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On Conditions of the Nonexistence of Solutions of Nonlinear Equations with Data from Classes Close to L^1

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Abstract. We establish conditions of the nonexistence of weak solutions of the Dirichlet problem for nonlinear elliptic equations of arbitrary even order with some right-hand sides from L^m where $m>1$. The conditions include the requirement of a certain closeness of the parameter m to 1.

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1 Introduction

In the well-known work [1], a theory of entropy solutions for nonlinear elliptic second-order equations with L^1-data was developed. According to the results of this work, if Ω is a bounded open set of $\mathbb{R}^n$ $(n \geqslant 2)$, $1<p<n$, and coefficients of the equations under consideration grow with respect to the gradient of unknown function u as $|\nabla u|^{p-1}$ and satisfy natural coercivity and strict monotonicity conditions, then the Dirichlet problem in Ω for these equations has a unique entropy solution for every L^1-right-hand side. In addition, if $p>2-1/n$, the entropy solution is a weak solution. At the same time in [1] it was shown that if $1<p\leqslant 2-1/n$, the Dirichlet problem for the equation $-\Delta_p u+u=f$ in Ω does not have weak solutions for some $f\in L^1(\Omega)$.

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In connection with the above, now we note the following two cases where the Dirichlet problem for equations of the class under consideration has a weak solution for every right-hand side in $L^m(\Omega)$ (see in [2, Theorems 1.5.5 and 1.5.6]):

(a) $p \geqslant 2-1/n$, $m>1$;

(b) $p<2-1/n$, $m \geqslant n/(np-n+1)$.

In the present article, we give two nonexistence results. The first one concerns the above-mentioned Dirichlet problem for second-order equations. We prove that if $p< 2-1/n$ and $1<m<n/(np-n+1)$, then for some $f\in L^m(\Omega)$ the problem with the datum f does not have weak solutions (see Theorem 3.3). The second result concerns the Dirichlet problem in the same open set Ω for a class of $2l$-order equations whose coefficients admit the growth of rate $p-1>0$ with respect to the derivatives of order l of unknown function. We establish that under the conditions $n>2$, $2\leqslant l<n$, $p<2-l/n$ and $1<m<n/(np-n+l)$ for some $f\in L^m(\Omega)$ the problem with the datum f does not have weak solutions (see Theorem 4.1). We remark that the proof of these results given in Sections 3 and 4 respectively is based on the use of an assertion which establishes a relation between the parameters n, l, p and m of an operator acting from $L^m(\Omega)$ into $(\overset{\circ}{W}{}^{l,p}(\Omega))^*$ (see Proposition 2.3 in Section 2).

We note that a condition of the nonexistence of weak solutions of the Dirichlet problem for high-order equations with L^1-data was established in the recent article [3].

As far as the solvability of nonlinear elliptic high-order equations with L^1-right-hand sides is concerned, to our knowledge, there are no results on this subject in the general case. Some results on the existence of entropy and weak solutions of the Dirichlet problem for nonlinear elliptic high-order equations with coefficients satisfying a strengthened coercivity condition and L^1-data were obtained for instance in [4] and [5]. In this connection see also [2, Chapter 2] where a theory of the existence and properties of entropy and weak solutions of the Dirichlet problem for nonlinear fourth-order equations with strengthened coercivity and data from L^1 and classes close to L^1 is presented.

2 Auxiliary assertions

Let $n\in\mathbb{N}$, $n\geqslant 2$, and let Ω be a bounded open set of $\mathbb{R}^n$.

Propositon 2.1. *Let $m>1$, $l\in\mathbb{N}$, $p>1$, and let $H: L^m(\Omega)\to(\overset{\circ}{W}{}^{l,p}(\Omega))^*$ be an operator such that*

$$f\in L^m(\Omega), \varphi\in C_0^\infty(\Omega) \Longrightarrow \langle Hf,\varphi\rangle = \int_\Omega f\varphi \mathrm{d}x. \tag{2.1}$$

Then $\overset{\circ}{W}{}^{l,p}(\Omega)\subset L^{m/(m-1)}(\Omega)$.

