb{R}^+,n\geq 1,$$

and define

$$G_n(t):=3\int_{\mathbb{R}}(\rho_n*u_x)(\rho_n*u^2)\mathrm{d}x-2\int_{\mathbb{R}}(\rho_n*u_x)\left[\rho_{n,x}*(uu_x)\right]\mathrm{d}x$$
$$+\int_{\mathbb{R}}(\rho_n*u_x)(\rho_n*u_x^2)\mathrm{d}x,\qquad t\in\mathbb{R}^+,\ \ n\geq 1.$$

Then we obtain by (2.18) that for a.e. $t\in\mathbb{R}^+$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_n(t)+2\lambda E_n(t)=G_n(t),\qquad n\geq 1. \tag{2.19}$$

From Lemma 2.4 applied to both ρ_n*u and ρ_n*u_x, we have by (2.19) that

$$E_n(t)-e^{-2\lambda t}E_n(0)=\int_0^t e^{-2\lambda(t-s)}G_n(s)\mathrm{d}s,\qquad t\in\mathbb{R}^+,\ \ n\geq 1. \tag{2.20}$$

It can be shown (see [23, pp.53-54]) that for a.e. $t\in\mathbb{R}^+$,

$$G_n(t)\to 0,\qquad n\to\infty,$$

and for all $T>0$ there exists a constant $K(T)>0$ such that

$$|G_n(t)|\leq K(T),\qquad t\in[0,T],\ \ n\geq 1.$$

On account of Lebesgue's dominated convergence theorem we have from (2.20) that

$$\lim_{n\to\infty}\left(E_n(t)-e^{-2\lambda t}E_n(0)\right)=0,\qquad t\in\mathbb{R}^+.$$

For fixed $t\in\mathbb{R}^+$, we therefore have

$$\|u(t)\|_{H^1}^2=e^{-2\lambda t}\|u_0\|_{H^1}^2. \tag{2.21}$$

We now infer from Eq. (1.3), Young's inequality and (2.21) that

$$\begin{aligned}\|u_t(t,\cdot)\|_{L^2}&\le\|uu_x\|_{L^2}+\|\partial_x[p*(u^2+\frac{1}{2}u_x^2)]\|_{L^2}+\lambda\|u_0\|_{L^2}\\&\le\|u\|_{H^1}^2+\|p_x\|_{L^2}\,\|u^2+\frac{1}{2}u_x^2\|_{L^1}+\lambda\|u_0\|_{L^2}\\&\le\|u_0\|_{H^1}^2+\|p_x\|_{L^2}\,\|u\|_{H^1}^2+\lambda\|u_0\|_{L^2}\\&\le(1+\|p_x\|_{L^2})\|u_0\|_{H^1}^2+\lambda\|u_0\|_{L^2},\qquad t\in\mathbb{R}^+,\end{aligned}$$

which means that $u\in C^1(\mathbb{R}^+;L^2(\mathbb{R}))$.

Finally, we prove that $y(t,\cdot)=u(t,\cdot)-u_{xx}(t,\cdot)\in L^\infty_{\rm loc}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}))$. From Lemma 2.3 and inequality (2.4), we have

$$\|u^n(t,\cdot)\|_{L^1}\le\|y^n(t,\cdot)\|_{L^1}\le e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0^n\|_{L^1}\le e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}}.$$

Let $n\to\infty$, we get for $t\in\mathbb{R}^+$

$$\|u(t,\cdot)\|_{L^1}\le e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}}. \tag{2.22}$$

Note that $L^1(\mathbb{R})\subset(L^\infty(\mathbb{R}))^*\subset(C_0(\mathbb{R}))^*=\mathcal{M}(\mathbb{R})$. We infer from (2.12) and (2.22) that for $t\in\mathbb{R}^+$

$$\begin{aligned}\|u(t,\cdot)-u_{xx}(t,\cdot)\|_{\mathcal{M}}&\le\|u(t,\cdot)\|_{\mathcal{M}}+\|u_{xx}(t,\cdot)\|_{\mathcal{M}}\\&\le\|u(t,\cdot)\|_{L^1}+\|u_{xx}(t,\cdot)\|_{\mathcal{M}}\\&\le 3e^{(\frac{1}{\sqrt{2}}\|u_0\|_{H^1}-\lambda)t}\|y_0\|_{\mathcal{M}}.\end{aligned}$$

This shows that $u(t,\cdot)-u_{xx}(t,\cdot)\in L^\infty_{\rm loc}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}))$. ☐

Next, we prove the uniqueness of the weak solution of Eq. (1.1).

