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Existence of Nontrivial Weak Solutions to Quasi-Linear Elliptic Equations with Exponential Growth

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Abstract. In this paper, we study the existence of nontrivial weak solutions to the following quasi-linear elliptic equations

$$-\triangle_n u+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^\beta},\quad x\in\mathbb{R}^n\ \ (n\geqslant 2),$$

where $-\triangle_n u=-\mathrm{div}(|\nabla u|^{n-2}\nabla u)$, $0\leqslant\beta<n$, $V:\mathbb{R}^n\to\mathbb{R}$ is a continuous function, $f(x,u)$ is continuous in $\mathbb{R}^n\times\mathbb{R}$ and behaves like $e^{\alpha u^{\frac{n}{n-1}}}$ as u $\to+\infty$.

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1 Introduction

Consider nonlinear elliptic equations of the form

$$-\triangle_p u=f(x,u),\qquad \text{in }\Omega, \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, and $-\triangle_p u=-\ \mathrm{div}\ (|\nabla u|^{p-2}\nabla u)$. Brézis [1], Brézis-Nirenberg [2] and Bartsh-Willem [3] studied this problem under the assumptions that $p=2$ and $|f(x,u)|\leqslant c(|u|+|u|^{q-1})$. Garcia-Alonso [4] studied this problem under the assumptions that $p\leqslant n$ and $p^2\leqslant n$. When $\Omega=\mathbb{R}^n$ and $p=2$, Kryszewski-Szulkin [5], Alama-Li [6], Ding-Ni [7] and Jeanjean [8] studied the following equations in stead of (1.1):

$$-\triangle u+V(x)u=f(x,u),\quad \text{in }\mathbb{R}^n.$$

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In this paper we consider quasi-linear elliptic equations in the whole Euclidean space

$$-\triangle_n u+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^\beta}, \qquad x\in\mathbb{R}^n \ \ (n\geqslant 2), \tag{1.2}$$

where $-\triangle_n u=-\mathrm{div}(|\nabla u|^{n-2}\nabla u)$, $0\leqslant\beta<n$, $V:\mathbb{R}^n\to\mathbb{R}$ is a continuous function, $f(x,u)$ is continuous in $\mathbb{R}^n\times\mathbb{R}$ and behaves like $e^{\alpha u^{\frac{n}{n-1}}}$ as $u\to+\infty$.

D. Cao [9] and Cao-Zhang [10] studied problem (1.2) in the case $n=2$ and $\beta=0$. Panda [11], do Ó et al. [12,13] and Alevs-Figueiredo [14] studied problem (1.2) in general dimension and $\beta=0$. When $\beta\neq 0$, (1.2) was studied by Adimurthi-Yang [15], do Ó et al. [16], Yang [17], Zhao [18], and others. Similar problems in $\mathbb{R}^4$ or complete noncompact Riemannian manifolds were also studied by Yang [19,20].

We define a function space

$$E\triangleq\left\{u\in W^{1,n}(\mathbb{R}^n):\int_{\mathbb{R}^n}V(x)|u|^n\mathrm{d}x<\infty\right\}$$

with the norm

$$\|u\|\triangleq\left\{\int_{\mathbb{R}^n}(|\nabla u|^n+V(x)|u|^n)\mathrm{d}x\right\}^{\frac{1}{n}}. \tag{1.3}$$

We say that $u\in E$ is a weak solution of problem (1.2) if for all $\varphi\in E$ we have

$$\int_{\mathbb{R}^n}(|\nabla u|^{n-2}\nabla u\nabla\varphi+V(x)|u|^{n-2}u\varphi)\mathrm{d}x=\int_{\mathbb{R}^n}\frac{f(x,u)}{|x|^\beta}\varphi\,\mathrm{d}x.$$

If a weak solution u satisfies $u(x)\geqslant 0$ for almost every $x\in\mathbb{R}^n$, we say u is positive.

Throughout this paper we assume the following two conditions on the potential $V(x)$:

(V_1) $V(x)\geqslant V_0>0$;

(V_2) The function $\frac{1}{V(x)}$ belongs to $L^{1/(n-1)}(\mathbb{R}^n)$.

We also assume that the nonlinearity $f(x,s)$ satisfies the following:

(H_1) There exist constants $\alpha_0,b_1,b_2>0$ such that for all $(x,s)\in\mathbb{R}^n\times\mathbb{R}^+$,

$$|f(x,s)|\leqslant b_1 s^{n-1}+b_2\left\{e^{\alpha_0|s|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha_0^k|s|^{\frac{kn}{n-1}}}{k!}\right\};$$

(H_2) There exists $\mu>n$, such that for all $x\in\mathbb{R}^n$ and $s>0$,

$$0<\mu F(x,s)\equiv\mu\int_0^s f(x,t)\mathrm{d}t\leqslant sf(x,s);$$

(H_3) There exist constants $R_0,M_0>0$, such that for all $x\in\mathbb{R}^n$ and $s>R_0$,

$$F(x,s)\leqslant M_0 f(x,s);$$

(H_4)

$$\lim_{s\to 0+} \sup \frac{n|F(x,s)|}{s^n} < \lambda_\beta$$

uniformly with respect to x $\in \mathbb{R}^n$, where

$$\lambda_\beta \triangleq \inf_{u\in E, u\neq 0} \frac{\| u \|^n}{\int_{\mathbb{R}^n} \frac{|u|^n}{|x|^\beta} \mathrm{d}x};$$