Proof. First of all we observe that the operator H is linear. In fact, let $f,g\in L^m(\Omega)$ and $\alpha,\beta\in\mathbb{R}$. We fix an arbitrary function $\varphi\in\overset{\circ}{W}{}^{l,p}(\Omega)$ and a sequence $\{\varphi_k\}\subset C_0^\infty(\Omega)$ such that

$\varphi_k \to \varphi$ strongly in $W^{l,p}(\Omega)$. Now, taking an arbitrary $k \in \mathbb{N}$, by virtue of (2.1), we have

$$\langle H(\alpha f+\beta g),\varphi_k\rangle = \int_\Omega (\alpha f+\beta g)\varphi_k \,\mathrm{d}x = \alpha \int_\Omega f\varphi_k \,\mathrm{d}x + \beta \int_\Omega g\varphi_k \,\mathrm{d}x$$
$$= \alpha\langle Hf,\varphi_k\rangle + \beta\langle Hg,\varphi_k\rangle = \langle \alpha Hf+\beta Hg,\varphi_k\rangle.$$

Hence, taking into account the continuity of the functionals Hf, Hg and $H(\alpha f+\beta g)$ along with the strong convergence of $\{\varphi_k\}$ to φ in $W^{l,p}(\Omega)$, we deduce the equality $\langle H(\alpha f+\beta g),\varphi\rangle = \langle \alpha Hf+\beta Hg,\varphi\rangle$. Then, due to the arbitrariness of φ, we get $H(\alpha f+\beta g) = \alpha Hf + \beta Hg$. Therefore, the operator H is linear.

Now, we pass to the immediate proof of the conclusion of the proposition.

We fix an arbitrary function $\varphi \in \overset{\circ}{W}{}^{l,p}(\Omega)$ and define the functional $F: L^m(\Omega) \to \mathbb{R}$ by

$$\langle F,f\rangle = \langle Hf,\varphi\rangle, \qquad f \in L^m(\Omega).$$

Owing to the linearity of the operator H, the functional F is linear.

Let us show that the functional F is continuous. To this purpose we fix a sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ such that

$$\|\varphi_k - \varphi\|_{W^{l,p}(\Omega)} \to 0, \tag{2.2}$$

and for every $k \in \mathbb{N}$ define the functional $F_k: L^m(\Omega) \to \mathbb{R}$ by

$$\langle F_k,f\rangle = \langle Hf,\varphi_k\rangle, \qquad f \in L^m(\Omega).$$

Due to the linearity of the operator H, for every $k \in \mathbb{N}$ the functional F_k is linear. Moreover, using (2.1), we establish that if $k \in \mathbb{N}$ and $f,g \in L^m(\Omega)$, then

$$|\langle F_k,f\rangle - \langle F_k,g\rangle| \leqslant (\max_\Omega |\varphi_k|)(\operatorname{meas}\Omega)^{(m-1)/m}\|f-g\|_{L^m(\Omega)}.$$

This implies that for every $k \in \mathbb{N}$ the functional F_k is continuous on $L^m(\Omega)$. Finally, it is obvious that for every function $f \in L^m(\Omega)$ the sequence of the numbers $\langle F_k,f\rangle$ is bounded. The given properties of the functionals F_k and the theorem on uniform boundedness (see, for instance [6, Charter 2]) allow us to conclude that there exists $M>0$ such that for every $k \in \mathbb{N}$ and for every $f \in L^m(\Omega)$,

$$|\langle F_k,f\rangle| \leqslant M\|f\|_{L^m(\Omega)}.$$

Hence, using the definition of the functionals F_k along with (2.2), we infer that

$$\forall f \in L^m(\Omega), \qquad |\langle Hf,\varphi\rangle| \leqslant M\|f\|_{L^m(\Omega)}.$$

Therefore, the functional F is continuous.

Thus, $F \in (L^m(\Omega))^*$. Then there exists a function $\psi \in L^{m/(m-1)}(\Omega)$ such that for every $f \in L^m(\Omega)$,

$$\langle F,f\rangle = \int_\Omega \psi f \,\mathrm{d}x.$$

This and the definition of the functional F imply that

$$\forall f\in L^m(\Omega),\qquad \langle Hf,\varphi\rangle=\int_\Omega \psi f\,\mathrm{d}x. \tag{2.3}$$

Let us show that $\varphi=\psi$ a. e. in Ω. Indeed, let $f\in L^m(\Omega)\cap L^{p/(p-1)}(\Omega)$. Since $f\in L^m(\Omega)$, by (2.1), for every $k\in\mathbb{N}$ we have

$$\langle Hf,\varphi_k\rangle=\int_\Omega f\varphi_k\,\mathrm{d}x. \tag{2.4}$$

Moreover, taking into account that $f\in L^{p/(p-1)}(\Omega)$ and using Hölder's inequality, for every $k\in\mathbb{N}$ we get

$$\left|\int_\Omega f\varphi_k\,\mathrm{d}x-\int_\Omega f\varphi\,\mathrm{d}x\right|\leqslant\|f\|_{L^{p/(p-1)}(\Omega)}\|\varphi_k-\varphi\|_{L^p(\Omega)}. \tag{2.5}$$