2.2 Proof of uniqueness

Let $u,v\in C^1(\mathbb{R}^+;L^2(\mathbb{R}^+))\cap C(\mathbb{R}^+;H^1(\mathbb{R}^+))$ be two global weak solutions of (1.1) with initial data u_0 such that $u-u_{xx}$ and $v-v_{xx}$ belong to $L^\infty_{\rm loc}(\mathbb{R}^+;\mathcal{M}(\mathbb{R}))$.

Fix $T>0$ and set

$$M(T):=\sup_{t\in[0,T]}\{\|u(t,\cdot)-u_{xx}(t,\cdot)\|_{\mathcal{M}}+\|v(t,\cdot)-v_{xx}(t,\cdot)\|_{\mathcal{M}}\}.$$

Then for all $(t,x)\in[0,T]\times\mathbb{R}$,

$$|u(t,x)|=|p*[u(t,x)-u_{xx}(t,x)]|\le\|p\|_{L^\infty}\|u(t,\cdot)-u_{xx}(t,\cdot)\|_{\mathcal{M}}\le\frac{1}{2}M(T), \tag{2.23}$$

and

$$|u_x(t,x)| = |p_x * [u(t,x) - u_{xx}(t,x)]| \leq \frac{1}{2}M(T). \tag{2.24}$$

Similarly, we can obtain

$$|v(t,x)| \leq \frac{1}{2}M(T), \quad |v_x(t,x)| \leq \frac{1}{2}M(T), \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{2.25}$$

Again following the same procedure as in [23, pp.48, (2.2)], we can get that for $t \in [0,T]$

$$\begin{cases} \|u(t,\cdot)\|_{L^1} = \|p*(u(t,\cdot) - u_{xx}(t,\cdot))\|_{L^1} \leq M(T), \\ \|u_x(t,\cdot)\|_{L^1} = \|p_x*(u(t,\cdot) - u_{xx}(t,\cdot))\|_{L^1} \leq M(T), \\ \|v(t,\cdot)\|_{L^1} \text{ and } \|v_x(t,\cdot\|_{L^1} \leq M(T). \end{cases} \tag{2.26}$$

Let us define

$$w(t,x) := u(t,x) - v(t,x), \ (t,x) \in [0,T] \times \mathbb{R}.$$

Convoluting Eq. (1.3) for u and v with ρ_n and using Lemma 2.4, we get that for a.e. $t \in [0,T]$ and all $n \geq 1$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\rho_n * w| \mathrm{d}x + \lambda \int_{\mathbb{R}} |\rho_n * w| \mathrm{d}x \\ =& -\int_{\mathbb{R}} [\rho_n * (wu_x)] \mathrm{sgn}(\rho_n * w) \mathrm{d}x - \int_{\mathbb{R}} [\rho_n * (vw_x)] \mathrm{sgn}(\rho_n * w) \mathrm{d}x \\ & \quad - \int_{\mathbb{R}} (\rho_n * p_x * [w(u+v)]) \mathrm{sgn}(\rho_n * w) \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} (\rho_n * p_x * [w_x(u_x + v_x)]) \mathrm{sgn}(\rho_n * w) \mathrm{d}x. \end{aligned}$$

