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On a Class of Neumann Boundary Value Equations Driven by a $(p_1,\cdots,p_n)$-Laplacian Operator

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Abstract. In this paper we prove the existence of an open interval (λ',λ'') for each λ in the interval a class of Neumann boundary value equations involving the $(p_1,\ldots,p_n)$-Laplacian and depending on λ admits at least three solutions. Our main tool is a recent three critical points theorem of Averna and Bonanno [Topol. Methods Nonlinear Anal. [1] (2003) 93-103].

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1 Introduction

Here and in what follows, $\Omega\subset\mathbb{R}^N (N\geq 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class C^1, $p_i > N$ for $1\leq i\leq n$ and λ is a positive parameter.

Let us consider the following quasilinear elliptic system

$$\begin{cases} \Delta_{p_1}u_1+\lambda F_{u_1}(x,u_1,\cdots,u_n)=a_1(x)|u_1|^{p_1-2}u_1 & \text{in } \Omega,\\ \Delta_{p_2}u_2+\lambda F_{u_2}(x,u_1,\cdots,u_n)=a_2(x)|u_2|^{p_2-2}u_2 & \text{in } \Omega,\\ \vdots & \\ \Delta_{p_n}u_n+\lambda F_{u_n}(x,u_1,\cdots,u_n)=a_n(x)|u_n|^{p_n-2}u_n & \text{in } \Omega,\\ \dfrac{\partial u_i}{\partial \nu}=0 \quad \text{for } 1\leq i\leq n & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Delta_{p_i}u_i:=\mathrm{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator and ν is the outer unit normal to $\partial\Omega$. Here, $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is a function such that the mapping $(t_1,t_2,\cdots,t_n)\to F(x,t_1,t_2,\cdots,t_n)$ is measurable in Ω for all $(t_1,\cdots,t_n)\in\mathbb{R}^n$ and is C^1 in $\mathbb{R}^n$ for almost every $x\in\Omega$ satisfying the condition

$$\sup_{\sum_{i=1}^n|t_i|^{p_i}/p_i\le\varrho}|F(\cdot,t_1,\cdots,t_n)|\in L^1(\Omega)$$

for every $\varrho>0$, F_{u_i} denotes the partial derivative of F with respect to u_i, and $a_i\in L^\infty(\Omega)$ with $\mathrm{essinf}_\Omega a_i\ge 0$ for $1\le i\le n$.

Throughout this paper, we let X be the Cartesian product of n spaces $W^{1,p_i}(\Omega)$ for $1\le i\le n$, i.e., $X=W^{1,p_1}(\Omega)\times W^{1,p_2}(\Omega)\times\cdots\times W^{1,p_n}(\Omega)$ equipped with the norm

$$\|(u_1,u_2,\cdots,u_n)\|:=\|u_1\|+\|u_2\|+\cdots+\|u_n\|,$$

where

$$\|u_i\|:=\Big(\int_\Omega|\nabla u_i(x)|^{p_i}\mathrm{d}x+\int_\Omega a_i(x)|u_i(x)|^{p_i}\mathrm{d}x\Big)^{\frac{1}{p_i}}$$

for $1\le i\le n$, which is equivalent to the usual one.

Put

$$c:=\max\left\{\sup_{u_i\in W^{1,p_i}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{\|u_i\|^{p_i}}:\text{ for }1\le i\le n\right\}. \tag{1.2}$$

Since $p_i>N$ for $1\le i\le n$, X is compactly embedded in $(C^0(\overline{\Omega}))^n$, so that $c<+\infty$. It follows from [2, Proposition 4.1] that

$$\sup_{u_i\in W^{1,p_i}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{\|u_i\|^{p_i}}>\frac{1}{\|a_i\|_1}\qquad\text{for }\ 1\le i\le n,$$

where $\|a_i\|_1:=\int_\Omega|a_i(x)|\mathrm{d}x$ for $1\le i\le n$, and so $1/\|a_i\|_1\le c$ for $1\le i\le n$. In addition, if Ω is convex, it is known [2] that

$$\begin{aligned}&\sup_{u_i\in W^{1,p_i}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|}{\|u_i\|}\\&\le 2^{\frac{p_i-1}{p_i}}\max\left\{\Big(\frac{1}{\|a_i\|_1}\Big)^{\frac{1}{p_i}},\ \frac{\mathrm{diam}(\Omega)}{N^{\frac{1}{p_i}}}\Big(\frac{p_i-1}{p_i-N}m(\Omega)\Big)^{\frac{p_i-1}{p_i}}\frac{\|a_i\|_\infty}{\|a_i\|_1}\right\}\end{aligned}$$

for $1\le i\le n$, where $m(\Omega)$ is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball.