(H_5) There exist constants $p>n$ and C_p such that

$$f(x,s) \geqslant C_p\, s^{p-1},$$

for all $s\geqslant 0$ and all $x\in\mathbb{R}^n$, where

$$C_p > \left(\frac{p-n}{p}\right)^{\frac{p-n}{n}} \left(\frac{n\alpha_0}{(n-\beta)\alpha_n}\right)^{\frac{(n-1)(p-n)}{n}} S_p^p,$$

$\alpha_n = n\omega_{n-1}{}^{\frac{1}{n-1}}$, ω_{n-1} is the volume of the unit sphere $\mathbb{S}^{n-1}$, and

$$S_p \triangleq \inf_{u\in E, u\neq 0} \frac{\left(\int_{\mathbb{R}^n} (|\nabla u|^n + V(x)|u|^n) \mathrm{d}x\right)^{\frac{1}{n}}}{\left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^\beta} \mathrm{d}x\right)^{\frac{1}{p}}} = \inf_{u\in E, u\neq 0} \frac{\| u \|}{\left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^\beta} \mathrm{d}x\right)^{\frac{1}{p}}};$$

(H_6) when $s\leqslant 0$, $f(x,s)=0$ for all $x\in\mathbb{R}^n$.

Our main result is the following theorem:

Theorem 1.1. *Assume that* V(x) *is a continuous function satisfying* (V_1) *and* (V_2). $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ *is a continuous function and the hypotheses* (H_1)-(H_6) *hold. Then Eq.* (1.2) *has a nontrivial positive weak solution.*

Here the assumption (H_5) is different from that of [17]. (H_5) was also used in [16] and [18]. An example of f satisfying (H_1)-(H_6) reads

$$f(t) = \begin{cases} 2^l l! C_p \sum_{k=l}^{\infty} \frac{(t^{\frac{n}{n-1}} - \chi(t) t^{\frac{1}{n-1}})^k}{k!}, & t\geq 0, \\ 0, & t<0, \end{cases}$$

where $l\geq N$ is an integer, C_p is as in (H_5), $\chi:[0,\infty)\to\mathbb{R}$ is a smooth function such that $0\leq\chi\leq 1$, $\chi\equiv 0$ on $[0,A]$, $\chi\equiv 1$ on $[2A,\infty)$, and $|\chi'|\leq 2/A$, where A is a large constant, say $A>4^{n-1}$. For details we refer the reader to in [20, Proposition 2.9]. Other examples were also given in [16] and [18] respectively.

2 Compactness analysis

We will give some preliminary results before proving Theorem 1.1. Define a function $\zeta:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ by

$$\zeta(n,s)=e^s-\sum_{k=0}^{n-2}\frac{s^k}{k!}=\sum_{k=n-1}^{\infty}\frac{s^k}{k!}. \tag{2.1}$$

Let $s\geqslant 0, p\geqslant 1$ be real numbers and $n\geqslant 2$ be an integer, then there holds (see [17])

$$\Big(\zeta(n,s)\Big)^p\leqslant\zeta(n,ps). \tag{2.2}$$

Problem (1.2) is closely related to a singular Trudinger-Moser type inequality [15]. That is, for all $\alpha>0$, $0\leqslant\beta<n$, and $u\in W^{1,n}(\mathbb{R}^n)$ $(n\geqslant 2)$, there holds

$$\int_{\mathbb{R}^n}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum\limits_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{kn}{n-1}}}{k!}}{|x|^\beta}\,\mathrm{d}x<\infty. \tag{2.3}$$

Furthermore, we have for all $\alpha\leqslant(1-\frac{\beta}{n})\,\alpha_n$ and $\tau>0$,

$$\sup_{\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)\mathrm{d}x\leqslant 1}\int_{\mathbb{R}^n}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum\limits_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{kn}{n-1}}}{k!}}{|x|^\beta}\,\mathrm{d}x<\infty. \tag{2.4}$$

In this paper, we also need the following result which is taken from Lemma 2.4 in [17]. That is, if $V:\mathbb{R}^n\to\mathbb{R}$ is continuous and (V_1), (V_2) are satisfied, then for any $q\geqslant 1$, there holds

$$E\hookrightarrow L^q(\mathbb{R}^n)\quad\text{compactly}. \tag{2.5}$$

Define a functional $J:E\to\mathbb{R}$ by

$$J(u)\triangleq\frac{1}{n}\,\|\,u\,\|^n-\int_{\mathbb{R}^n}\frac{F(x,u)}{|x|^\beta}\,\mathrm{d}x,$$

where $0\leqslant\beta<n$, $\|\,u\,\|$ is the norm of $u\in E$ defined by (1.3), $F(x,s)=\int_0^s f(x,t)\mathrm{d}t$ is the primitive of $f(x,s)$. Assume $f(x,u)$ satisfies the hypotheses (H_1), then there exist some positive constants $\alpha_1>\alpha_0$ and b_3 such that for all $(x,s)\in\mathbb{R}^n\times\mathbb{R}$,

$$F(x,s)\leqslant b_3\,\zeta\Big(n,\alpha_1|s|^{\frac{n}{n-1}}\Big),$$

where $\zeta(n,s)$ is defined by (2.1). Thus J is well defined thanks to (2.3).