Using (2.23)–(2.26) and Young's inequality, we infer following the procedure described in [23, pp.56-57] that for a.e. $t \in [0,T]$ and all $n \geq 1$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\rho_n * w| \mathrm{d}x + \lambda \int_{\mathbb{R}} |\rho_n * w| \mathrm{d}x \\ \leq & 2M(T) \int_{\mathbb{R}} |\rho_n * w| \mathrm{d}x + 2M(T) \int_{\mathbb{R}} |\rho_n * w_x| \mathrm{d}x + R_n(t), \end{aligned} \tag{2.27}$$

where

$$\begin{cases} \lim_{n \to \infty} R_n(t) = 0, \\ |R_n(t)| \leq K(T), \qquad n \geq 1, \ \ t \in [0,T]. \end{cases} \tag{2.28}$$

Here and henceforth $K(T) > 0$ is a constant depending on $M(T)$ and the $H^1(\mathbb{R})$-norms of $u(0)$ and $v(0)$.

Similarly, convoluting Eq. (1.3) for u and v with $\rho_{n,x}$ and using Lemma 2.4, we get for a.e. $t\in[0,T]$ and all $n\geq 1$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\rho_n*w_x|\mathrm{d}x+\lambda\int_{\mathbb{R}}|\rho_n*w_x|\mathrm{d}x\\
=&-\int_{\mathbb{R}}[\rho_n*(w_x(u_x+v_x))]\mathrm{sgn}(\rho_{n,x}*w)\mathrm{d}x-\int_{\mathbb{R}}[\rho_n*(wu_{xx})]\mathrm{sgn}(\rho_{n,x}*w)\mathrm{d}x\\
&-\int_{\mathbb{R}}[\rho_n*(vw_{xx})]\mathrm{sgn}(\rho_{n,x}*w)\mathrm{d}x-\int_{\mathbb{R}}\rho_n*p_{xx}*\left(u^2-v^2+\frac{1}{2}u_x^2-\frac{1}{2}v_x^2\right)\mathrm{sgn}(\rho_{n,x}*w)\mathrm{d}x.
\end{aligned}$$

Using (2.23)-(2.26), Young's inequality and the identity $\partial_x^2(p*f)=p*f-f$, we infer following the arguments given in [23, pp.57–59] that for a.e. $t\in[0,T]$ and all $n\geq 1$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|\rho_n*w_x|\mathrm{d}x+\lambda\int_{\mathbb{R}}|\rho_n*w_x|\mathrm{d}x\\
\leq&4M(T)\int_{\mathbb{R}}|\rho_n*w|\mathrm{d}x+4M(T)\int_{\mathbb{R}}|\rho_n*w_x|\mathrm{d}x+R_n(t),
\end{aligned}\tag{2.29}$$

with $R_n(t)$ in the class (2.28).

Adding (2.27) and (2.29) we obtain by Gronwall's inequality that for all $t\in[0,T]$ and $n\geq 1$,

$$\begin{aligned}
\int_{\mathbb{R}}(|\rho_n*w|+|\rho_n*w_x|)(t,x)\mathrm{d}x\leq& e^{(6M(T)-\lambda)t}\int_{\mathbb{R}}(|\rho_n*w|+|\rho_n*w_x|)(0,x)\mathrm{d}x\\
&+2\int_0^t e^{(6M(T)-\lambda)(t-s)}R_n(s)\mathrm{d}s.
\end{aligned}$$

Fix $t\in[0,T]$ and let $n\to\infty$ in the above inequality. Note that $w,w_x\in L^1(\mathbb{R})$ and relation (2.28) holds, we get by Lebesgue's dominated convergence theorem that

$$\int_{\mathbb{R}}(|w|+|w_x|)(t,x)\mathrm{d}x\leq e^{(6M(T)-\lambda)t}\int_{\mathbb{R}}(|w|+|w_x|)(0,x)\mathrm{d}x,\qquad t\in[0,T].$$

When $w(0)=w_x(0)=0$ it follows from the above inequality that $u(t,x)=v(t,x)$ for a.e.$(t,x)\in[0,T]\times\mathbb{R}$. Recalling that T was chosen arbitrarily, the proof of the uniqueness is complete. $\square$

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