By a (weak) solution of the system (1.1), we mean any $u=(u_1,u_2,\cdots,u_n)\in X$ such that

$$\begin{aligned}&\int_\Omega\sum_{i=1}^n|\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x)\mathrm{d}x\\&\quad-\lambda\int_\Omega\sum_{i=1}^n F_{u_i}(x,u_1(x),\cdots,u_n(x))v_i(x)\mathrm{d}x+\int_\Omega\sum_{i=1}^n a_i(x)|u_i(x)|^{p_i-2}u_i(x)v_i(x)\mathrm{d}x=0\end{aligned}$$

for all $v=(v_1,v_2,\cdots,v_n)\in X$.

We shall establish the existence of a definite interval, in which λ lies, the system (1.1) admits at least three weak solutions in X, by means of a recent abstract critical points result of Averna and Bonanno [1] which is actually a refinement of a general principle of Ricceri [3]. Various applications and extensions of this principle are already available; see, for instance, [4–16]. For other basic notations and definitions we refer to [17].

2 Main results

First we here recall for the reader's convenience the three critical points theorem of [1] which is our main tool to prove the results. Here, Y^* denotes the dual space of Y.

Theorem 2.1. ([1, Theorem B]) *Let Y be a real reflexive Banach space; $\Phi:Y\to\mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on Y^*; $\Psi:Y\to\mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u))=+\infty$ *for all* $\lambda\in[0,+\infty[$;

(ii) *there is $r\in\mathbb{R}$ such that:*

$$\inf_Y\Phi<r, \qquad \text{and} \qquad \varphi_1(r)<\varphi_2(r),$$

where

$$\varphi_1(r):=\inf_{u\in\Phi^{-1}(]-\infty,r[)}\frac{\Psi(u)-\inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\Psi}{r-\Phi(u)},$$
$$\varphi_2(r):=\inf_{u\in\Phi^{-1}(]-\infty,r[)}\sup_{v\in\Phi^{-1}([r,+\infty[)}\frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)},$$

and $\overline{\Phi^{-1}(]-\infty,r[)}^w$ is the closure of $\Phi^{-1}(]-\infty,r[)$ in the weak topology.

Then, for each $\lambda\in]1/\varphi_2(r),1/\varphi_1(r)[$ the functional $\Phi+\lambda\Psi$ has at least three critical points in Y.

For all $\gamma>0$ we denote by $K(\gamma)$ the set

$$\left\{(t_1,\cdots,t_n)\in\mathbb{R}^n:\ \sum_{i=1}^{n}\frac{|t_i|^{p_i}}{p_i}\le\gamma\right\}. \tag{2.1}$$

We formulate our main result as follows:

Theorem 2.2. *Assume that there exist two positive constants γ and δ with $\sum_{i=1}^{n}(\delta^{p_i}/p_i)>(\gamma/\prod_{i=1}^{n}p_i)$ such that*

(j)

$$\frac{1}{\gamma}\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x < \frac{1}{2}\frac{\int_\Omega F(x,\delta,\cdots,\delta)\mathrm{d}x}{c\sum_{i=1}^n(\prod_{j=1,j\neq i}^n p_j)\|a_i\|_1\delta^{p_i}},$$

where

$$K\Big(\frac{\gamma}{\prod_{i=1}^n p_i}\Big)=\Big\{(t_1,\cdots,t_n):\ \sum_{i=1}^n\frac{|t_i|^{p_i}}{p_i}\le\frac{\gamma}{\prod_{i=1}^n p_i}\Big\} \tag{2.2}$$

(*see* (2.1)) *and c is given by* (1.2);

(jj)

$$\limsup_{|t_1|\to+\infty,\cdots,|t_n|\to+\infty}\frac{F(x,t_1,\cdots,t_n)}{\sum_{i=1}^n\frac{|t_i|^{p_i}}{p_i}}\le 0;$$

(jjj) $F(x,0,\cdots,0)=0$ *for every* $x\in\Omega$.