Lemma 2.1. *Assume $V(x)\geqslant V_0$ in $\mathbb{R}^n$, (H_1), (H_2) and (H_3) hold. Then for any nonnegative, compactly supported function $u\in W^{1,n}(\mathbb{R}^n)\setminus\{0\}$, there holds $J(tu)\to-\infty$ as $t\to+\infty$.*

Proof. We follow the line of [15]. (H_2) and (H_3) imply that there exists $R_0>0$ such that for all $s>R_0$,

$$\frac{\mu}{s}\leqslant\frac{\frac{\partial}{\partial s}F(x,s)}{F(x,s)}=\frac{\partial}{\partial s}\Big(lnF(x,s)\Big),$$

therefore,

$$\left(\frac{s}{R_0}\right)^{\mu}\leqslant\frac{F(x,s)}{F(x,R_0)}.$$

It follows that

$$F(x,s)\geqslant F(x,R_0)\ R_0^{-\mu}\cdot s^{\mu}.$$

Let $c_1=F(x,R_0)\ R_0^{-\mu}$, then we have for all $(x,s)\in\Omega\times[0,+\infty)$, $F(x,s)\geqslant c_1s^{\mu}-c_2$, which is under the assumption that u is supported in a bounded domain Ω and c_2 is a positive constant. This implies that

$$J(tu)\leqslant\frac{t^n}{n}\|u\|^n-\int_{\Omega}\frac{c_1t^{\mu}u^{\mu}}{|x|^{\beta}}\mathrm{d}x=t^n\left(\frac{\|u\|^n}{n}-c_1t^{\mu-n}\int_{\Omega}\frac{u^{\mu}}{|x|^{\beta}}\mathrm{d}x\right).$$

Since $\mu>n$, this implies $J(tu)\to-\infty$ as $t\to+\infty$. □

Lemma 2.2. *Assume that $V(x)\geqslant V_0$ in $\mathbb{R}^n$, (H_1) and (H_4) are satisfied. Then there exist $\delta>0$ and $r>0$ such that $J(u)\geqslant\delta$ for all $\|u\|=r$.*

Proof. According to (H_4), there exist $\tau,\delta>0$ such that if $|s|\leqslant\delta$,

$$\frac{n|F(x,s)|}{|s|^n}<\lambda_{\beta}-\tau.$$

Therefore for all $x\in\mathbb{R}^n,|s|\leqslant\delta$, we have

$$F(x,s)\leqslant\frac{\lambda_{\beta}-\tau}{n}|s|^n. \tag{2.6}$$

On the other hand, according to (H_1), we can obtain that for any $|s|\geqslant\delta$,

$$F(x,s)\leqslant C_{\delta}|s|^{n+1}R(\alpha_0,s), \tag{2.7}$$

where

$$C_{\delta}=\frac{b_1}{n|\delta|\cdot\sum\limits_{k=n-1}^{\infty}\frac{(\alpha_0|\delta|^{\frac{n}{n-1}})^k}{k!}}+\frac{b_2}{|\delta|^n},\qquad R(\alpha_0,s)=\sum_{k=n-1}^{\infty}\frac{(\alpha_0|s|^{\frac{n}{n-1}})^k}{k!}.$$

Combining (2.6) and (2.7), we have for all $(x,s)\in\mathbb{R}^n\times\mathbb{R}^n$,

$$F(x,s)\leqslant\frac{\lambda_{\beta}-\tau}{n}|s|^n+C|s|^{n+1}R(\alpha_0,s), \tag{2.8}$$

where $C=C_\delta$. Here we also use the inequality

$$\int_{\mathbb{R}^n}\frac{|u|^{n+1}R(\alpha_0,u)}{|x|^\beta}\,\mathrm{d}x\leqslant C\|u\|^{n+1}, \tag{2.9}$$

which is taken from Lemma 4.2 in [15]. According to the definition of λ_β, we get

$$\int_{\mathbb{R}^n}\frac{|u|^n}{|x|^\beta}\mathrm{d}x\leqslant\frac{\|u\|^n}{\lambda_\beta}. \tag{2.10}$$

Thanks to (2.8), (2.9), and (2.10), we obtain

$$\begin{aligned}J(u)&\geqslant\frac{1}{n}\|u\|^n-\int_{\mathbb{R}^n}\frac{\lambda_\beta-\tau}{n}\frac{|u|^n}{|x|^\beta}\,\mathrm{d}x-\int_{\mathbb{R}^n}C\frac{|u|^{n+1}R(\alpha_0,u)}{|x|^\beta}\,\mathrm{d}x\\&\geqslant\frac{1}{n}\|u\|^n-\frac{\lambda_\beta-\tau}{n}\cdot\frac{\|u\|^n}{\lambda_\beta}-C\|u\|^{n+1}\\&=\|u\|\cdot\left(\frac{\tau}{n\lambda_\beta}\|u\|^{n-1}-C\|u\|^n\right).\end{aligned} \tag{2.11}$$