From (2.4), (2.5) and (2.2) it follows that

$$\langle Hf,\varphi_k\rangle\to\int_\Omega f\varphi\,\mathrm{d}x. \tag{2.6}$$

On the other hand, by virtue of (2.2) and the continuity of the functional Hf, we obtain

$$\langle Hf,\varphi_k\rangle\to\langle Hf,\varphi\rangle. \tag{2.7}$$

From (2.3), (2.6) and (2.7) we derive that

$$\int_\Omega f(\varphi-\psi)\,\mathrm{d}x=0. \tag{2.8}$$

Hence, owing to the arbitrariness of the function f in $L^m(\Omega)\cap L^{p/(p-1)}(\Omega)$, we get that $\varphi=\psi$ a. e. in Ω. Then, since $\psi\in L^{m/(m-1)}(\Omega)$, we have $\varphi\in L^{m/(m-1)}(\Omega)$.

Thus, we conclude that $\mathring{W}^{l,p}(\Omega)\subset L^{m/(m-1)}(\Omega)$. □

Propositon 2.2. *Let $l\in\mathbb{N}$, $l<n$, $1<p<n/l$ and $t>np/(n-lp)$. Then $\mathring{W}^{l,p}(\Omega)\setminus L^t(\Omega)\neq\emptyset$.*

Proof. We fix $y\in\Omega$, and let $v:\Omega\to\mathbb{R}$ be the function such that $v(y)=0$ and $v(x)=|x-y|^{-1}$ if $x\in\Omega\setminus\{y\}$. It is easy to verify that the following assertions hold:

$$0<\gamma<n\Longrightarrow v\in L^\gamma(\Omega), \tag{2.9}$$

and

$$\gamma\geqslant n\Longrightarrow v\notin L^\gamma(\Omega). \tag{2.10}$$

Next, let B be a closed ball in $\mathbb{R}^n$ with center y such that $B\subset\Omega$. We fix a function $\varphi\in C_0^\infty(\Omega)$ such that $0\leqslant\varphi\leqslant 1$ in Ω and $\varphi=1$ in B.

Now we set $\lambda = n/t$ and $w = v^{\lambda}\varphi$. We have

$$w \in L^p(\Omega)\setminus L^t(\Omega). \tag{2.11}$$

In fact, since $t > p$, then $\lambda p < n$. This along with the obvious estimate $w^p \leqslant v^{\lambda p}$ in Ω and (2.9) implies that $w \in L^p(\Omega)$. Clearly, $w^t = v^n$ in B, and, by (2.10), $v \notin L^n(\Omega)$. Then $w \notin L^t(\Omega)$. Thus, inclusion (2.11) holds.

Let us show that $w \in \overset{\circ}{W}{}^{l,p}(\Omega)$. To this purpose for every $j \in \mathbb{N}$ we define the function $w_j : \Omega \to \mathbb{R}$ by

$$w_j(x) = (|x-y|^2 + 1/j)^{-\lambda/2}\varphi(x), \quad x \in \Omega.$$

Obviously, $\{w_j\} \subset C_0^\infty(\Omega)$. Moreover, $w_j \to w$ in $\Omega\setminus\{y\}$ and for every $j \in \mathbb{N}$, $w_j \leqslant w$ in $\Omega\setminus\{y\}$. Therefore, taking into account the inclusion $w \in L^p(\Omega)$, we get

$$w_j \to w, \qquad \text{strongly in } L^p(\Omega). \tag{2.12}$$

Using Leibniz' formula of differentiation of the product of two functions, we establish that there exists $C > 0$ such that for every $j \in \mathbb{N}$ and for every n-dimensional multi-index α, $|\alpha| \leqslant l$,

$$|D^\alpha w_j|^p \leqslant C v^{(\lambda + l)p}, \qquad \text{in } \Omega\setminus\{y\}. \tag{2.13}$$