Then, setting

$$\lambda^{'}:=\frac{\sum_{i=1}^n\frac{\delta^{p_i}}{p_i}\|a_i\|_1}{\int_\Omega F(x,\delta,\cdots,\delta)-\int_\Omega\sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})}F(x,t_1,\cdots,t_n)\mathrm{d}x}, \tag{2.3a}$$

$$\lambda^{''}:=\frac{\gamma}{(c\prod_{i=1}^n p_i)\int_\Omega\sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})}F(x,t_1,\cdots,t_n)\mathrm{d}x}, \tag{2.3b}$$

for each $\lambda\in]\lambda^{'},\lambda^{''}[$ *the system* (1.1) *admits at least three weak solutions in* X.

Proof. For each $u=(u_1,\cdots,u_n)\in X$, put

$$\Phi(u):\ =\sum_{i=1}^n\frac{\|u_i\|^{p_i}}{p_i},\qquad \Psi(u):\ =-\int_\Omega F(x,u_1(x),\cdots,u_n(x))\mathrm{d}x. \tag{2.4}$$

It is well known that Φ and Ψ are well defined and continuously Gâteaux differentiable functionals with

$$\Phi'(u)(v)=\int_\Omega\sum_{i=1}^n|\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x)\mathrm{d}x+\int_\Omega\sum_{i=1}^n a_i(x)|u_i(x)|^{p_i-2}u_i(x)v_i(x)\mathrm{d}x,$$
$$\Psi'(u)(v)=-\int_\Omega\sum_{i=1}^n F_{u_i}(x,u_1(x),\cdots,u_n(x))v_i(x)\mathrm{d}x$$

for every $u=(u_1,\cdots,u_n)$, $v=(v_1,\cdots,v_n)\in X$, as well as $\Psi':X\to X^*$ is continuous and compact operator (see [17, Proposition 26.2]). Also, $\Phi':X\to X^*$ is an uniformly monotone operator in X, and since Φ' is coercive and semicontinuous in X, by applying [17, Theorem 26.A], Φ' admits a continuous inverse on X^*. Furthermore, by [17, Proposition 25.20], Φ is sequentially weakly lower semicontinuous.

Thanks to the assumption (jj), for each $\lambda > 0$ one has that

$$\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u)) = +\infty.$$

Put $r := \gamma/(c\prod_{i=1}^{n} p_i)$. From the hypothesis (j), we get

$$\begin{aligned}&\frac{1}{\gamma}\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x\\ &< \frac{\displaystyle\int_\Omega F(x,\delta,\cdots,\delta)\mathrm{d}x}{\displaystyle c\sum_{i=1}^n\Big(\prod_{j=1,j\neq i}^n p_j\Big)\|a_i\|_1\delta^{p_i}} - \frac{1}{\gamma}\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x,\end{aligned} \tag{2.5}$$

thus, since $\sum_{i=1}^n \delta^{p_i}/p_i > \gamma/\prod_{i=1}^n p_i$, and $c\|a_i\|_1 \geq 1$ for $1 \leq i \leq n$, we have

$$\begin{aligned}&\frac{1}{\gamma}\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x\\ &< \frac{\displaystyle\int_\Omega F(x,\delta,\cdots,\delta)\mathrm{d}x - \int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}{\displaystyle c\sum_{i=1}^n\Big(\prod_{j=1,j\neq i}^n p_j\Big)\|a_i\|_1\delta^{p_i}},\end{aligned} \tag{2.6}$$

from which, multiplying by $c\prod_{i=1}^n p_i$, we obtain

$$\begin{aligned}&\frac{1}{\gamma}\Big(c\prod_{i=1}^n p_i\Big)\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x\\ &< \frac{\displaystyle\int_\Omega F(x,\delta,\cdots,\delta)\mathrm{d}x - \int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}{\displaystyle\sum_{i=1}^n \frac{\delta^{p_i}}{p_i}\|a_i\|_1}.\end{aligned} \tag{2.7}$$