For sufficiently small $r>0$, we have

$$\frac{\tau}{n\lambda_\beta}r^{n-1}-Cr^n\geqslant\frac{\tau}{2n\lambda_\beta}r^{n-1}, \tag{2.12}$$

which is due to $\tau>0$. Therefore, according to (2.11) and (2.12), for all $\|u\|=r$,

$$J(u)\geqslant r\cdot\frac{\tau}{2n\lambda_\beta}\cdot r^{n-1}=\frac{\tau}{2n\lambda_\beta}\cdot r^n.$$

Finally, let $\delta=\frac{\tau}{2n\lambda_\beta}\cdot r^n$, we have $J(u)\geqslant\delta$ for all $\|u\|=r$. □

Lemma 2.3. *Critical points of J are weak solutions of* (1.2).

Proof. Though the proof is standard, we write it for completeness. Define a function $g(t)=J(u+t\varphi)$, namely

$$g(t)=\frac{1}{n}\int_{\mathbb{R}^n}(|\nabla(u+t\varphi)|^n+V(x)|u+t\varphi|^n)\mathrm{d}x-\int_{\mathbb{R}^n}\frac{F(x,u+t\varphi)}{|x|^\beta}\mathrm{d}x.$$

By a simple calculation,

$$g'(t)\Big|_{t=0}=J'(u+t\varphi)\cdot\varphi\Big|_{t=0}=J'(u)\cdot\varphi.$$

Let $f_1(t)=|\nabla(u+t\varphi)|^n$, $f_2(t)=|u+t\varphi|^n$, and

$$f_3(t)=\int_{\mathbb{R}^n}\frac{F(x,u+t\varphi)}{|x|^\beta}\,\mathrm{d}x.$$

Clearly we have

$$f_1'(t)\Big|_{t=0}=\frac{n}{2}\times 2\times|\nabla u|^{n-2}\cdot\nabla u\cdot\nabla\varphi=n|\nabla u|^{n-2}\cdot\nabla u\cdot\nabla\varphi,$$
$$f_2'(t)\Big|_{t=0}=\frac{n}{2}|u|^{n-2}\times 2u\times\varphi=n|u|^{n-2}u\varphi,$$
$$f_3'(t)\Big|_{t=0}=\int_{\mathbb{R}^n}\frac{f(x,u)}{|x|^\beta}\cdot\varphi\,\mathrm{d}x.$$

Combining the above, we have for all $\varphi\in E$,

$$J'(u)\cdot\varphi=\int_{\mathbb{R}^n}(|\nabla u|^{n-2}\cdot\nabla u\cdot\nabla\varphi+V(x)|u|^{n-2}u\varphi)\mathrm{d}x-\int_{\mathbb{R}^n}\frac{f(x,u)}{|x|^\beta}\cdot\varphi\,\mathrm{d}x. \tag{2.13}$$

Therefore, $J'(u)\cdot\varphi=0$ is equivalent to

$$\int_{\mathbb{R}^n}(|\nabla u|^{n-2}\cdot\nabla u\cdot\nabla\varphi+V(x)|u|^{n-2}u\varphi)\mathrm{d}x-\int_{\mathbb{R}^n}\frac{f(x,u)}{|x|^\beta}\cdot\varphi\,\mathrm{d}x=0.$$

Hence we get the desired result. □

Lemma 2.4. *Assume* (H_5) *is satisfied, then there exists a function* $u_p\in E$ *which satisfies* $\|u_p\|=S_p$*, and for* $t\in[0,+\infty)$*, we define*

$$J(tu_p)\triangleq\frac{t^n}{n}\|u_p\|^n-\int_{\mathbb{R}^n}\frac{F(x,tu_p)}{|x|^\beta}\mathrm{d}x.$$

There holds

$$\max_{t\geqslant 0}J(tu_p)<\frac{1}{n}\left(\frac{n-\beta}{n}\frac{\alpha_n}{\alpha_0}\right)^{n-1}. \tag{2.14}$$

Proof. Similar to [18], assume $\{u_k\}$ is a bounded positive sequence of functions in E which satisfies

$$\int_{\mathbb{R}^n}\frac{|u_k|^p}{|x|^\beta}\mathrm{d}x=1\quad\text{and}\quad\|u_k\|\to S_p.$$

Meanwhile we assume that $u_k\rightharpoonup u_p$ in E, $u_k\to u_p$ in $L^q(\mathbb{R}^n)$ for all $q\geqslant 1$, $u_k(x)\to u_p(x)$ almost everywhere. Using the Hölder inequality and the Mean Value Theorem, we can easily prove that for any $\varepsilon>0$, there exists a constant K such that when $k>K$,

$$\left|\int_{\mathbb{R}^n}\frac{|u_k|^p-|u_p|^p}{|x|^\beta}\,\mathrm{d}x\right|<\varepsilon.$$

Therefore,

$$\int_{\mathbb{R}^n}\frac{|u_k|^p}{|x|^\beta}\,\mathrm{d}x\to\int_{\mathbb{R}^n}\frac{|u_p|^p}{|x|^\beta}\,\mathrm{d}x=1. \tag{2.15}$$