Since $t > np/(n - lp)$, we have $(\lambda + l)p < n$. Then, by (2.9), $v^{(\lambda+l)p} \in L^1(\Omega)$. From this and (2.13) it follows that the sequence $\{w_j\}$ is bounded in $\overset{\circ}{W}{}^{l,p}(\Omega)$. Then, by virtue of the reflexivity of the space $\overset{\circ}{W}{}^{l,p}(\Omega)$, there exist an increasing sequence $\{j_k\} \subset \mathbb{N}$ and a function $u \in \overset{\circ}{W}{}^{l,p}(\Omega)$ such that $w_{j_k} \to u$ weakly in $\overset{\circ}{W}{}^{l,p}(\Omega)$. This and (2.12) imply that $w = u$ a.e. in Ω. Therefore, $w \in \overset{\circ}{W}{}^{l,p}(\Omega)$. The result obtained and (2.11) lead to the conclusion of the proposition. □

Propositon 2.3. *Let $m > 1$, $l \in \mathbb{N}$, $l < n$, $1 < p < n/l$, and let $H : L^m(\Omega) \to (\overset{\circ}{W}{}^{l,p}(\Omega))^*$ be an operator such that*

$$f \in L^m(\Omega), \varphi \in C_0^\infty(\Omega) \Longrightarrow \langle Hf, \varphi\rangle = \int_\Omega f\varphi \,\mathrm{d}x.$$

Then

$$\frac{1}{m} \leqslant \frac{p-1}{p} + \frac{l}{n}. \tag{2.14}$$

Proof. Suppose that inequality (2.14) is not valid. Then $m/(m-1) > np/(n - lp)$. Hence, by Proposition 2.2,

$$\overset{\circ}{W}{}^{l,p}(\Omega)\setminus L^{m/(m-1)}(\Omega) \neq \emptyset. \tag{2.15}$$

On the other hand, by Proposition 2.1, $\overset{\circ}{W}{}^{l,p}(\Omega) \subset L^{m/(m-1)}(\Omega)$. However, this contradicts inequality (2.15). The contradiction obtained proves that inequality (2.14) is valid. □

3 Existence and nonexistence of solutions of second-order equations

Let $1<p<n$, $c_1,c_2>0$, $g\in L^{p/(p-1)}(\Omega)$, $g\geqslant 0$ in Ω, and let for every $i\in\{1,\cdots,n\}$, $a_i:\Omega\times\mathbb{R}^n\to\mathbb{R}$ be a Carathéodory function. We shall suppose that for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^n$,

$$\sum_{i=1}^n|a_i(x,\xi)|\leqslant c_1|\xi|^{p-1}+g(x), \tag{3.1}$$

$$\sum_{i=1}^n a_i(x,\xi)\xi_i\geqslant c_2|\xi|^p. \tag{3.2}$$

Moreover, we shall assume that for almost every $x\in\Omega$ and for every $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$,

$$\sum_{i=1}^n[a_i(x,\xi)-a_i(x,\xi')](\xi_i-\xi'_i)>0. \tag{3.3}$$

For every $f\in L^1(\Omega)$ by (P_f) we denote the following problem:

$$-\sum_{i=1}^n\frac{\partial}{\partial x_i}a_i(x,\nabla u)=f \qquad \text{in } \Omega,$$
$$u=0 \qquad \text{on } \partial\Omega.$$

Definition 3.1. *Let $f\in L^1(\Omega)$. A weak solution of problem (P_f) is a function $u\in\overset{\circ}{W}{}^{1,1}(\Omega)$ such that:*

(i) for every $i\in\{1,\cdots,n\}$, $a_i(x,\nabla u)\in L^1(\Omega)$;

(ii) for every function $\varphi\in C_0^\infty(\Omega)$,

$$\int_\Omega\Big\{\sum_{i=1}^n a_i(x,\nabla u)D_i\varphi\Big\}\mathrm{d}x=\int_\Omega f\varphi\mathrm{d}x.$$

Let us recall some known results on the solvability of problem (P_f) in the case where $f\in L^m(\Omega)$ with $m>1$.

For every $\lambda\in[1,n)$ we set $\lambda^*=n\lambda/(n-\lambda)$.

If $f\in L^{p^*/(p^*-1)}(\Omega)$, in view of conditions (3.1)-(3.3) and Sobolev inequality and according to well known results of the theory of monotone operators (see for instance [7, Chapter 2]), there exists a weak solution of problem (P_f) which belongs to $\overset{\circ}{W}{}^{1,p}(\Omega)$.

Now consider the case where $f\in L^m(\Omega)$ with m lying in the interval $(1,p^*/(p^*-1))$.