We claim that

$$\varphi_1(r)\le\frac{1}{\gamma}\Big(c\prod_{i=1}^n p_i\Big)\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x \tag{2.8a}$$

$$\varphi_2(r)\ge\frac{\int_\Omega F(x,\delta,\cdots,\delta)\mathrm{d}x-\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}{\sum_{i=1}^n\frac{\delta^{p_i}}{p_i}\|a_i\|_1}, \tag{2.8b}$$

from which (ii) of Theorem 2.1 follows. In fact, taking into account that the function identically 0 obviously belongs to $\Phi^{-1}(]-\infty,r[)$, and that $\Psi(0)=0$, we get

$$\varphi_1(r)\le\frac{1}{r}\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\int_\Omega F(x,u_1(x),\cdots,u_n(x))\mathrm{d}x, \tag{2.9}$$

and, since $\overline{\Phi^{-1}(]-\infty,r[)}^w=\Phi^{-1}(]-\infty,r])$, we have

$$\frac{1}{r}\sup_{\overline{\Phi^{-1}(]-\infty,r[)}^w}\int_\Omega F(x,u_1(x),\cdots,u_n(x))\mathrm{d}x=\frac{1}{r}\sup_{\Phi^{-1}(]-\infty,r])}\int_\Omega F(x,u_1(x),\cdots,u_n(x))\mathrm{d}x.$$

Since for each $u_i\in W^{1,p_i}(\Omega)$

$$\sup_{x\in\Omega}|u_i(x)|^{p_i}\le c\|u_i\|^{p_i}$$

for $1\le i\le n$ (see (1.2)), we have that

$$\sup_{x\in\Omega}\sum_{i=1}^n\frac{|u_i(x)|^{p_i}}{p_i}\le c\sum_{i=1}^n\frac{\|u_i\|^{p_i}}{p_i}=c\Phi(u) \tag{2.10}$$

for every $u=(u_1,\cdots,u_n)\in X$. Thus, taking into account that $\sum_{i=1}^n|u_i(x)|^{p_i}/p_i\le\gamma/\prod_{i=1}^n p_i$, for every $u=(u_1,\cdots,u_n)\in X$ such that $\Phi(u)\le r$ and for each $x\in\Omega$, we obtain

$$\frac{1}{r}\sup_{\Phi^{-1}(]-\infty,r])}\int_\Omega F(x,u_1(x),\cdots,u_n(x))\mathrm{d}x\le\frac{1}{r}\int_\Omega\sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})}F(x,t_1,\cdots,t_n)\mathrm{d}x.$$

So, (2.8a) follows at once by the definition of r.

Moreover, for each $v=(v_1,\cdots,v_n)\in X$ such that $\Phi(v)\ge r$, we have

$$\varphi_2(r)\ge\inf_{u\in\Phi^{-1}(]-\infty,r[)}\frac{\int_\Omega F(x,v_1(x),\cdots,v_n(x))\mathrm{d}x-\int_\Omega F(x,u_1(x),\cdots,u_n(x))\mathrm{d}x}{\Phi(v)-\Phi(u)}.$$

Thus, from $\sum_{i=1}^n |u_i(x)|^{p_i}/p_i \le \gamma/\prod_{p=1}^n p_i$, for every $u=(u_1,\cdots,u_n)\in X$ such that $\Phi(u)<r$ and for each $x\in\Omega$, we obtain

$$\inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_\Omega F(x,v_1(x),\cdots,v_n(x))\mathrm{d}x - \int_\Omega F(x,u_1(x),\cdots,u_n(x))\mathrm{d}x}{\Phi(v)-\Phi(u)}$$
$$\ge \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_\Omega F(x,v_1(x),\cdots,v_n(x))\mathrm{d}x - \int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}{\Phi(v)-\Phi(u)},$$

from which, being $0<\Phi(v)-\Phi(u)\le\Phi(v)$ for every $u\in\Phi^{-1}(]-\infty,r[)$, and under further condition

$$\int_\Omega F(x,v_1(x),\cdots,v_n(x))\mathrm{d}x \ge \int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x, \tag{2.11}$$

we can write

$$\inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\int_\Omega F(x,v_1(x),\cdots,v_n(x))\mathrm{d}x - \int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}{\Phi(v)-\Phi(u)}$$
$$\ge \frac{\int_\Omega F(x,v_1(x),\cdots,v_n(x))\mathrm{d}x - \int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}{\sum_{i=1}^n \frac{\|v_i\|^{p_i}}{p_i}}.$$

If we put $v(x):=(\delta,\cdots,\delta)$, for each $x\in\Omega$, we have $\|v_i\|=\|a_i\|_1^{1/p_i}\delta$ for $1\le i\le n$.