Next we will prove

$$\|u_p\| \leqslant \liminf_{k\to\infty} \| u_k \| = S_p. \tag{2.16}$$

Since $u_k \rightharpoonup u_p$ weakly in E, we know $\nabla u_k \rightharpoonup \nabla u_p$ weakly in $L^n(\mathbb{R}^n)$. According to the definition of weak convergence and the Hölder inequality, we get

$$\int_{\mathbb{R}^n} |\nabla u_p|^n \, \mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^n} |\nabla u_k|^n \, \mathrm{d}x. \tag{2.17}$$

Similarly to the proof of (2.15), we know

$$\int_{\mathbb{R}^n} V(x)|u_k|^n \, \mathrm{d}x \to \int_{\mathbb{R}^n} V(x)|u_p|^n \, \mathrm{d}x. \tag{2.18}$$

Thanks to (2.17) and (2.18), (2.16) holds. Meanwhile, by the definition of S_p , we know $S_p \leqslant \| u_p \|$. Therefore, we know $\| u_p \| = S_p$. According to (H_5), we have

$$\int_{\mathbb{R}^n} \frac{F(x,tu_p)}{|x|^\beta} \mathrm{d}x \geqslant C_p \frac{t^p}{p}. \tag{2.19}$$

Due to the definition of $J(tu_p)$ and (2.19), we have

$$J(tu_p) \leqslant \frac{t^n}{n} S_p^{\,n} - C_p \frac{t^p}{p}.$$

Let

$$f(t) = \frac{t^n}{n} S_p^{\,n} - C_p \frac{t^p}{p},$$

and by calculation we know for any real number t,

$$f(t) \leqslant f\left(\left(\frac{S_p^{\,n}}{C_p} \right)^{\frac{1}{p-n}} \right).$$

This means

$$\frac{t^n}{n} S_p^{\,n} - C_p \cdot \frac{t^p}{p} \leqslant \frac{p-n}{np} \cdot \frac{S_p^{\frac{np}{p-n}}}{C_p^{\frac{n}{p-n}}}.$$

If we set

$$C_p > \left(\frac{p-n}{p}\right)^{\frac{p-n}{n}} \left(\frac{n\alpha_0}{(n-\beta)\alpha_n} \right)^{\frac{(n-1)(p-n)}{n}} S_p^{\,p},$$

then we have

$$\frac{p-n}{np} \cdot \frac{S_p^{\frac{np}{p-n}}}{C_p^{\frac{n}{p-n}}} < \frac{1}{n} \left(\frac{n-\beta}{n} \frac{\alpha_n}{\alpha_0} \right)^{n-1}.$$

In view of (H_5), we get (2.14) immediately. $\square$

Lemma 2.5. *Assume that* (V_1), (V_2), (H_1), (H_2) *and* (H_3) *hold and* $\{u_k\}\subset E$ *be an arbitrary Palais-Smale sequence of* J, *i.e.,*

$$J(u_k)\to c,\quad J^{'}(u_k)\to 0$$

in E^* *as* $k\to\infty$, *where* E^* *denotes the dual space of* E. *Then there exists a subsequence of* $\{u_k\}$ *(still denoted by* $\{u_k\}$ *) and* $u\in E$ *such that* $u_k\rightharpoonup u$ *weakly in* E, $u_k\to u$ *strongly in* $L^q(\mathbb{R}^n)$ *for all* $q\geqslant 1$, *and*

$$\begin{cases}\nabla u_k(x)\to\nabla u(x), & \text{a. e.in } \mathbb{R}^n,\\ \dfrac{f(x,u_k)}{|x|^{\beta}}\to\dfrac{f(x,u)}{|x|^{\beta}}, & \text{stronglyin } L^1(\mathbb{R}^n),\\ \dfrac{F(x,u_k)}{|x|^{\beta}}\to\dfrac{F(x,u)}{|x|^{\beta}}, & \text{stronglyin } L^1(\mathbb{R}^n).\end{cases}$$

Furthermore, u *is a weak solution of* (1.2).

Proof. Assume $\{u_k\}$ is a Palais-Smale sequence of J. Since $J(u_k)\to c$, we obtain

$$\frac{1}{n}\|u_k\|^n-\int_{\mathbb{R}^n}\frac{F(x,u_k)}{|x|^{\beta}}\,\mathrm{d}x\to c,\quad\text{as}\quad k\to\infty. \tag{2.20}$$

According to (2.13), we know

$$\begin{aligned}|J^{'}(u_k)\cdot\varphi|=&\left|\int_{\mathbb{R}^n}(|\nabla u_k|^{n-2}\cdot\nabla u_k\cdot\nabla\varphi+V(x)|u_k|^{n-2}u_k\varphi)\mathrm{d}x-\int_{\mathbb{R}^n}\frac{f(x,u_k)}{|x|^{\beta}}\cdot\varphi\,\mathrm{d}x\right|\\ \leqslant&\tau_k\|\varphi\|,\end{aligned} \tag{2.21}$$

for all $\varphi\in E$, where $\tau_k=\|J^{'}(u_k)\|$, and $\tau_k\to 0$ as $k\to\infty$. Taking $\varphi=u_k$ in (2.21), we have

$$\left|\|u_k\|^n-\int_{\mathbb{R}^n}\frac{f(x,u_k)}{|x|^{\beta}}\cdot u_k\,\mathrm{d}x\right|\leqslant\tau_k\|u_k\|. \tag{2.22}$$