Theorem 3.1. *Let $p>2-1/n$, $1<m<p^*/(p^*-1)$, and let $f\in L^m(\Omega)$. Then there exists a weak solution of problem (P_f) which belongs to $\overset{\circ}{W}{}^{1,(p-1)m^*}(\Omega)$.*

This result was proved in [8]. In this connection we observe that actually the conclusion of Theorem 3.1 holds if in the conditions of the theorem we assume that the inequality $p \geqslant 2-1/n$ is satisfied instead of the inequality $p > 2-1/n$ (see [2, Theorem 1.5.5].

Theorem 3.2. *Let $p < 2-1/n$, $n/(np-n+1) \leqslant m < p^*/(p^*-1)$, and let $f \in L^m(\Omega)$. Then there exists a weak solution of problem (P_f) which belongs to $\overset{\circ}{W}{}^{1,(p-1)m^*}(\Omega)$.*

This result was established by the first author in [2, Theorem 1.5.6]. The same conclusion as in the given theorem under the conditions $p \leqslant 2-1/n$ and $n/(np-n+1) < m < p^*/(p^*-1)$ has already been obtained in [9].

The main result of this section given in the following theorem shows that the condition on m in Theorem 3.2 cannot be weakened.

Theorem 3.3. *Let $p < 2-1/n$, and let*

$$1 < m < \frac{n}{np-n+1}. \tag{3.4}$$

Then there exists $f \in L^m(\Omega)$ such that problem (P_f) does not have weak solutions.

Proof. Let us suppose that for every $f \in L^m(\Omega)$ there exists a weak solution of problem (P_f). Therefore, if $f \in L^m(\Omega)$, then there exists a function $u_f \in \overset{\circ}{W}{}^{1,1}(\Omega)$ such that for every $i \in \{1,\cdots,n\}$, $a_i(x,\nabla u_f) \in L^1(\Omega)$, and for every function $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \left\{ \sum_{i=1}^n a_i(x,\nabla u_f) D_i\varphi \right\} \mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x. \tag{3.5}$$

We set $p_1 = 1/(2-p)$. Since $1 < p < 2-1/n$, we have $1 < p_1 < n$. Using (3.1), we establish that for every $f \in L^m(\Omega)$ and for every $i \in \{1,\cdots,n\}$, $a_i(x,\nabla u_f) \in L^{p_1/(p_1-1)}(\Omega)$. Taking this fact into account, for every $f \in L^m(\Omega)$ we define the functional $G_f : \overset{\circ}{W}{}^{1,p_1}(\Omega) \to \mathbb{R}$ by

$$\langle G_f, \varphi \rangle = \int_\Omega \left\{ \sum_{i=1}^n a_i(x,\nabla u_f) D_i\varphi \right\} \mathrm{d}x, \qquad \varphi \in \overset{\circ}{W}{}^{1,p_1}(\Omega).$$

It is easy to see that for every $f \in L^m(\Omega)$, $G_f \in (\overset{\circ}{W}{}^{1,p_1}(\Omega))^*$.

Now let $H : L^m(\Omega) \to (\overset{\circ}{W}{}^{1,p_1}(\Omega))^*$ be the operator such that for every $f \in L^m(\Omega)$, $Hf = G_f$. By virtue of (3.5), for every $f \in L^m(\Omega)$ and for every $\varphi \in C_0^\infty(\Omega)$ we have

$$\langle Hf, \varphi \rangle = \int_\Omega f\varphi \,\mathrm{d}x.$$

Then, applying Proposition 2.3, we get the inequality

$$\frac{1}{m} \leqslant \frac{p_1-1}{p_1} + \frac{1}{n}.$$

Hence, by the definition of p_1, we obtain that $m \geqslant n/(np-n+1)$. However, this contradicts (3.4). The contradiction obtained proves that the conclusion of the theorem is valid. $\square$

4 Nonexistence of solutions of high-order equations

Suppose that $n>2$, and let $l\in\mathbb{N}$, $2\leqslant l<n$. We shall use the following notation: Λ is the set of all n-dimensional multi-indices α such that $|\alpha|=l$; $\mathbb{R}^n_l$ is the space of all functions $\xi:\Lambda\to\mathbb{R}$; if $u\in L^1_{\rm loc}(\Omega)$ and the function u has the weak derivatives $D^\alpha u$, $\alpha\in\Lambda$, then $\nabla_l u$: $\Omega\to\mathbb{R}^n_l$ is the mapping such that for every $x\in\Omega$ and for every $\alpha\in\Lambda$, $(\nabla_l u(x))_\alpha=D^\alpha u(x)$.