Now since $\sum_{i=1}^n \delta^{p_i}/p_i > \gamma/\prod_{i=1}^n p_i$, bearing in mind that $1/\|a_i\|_1 \le c$ for $1\le i\le n$, we get $\Phi(v)=\sum_{i=1}^n(\delta^{p_i}\|a_i\|_1)/p_i > r$. Moreover, with this choice of v, (2.7) ensures (2.11), thus (2.8b) is also proved.

Taking into account that the weak solutions of the system (1.1) are exactly the solutions of the equation $\Phi'(u)+\lambda\Psi'(u)=0$, we have the conclusion by using of Theorem 2.1. Namely, by observing that

$$\frac{1}{\varphi_2(r)} \le \frac{\sum_{i=1}^n \frac{\delta^{p_i}}{p_i}\|a_i\|_1}{\int_\Omega F(x,\delta,\cdots,\delta)\mathrm{d}x - \int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}, \tag{2.12a}$$

$$\frac{1}{\varphi_1(r)} \ge \frac{\gamma}{\left(c\prod_{i=1}^n p_i\right)\int_\Omega \sup_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(x,t_1,\cdots,t_n)\mathrm{d}x}, \tag{2.12b}$$

for each $\lambda\in]\lambda',\lambda''[$ the system (1.1) admits at least three weak solutions in X. □

Since $\int_\Omega F(\delta,\cdots,\delta)\mathrm{d}x=m(\Omega)F(\delta,\cdots,\delta)$, we have the following remarkable consequence of Theorem 2.2.

Theorem 2.3. *Let $F:\mathbb{R}^n\to\mathbb{R}$ be a C^1-function and assume that there exist two positive constants γ and δ with $\sum_{i=1}^n\delta^{p_i}/p_i>\gamma/\prod_{i=1}^n p_i$ such that*

(j′)
$$\frac{1}{\gamma}\max_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})}F(t_1,\cdots,t_n)<\frac{1}{2}\frac{F(\delta,\cdots,\delta)}{c\sum_{i=1}^n\Big(\prod_{j=1,j\neq i}^n p_j\Big)\|a_i\|_1\delta^{p_i}},$$

where K is defined by (2.2) *and c is given by* (1.2)*;*

(jj′)
$$\limsup_{|t_1|\to+\infty,\cdots,|t_n|\to+\infty}\frac{F(t_1,\cdots,t_n)}{\sum_{i=1}^n\frac{|t_i|^{p_i}}{p_i}}\le 0;$$

(jjj′) $F(0,\cdots,0)=0$.

Then, setting

$$\lambda':=\frac{\sum_{i=1}^n\frac{\delta^{p_i}}{p_i}\|a_i\|_1}{m(\Omega)\Big(F(\delta,\cdots,\delta)-\max_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})}F(t_1,\cdots,t_n)\Big)},\tag{2.13a}$$

$$\lambda'':=\frac{\gamma}{m(\Omega)\Big(c\prod_{i=1}^n p_i\Big)\max_{(t_1,\cdots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})}F(t_1,\cdots,t_n)},\tag{2.13b}$$

for each $\lambda\in]\lambda',\lambda''[$ the system

$$\begin{cases}\Delta_{p_1}u_1+\lambda F_{u_1}(u_1,\cdots,u_n)=a_1(x)|u_1|^{p_1-2}u_1 & in\ \Omega,\\ \Delta_{p_2}u_2+\lambda F_{u_2}(u_1,\cdots,u_n)=a_2(x)|u_2|^{p_2-2}u_2 & in\ \Omega,\\ \vdots & \\ \Delta_{p_n}u_n+\lambda F_{u_n}(u_1,\cdots,u_n)=a_n(x)|u_n|^{p_n-2}u_n & in\ \Omega,\\ \dfrac{\partial u_i}{\partial\nu}=0\ for\ 1\le i\le n & on\ \partial\Omega,\end{cases}\tag{2.14}$$

admits at least three weak solutions in X.

Now, we give an example to illustrate Theorem 2.3.