By (H_2), we obtain

$$\int_{\mathbb{R}^n}\frac{\mu F(x,u_k)}{|x|^{\beta}}\,\mathrm{d}x\leqslant\int_{\mathbb{R}^n}\frac{u_kf(x,u_k)}{|x|^{\beta}}\,\mathrm{d}x. \tag{2.23}$$

Then considering $(\frac{\mu}{n}-1)\|u_k\|^n$, according to (2.23), we have

$$\begin{aligned}\Big(\frac{\mu}{n}-1\Big)\|u_k\|^n\leqslant&\Big(\frac{\mu}{n}-1\Big)\|u_k\|^n-\int_{\mathbb{R}^n}\frac{\mu F(x,u_k)-u_kf(x,u_k)}{|x|^{\beta}}\,\mathrm{d}x\\ =&\mu J(u_k)+\Big(\int_{\mathbb{R}^n}\frac{u_kf(x,u_k)}{|x|^{\beta}}\,\mathrm{d}x-\|u_k\|^n\Big)\\ \leqslant&\mu\cdot 2|c|+\tau_k\|u_k\|.\end{aligned} \tag{2.24}$$

According to (2.24), it's easy to prove that $\| u_k \|$ is bounded. Due to (2.20) and (2.22), we get

$$\int_{\mathbb{R}^n} \frac{u_k f(x,u_k)}{|x|^\beta}\,\mathrm{d}x \leqslant C, \qquad \int_{\mathbb{R}^n} \frac{F(x,u_k)}{|x|^\beta}\,\mathrm{d}x \leqslant C, \tag{2.25}$$

where C is a constant which depends only on μ and n. According to (2.5) we obtain that for some $u\in E$ and any $q\geqslant 1$, up to a subsequence, $u_k\to u$ strongly in $L^q(\mathbb{R}^n)$. Then we know $u_k\to u$ almost everywhere in $\mathbb{R}^n$. Next we will prove that up to a subsequence

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{|f(x,u_k)-f(x,u)|}{|x|^\beta}\mathrm{d}x=0. \tag{2.26}$$

Due to $f(x,\cdot)\geqslant 0$, it is sufficient for us to prove that up to a subsequence

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{f(x,u_k)}{|x|^\beta}\mathrm{d}x=\int_{\mathbb{R}^n}\frac{f(x,u)}{|x|^\beta}\mathrm{d}x. \tag{2.27}$$

Due to

$$\frac{f(x,u)}{|x|^\beta}\in L^1(\mathbb{R}^n),$$

we know

$$\lim_{\eta\to+\infty}\int_{|u|\geqslant\eta}\frac{f(x,u)}{|x|^\beta}\,\mathrm{d}x=0.$$

For any $\delta>0$, there exists $M>\frac{C}{\delta}$ such that

$$\int_{|u|\geqslant M}\frac{f(x,u)}{|x|^\beta}\,\mathrm{d}x<\delta, \tag{2.28}$$

where C is the constant in (2.25). According to (2.25), we know

$$\int_{|u_k|\geqslant M}\frac{f(x,u_k)}{|x|^\beta}\,\mathrm{d}x\leqslant\frac{1}{M}\int_{|u_k|\geqslant M}\frac{f(x,u_k)u_k}{|x|^\beta}\,\mathrm{d}x<\delta. \tag{2.29}$$

For all $x\in\{x\in\mathbb{R}^n:|u_k|<M\}$, by our assumption (H_1), we can deduce that

$$|f(x,s)|\leqslant\left(b_1+b_2\,e^{\alpha_0 M^{\frac{n}{n-1}}}\right)|s|^{n-1}. \tag{2.30}$$

Let $C_1=b_1+b_2\,e^{\alpha_0 M^{\frac{n}{n-1}}}$, according to (2.30), we know

$$|f(x,u_k(x))|\leqslant C_1|u_k(x)|^{n-1}.$$

Since

$$\frac{|u_k|^{n-1}}{|x|^\beta}\to\frac{|u|^{n-1}}{|x|^\beta}\ \text{ strongly in } L^1(\mathbb{R}^n),\quad\text{and } u_k\to u\ \text{ almost everywhere in } \mathbb{R}^n,$$

according to the generalized Lebesgue's dominated convergence theorem, we know

$$\lim_{k\to\infty}\int_{|u_k|<M}\frac{f(x,u_k)}{|x|^\beta}\,\mathrm{d}x=\int_{|u|<M}\frac{f(x,u)}{|x|^\beta}\,\mathrm{d}x. \tag{2.31}$$

According to (2.28), (2.29) and (2.31), we can prove that (2.27) holds. Therefore we get (2.26). By (H_3) and (H_1), we obtain that

$$F(x,u_k)\leqslant C_1\cdot|u_k|^n+C_2 f(x,u_k),$$

where $C_1=(b_1/n)+b_2\,e^{\alpha_0 R_0^{\frac{n}{n-1}}}$ and $C_2=M_0$. According to (2.5), (2.26), and the generalized Lebesgue's Dominated Convergence Theorem, we know

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,\mathrm{d}x=0.$$

Using the knowledge of (4.26) in [15], we know $\nabla u_k(x)\to\nabla u(x)$ almost everywhere in $\mathbb{R}^n$ and $|\nabla u_k|^{n-2}\nabla u_k\rightharpoonup|\nabla u|^{n-2}\nabla u$ weakly in $\left(L^{\frac{n}{n-1}}(\mathbb{R}^n)\right)^n$. Let $k\to\infty$ in (2.21), and then we obtain that

$$\left|\int_{\mathbb{R}^n}(|\nabla u|^{n-2}\cdot\nabla u\cdot\nabla\varphi+V(x)|u|^{n-2}u\varphi)\mathrm{d}x-\int_{\mathbb{R}^n}\frac{f(x,u)}{|x|^\beta}\cdot\varphi\,\mathrm{d}x\right|=0,$$

for all $\varphi\in C_0^\infty(\mathbb{R}^n)$. This demonstrates that u is a weak solution of (1.2). □

3 Proof of Theorem 1.1

Next we will prove Theorem 1.1. By Lemmas 2.1 and 2.2, we know J satisfies all the hypotheses of the Mountain-pass Theorem without the Palais-Smale condition:

$$\begin{cases} J\in C^1(E,\mathbb{R});\\ J(0)=0;\\ J(u)\geqslant\delta>0,\quad\text{when}\quad\|u\|=r;\\ J(e)<0,\quad\text{for some } e\in E \text{ with } \|e\|>r.\end{cases}$$

According to the Mountain-pass Theorem except for the Palais-Smale Condition [21], there exists a sequence $\{u_k\}\subset E$ such that

$$J(u_k)\to c>0,\quad J'(u_k)\to 0,$$

in E^*, where

$$c=\min_{\gamma\in\Gamma}\max_{u\in\gamma}J(u)\geqslant\delta\quad\text{and}\quad\Gamma=\Big\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=e\Big\}.$$

According to Lemma 2.5, we know that up to a subsequence

$$\begin{cases} u_k \rightharpoonup u \ \text{ weakly in } E, \\ u_k \to u \ \text{ strongly in } L^q(\mathbb{R}^n), \text{ for any } q \geqslant 1, \\ \displaystyle\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{F(x,u_k)}{|x|^\beta}\,\mathrm{d}x = \int_{\mathbb{R}^n} \frac{F(x,u)}{|x|^\beta}\,\mathrm{d}x, \\ u \ \text{ is a weak solution of (1.2).} \end{cases}$$

Next we will prove that the solution u which we get in the above is nontrivial. Suppose $u\equiv 0$. Due to $F(x,u)\equiv 0$ for all $x\in\mathbb{R}^n$, we get

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{F(x,u_k)}{|x|^\beta}\,\mathrm{d}x = \int_{\mathbb{R}^n} \frac{F(x,u)}{|x|^\beta}\,\mathrm{d}x = 0. \tag{3.1}$$

According to (2.20), we have

$$\lim_{k\to\infty} \left(\frac{1}{n} \| u_k \|^n - \int_{\mathbb{R}^n} \frac{F(x,u_k)}{|x|^\beta}\,\mathrm{d}x \right) = c > 0. \tag{3.2}$$

Combining (3.1) and (3.2), we can obtain that

$$\lim_{k\to\infty} \| u_k \|^n = n\cdot c > 0. \tag{3.3}$$

By Lemma 2.4, we get

$$0 < n\cdot c < \left(\frac{n-\beta}{n} \frac{\alpha_n}{\alpha_0} \right)^{n-1}. \tag{3.4}$$

According to (3.3) and (3.4), we know there exists some $\eta_0 > 0$ and $K > 0$, such that

$$\| u_k \|^n \leqslant \left(\frac{n-\beta}{n} \frac{\alpha_n}{\alpha_0} - \eta_0 \right)^{n-1}, \tag{3.5}$$

for all $k > K$. According to (3.5), we can choose $q > 1$ sufficiently close to 1 such that

$$q\,\alpha_0 \| u_k \|^{\frac{n}{n-1}} \leqslant \left(1 - \frac{\beta}{n} \right) \alpha_n - \frac{\alpha_0 \eta_0}{2}, \tag{3.6}$$

for all $k > K$. By (H_1) and (2.1), we have

$$|f(x,u_k)u_k| \leqslant b_1 |u_k|^n + b_2\, |u_k|\, \zeta(n,\alpha_0\, |u_k|^{\frac{n}{n-1}}).$$

It follows that

$$\int_{\mathbb{R}^n} \frac{|f(x,u_k)u_k|}{|x|^\beta}\,\mathrm{d}x \leqslant b_1 \int_{\mathbb{R}^n} \frac{|u_k|^n}{|x|^\beta}\,\mathrm{d}x + b_2 \int_{\mathbb{R}^n} \frac{|u_k|\, \zeta(n,\alpha_0\, |u_k|^{\frac{n}{n-1}})}{|x|^\beta}\,\mathrm{d}x. \tag{3.7}$$