Example 2.1. Consider the system

$$\begin{cases} \Delta_3 u_1 + \lambda e^{-u_1} u_1^{11}(12-u_1) = \dfrac{2(x^2+y^2)}{\pi}|u_1|u_1 & \text{in } \Omega, \\ \Delta_3 u_2 + \lambda e^{-u_2} u_2^{13}(14-u_2) = \dfrac{2(x^2+y^2)}{\pi}|u_2|u_2 & \text{in } \Omega, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 < 9\}$. Note that $c = 1536/\pi$ and we choose $\delta = 10$, $\gamma = 3$, $a_i(x,y) = 2(x^2+y^2)/\pi$ for $i = 1,2$ and

$$F(t_1,t_2) = e^{-t_1} t_1^{12} + e^{-t_2} t_2^{14}$$

for each $(t_1,t_2) \in \mathbb{R}^2$. We see that

$$\max_{|t_1|^3+|t_2|^3 \le 1} (e^{-t_1} t_1^{12} + e^{-t_2} t_2^{14}) \le \max_{|t_1|\le 1} e^{-t_1} t_1^{12} + \max_{|t_2|\le 1} e^{-t_2} t_2^{14} = 2e,$$

which gives that

$$\begin{aligned} &\frac{1}{2c} \frac{F(\delta,\delta)}{p_2\|a_1\|_1 \delta^{p_1} + p_1\|a_2\|_1 \delta^{p_2}} - \frac{1}{\gamma} \max_{(t_1,t_2)\in K(\frac{\gamma}{p_1 p_2})} F(t_1,t_2) \\ \ge& \frac{\pi}{2\times 1536} \frac{e^{-10}10^{12} + e^{-10}10^{14}}{6\times 81\times 10^3} - \frac{\max_{|t_1|\le 1} e^{-t_1} t_1^{12} + \max_{|t_2|\le 1} e^{-t_2} t_2^{14}}{3} \\ =& \frac{\pi}{1536} \frac{e^{-10}10^9 + e^{-10}10^{11}}{972} - \frac{2e}{3} > 0, \end{aligned} \tag{2.16}$$

and

$$\limsup_{(|t_1|,|t_2|)\to(+\infty,+\infty)} \frac{F(t_1,t_2)}{\frac{1}{3}|t_1|^3 + \frac{1}{3}|t_2|^3} = 0. \tag{2.17}$$

Hence, Theorem 2.3 is applicable to the system (2.15) for every

$$\lambda \in \left] \frac{54\times 10^3}{9\pi(e^{-10}10^{12} + e^{-10}10^{14} - 2e)}, \quad \frac{1}{1536\times 108e} \right[. \tag{2.18}$$

Finally, we conclude this paper by giving an immediate consequence of Theorem 2.3 when $n = 1$.

Corollary 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) = \int_0^t f(\xi)\mathrm{d}\xi$ for each $t \in \mathbb{R}$ and assume that there exist two positive constants γ and δ with $\delta^p > \gamma$ such that*

(j$''$)
$$\frac{1}{\gamma}\max_{t\in[-\sqrt[p]{\gamma},\sqrt[p]{\gamma}]}F(t)<\frac{1}{2}\frac{F(\delta)}{c\|a\|_1\delta^p},\ \textit{with}\ c=\sup_{u\in W^{1,p}(\Omega)\setminus\{0\}}\left(\frac{\|u\|_\infty}{\|u\|}\right)^p;$$

(jj$''$)
$$\limsup_{|t|\to+\infty}\frac{F(t)}{|t|^p}\le 0.$$

Then, setting

$$\lambda':=\frac{\|a\|_1\delta^p}{p\Big(m(\Omega)\big(F(\delta)-\max_{t\in[-\sqrt[p]{\gamma},\sqrt[p]{\gamma}]}F(t)\big)\Big)}, \tag{2.19a}$$
$$\lambda'':=\frac{\gamma}{m(\Omega)(pc)\max_{t\in[-\sqrt[p]{\gamma},\sqrt[p]{\gamma}]}F(t)}, \tag{2.19b}$$

for each $\lambda\in]\lambda',\lambda''[$ *the problem*

$$\begin{cases}\Delta_p u+\lambda f(u)=a(x)|u|^{p-2}u & \textit{in}\ \Omega,\\ \dfrac{\partial u}{\partial\nu}=0 & \textit{on}\ \partial\Omega,\end{cases} \tag{2.20}$$

admits at least three weak solutions in $W^{1,p}(\Omega)$.

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