Letting $1/q'+1/q=1$, and according to (3.6), (3.7), the Hölder Inequality, (2.2), and (2.4), we have

$$\int_{\mathbb{R}^n} \frac{|f(x,u_k)u_k|}{|x|^\beta} \mathrm{d}x \leqslant b_1 \int_{\mathbb{R}^n} \frac{|u_k|^n}{|x|^\beta} \mathrm{d}x + b_2 \left(\int_{\mathbb{R}^n} \frac{|u_k|^{q'}}{|x|^\beta} \mathrm{d}x \right)^{\frac{1}{q'}} \cdot \left(\int_{\mathbb{R}^n} \frac{\zeta(n,q\alpha_0|u_k|^{\frac{n}{n-1}})}{|x|^\beta} \mathrm{d}x \right)^{\frac{1}{q}}$$
$$\leqslant b_1 \int_{\mathbb{R}^n} \frac{|u_k|^n}{|x|^\beta} \mathrm{d}x + C \left(\int_{\mathbb{R}^n} \frac{|u_k|^{q'}}{|x|^\beta} \mathrm{d}x \right)^{\frac{1}{q'}} \to 0, \quad \text{as} \quad k \to \infty. \tag{3.8}$$

According to (2.22) and (3.8), we have

$$\| u_k \| \to 0, \qquad \text{as} \quad k \to \infty.$$

This is in contradiction with (3.3). Thus the solution u of (1.2) is nontrivial.

Testing Eq. (1.2) with u^-, the negative part of u, we conclude that $u^- \equiv 0$. Hence $u \geq 0$. □

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References

[1] Bréis H., Elliptic equations with limiting Sobolev exponents. *Comm. Pure Appl. Math.*, **39** (1986), 517-539.

[2] Bréis H., Nirenberg L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, **36** (1983), 437-477.

[3] Bartsh T., Willem M., On an elliptic equation with concave and convex nonlinearities. *Proc. Amer. Math. Soc.*, **123** (1995), 3555-3561.

[4] Garcia A., Alonso P., Existence and non-uniqueness for the p-Laplacian. *Commun. Partial Differential Equations*, **12** (1987), 1389-1430.

[5] Kryszewski W., Szulkin A., Generalized linking theorem with an application to semilinear Schrödinger equation. *Adv. Diff. Equ.*, **3** (1998), 441-472.

[6] Alama S., Li Y., Existence of solutions for semilinear elliptic equations with indefinite linear part. *J. Diff. Equ.*, **96** (1992), 89-115.

[7] Ding W., Ni W., On the existence of positive entire solutions of a semilinear elliptic equation. *Arch. Rat. Math. Anal.*, **31** (1986), 283-308.

[8] Jeanjean L., Solutions in spectral gaps for a nonlinear equation of Schrödinger type. *J. Diff. Equ.*, **112** (1994), 53-80.

[9] Cao D., Nontrivial solution of semilinear elliptic equations with critical exponent in $\mathbb{R}^2$. *Commun. Partial Differential Equations*, **17** (1992), 407-435.

[10] Cao D., Zhang Z., Eigenfunctions of the nonlinear elliptic equation with critical exponent in $\mathbb{R}^2$. *Acta Math. Sci.*, (English Ed.) **13** (1993), 74-88.

[11] Panda R., Nontrivial solution of a quasilinear elliptic equation with critical growth in $\mathbb{R}^n$. *Proc. Indian Acad. Sci.* (*Math. Sci.*), **105** (1995), 425-444.

[12] do Ó J. M., *n*-Laplacian equations in $\mathbb{R}^n$ with critical growth. *Abstr. Appl. Anal.*, **2** (1997), 301-315.

[13] do Ó J. M., Medeiros E. and U. Severo, On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^n$. *J. Diff. Equ.*, **246** (2009), 1363-1386.

[14] Alves C. O., Figueiredo G. M., On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^n$. *J. Diff. Equ.*, **246** (2009), 1288-1311.

[15] Adimurthi, Yang Y., An interpolation of Hardy inequality and Trudinger-Moser inequaltiy in $\mathbb{R}^n$ and its applicaitons. *International Mathematics Research Notices*, **13** (2010), 2394-2426.

[16] do Ó J. M., Souza M. de, On a class of singular Trudinger-Moser type inequalities and its applications. *Math. Narchr.*, **284** (2011), 1754-1776.

[17] Yang Y., Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space. *Journal of Functional Analysis*, **262** (2012), 1679-1704.

[18] Zhao L., A multiplicity result for a singular and nonhomogeneous elliptic problem in $\mathbb{R}^n$. *Journal of Partial Diff. Equ.*, **25** (2012), 90-102.

[19] Yang Y., Adams type inequalities and related elliptic partial differential equations in dimension four. *J. Diff. Equ.*, **252** (2012), 2266-2295.

[20] Yang Y., Trudinger-Moser inequalities on complete noncompact Riemannian manifolds. *Journal of Functional Analysis*, **263** (2012), 1894-1938.

[21] Rabinowitz P. H., On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.*, **43** (1992), 270-291.