Let $p>1$, $c>0$ and $h\in L^{1/(p-1)}(\Omega)$, $h\geqslant 0$ in Ω. Let for every $\alpha\in\Lambda$, $A_\alpha:\Omega\times\mathbb{R}^n_l\to\mathbb{R}$ be a Carathéodory function. We shall assume that for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^n_l$,

$$\sum_{\alpha\in\Lambda}|A_\alpha(x,\xi)|\leqslant c\sum_{\alpha\in\Lambda}|\xi_\alpha|^{p-1}+h(x). \tag{4.1}$$

For every $f\in L^1(\Omega)$ by $(\mathcal{P}_f)$ we denote the following problem:

$$\begin{aligned}&\sum_{\alpha\in\Lambda}(-1)^{|\alpha|}D^\alpha A_\alpha(x,\nabla_l u)=f \quad \text{in } \Omega,\\ &D^\alpha u=0 \quad |\alpha|\leqslant l-1, \quad \text{on } \partial\Omega.\end{aligned}$$

Definition 4.1. *Let $f\in L^1(\Omega)$. A weak solution of problem $(\mathcal{P}_f)$ is a function $u\in \overset{\circ}{W}{}^{l,1}(\Omega)$ such that:*

(i) for every $\alpha\in\Lambda$, $A_\alpha(x,\nabla_l u)\in L^1(\Omega)$;

(ii) for every function $\varphi\in C_0^\infty(\Omega)$,

$$\int_\Omega\Big\{\sum_{\alpha\in\Lambda}A_\alpha(x,\nabla_l u)D^\alpha\varphi\Big\}{\rm d}x=\int_\Omega f\varphi{\rm d}x.$$

Theorem 4.1. *Let $p<2-l/n$, and let*

$$1<m<\frac{n}{np-n+l}. \tag{4.2}$$

Then there exists $f\in L^m(\Omega)$ such that problem $(\mathcal{P}_f)$ does not have weak solutions.

Proof. Let us suppose that for every $f\in L^m(\Omega)$ there exists a weak solution of problem $(\mathcal{P}_f)$. Therefore, if $f\in L^m(\Omega)$, then there exists a function $u_f\in\overset{\circ}{W}{}^{l,1}(\Omega)$ such that for every $\alpha\in\Lambda$, $A_\alpha(x,\nabla_l u_f)\in L^1(\Omega)$, and for every function $\varphi\in C_0^\infty(\Omega)$,

$$\int_\Omega\Big\{\sum_{\alpha\in\Lambda}A_\alpha(x,\nabla_l u_f)D^\alpha\varphi\Big\}{\rm d}x=\int_\Omega f\varphi{\rm d}x. \tag{4.3}$$

We set $p_1=1/(2-p)$. Since $1<p<2-l/n$, we have $1<p_1<n/l$. Using (4.1), we establish that for every $f\in L^m(\Omega)$ and for every $\alpha\in\Lambda$, $A_\alpha(x,\nabla_l u_f)\in L^{p_1/(p_1-1)}(\Omega)$. Taking this fact into account, for every $f\in L^m(\Omega)$ we define the functional $I_f:\overset{\circ}{W}{}^{l,p_1}(\Omega)\to\mathbb{R}$ by

$$\langle I_f,\varphi\rangle=\int_\Omega\Big\{\sum_{\alpha\in\Lambda}A_\alpha(x,\nabla_l u_f)D^\alpha\varphi\Big\}{\rm d}x,\qquad \varphi\in\overset{\circ}{W}{}^{l,p_1}(\Omega).$$

It is obvious that for every $f\in L^m(\Omega)$, $I_f\in(\overset{\circ}{W}{}^{l,p_1}(\Omega))^*$.

Now let $H{:}L^m(\Omega)\to(\overset{\circ}{W}{}^{l,p_1}(\Omega))^*$ be the operator such that for every $f\in L^m(\Omega)$, $Hf=I_f$. In view of (4.3), for every $f\in L^m(\Omega)$ and for every $\varphi\in C_0^\infty(\Omega)$ we have

$$\langle Hf,\varphi\rangle=\int_\Omega f\varphi\mathrm{d}x.$$

Then, applying Proposition 2.3, we get the inequality

$$\frac{1}{m}\leqslant\frac{p_1-1}{p_1}+\frac{l}{n}.$$

Hence, taking into account the definition of p_1, we obtain that $m\geqslant n/(np-n+l)$. However, this contradicts (4.2). The contradiction obtained proves that the conclusion of the theorem is valid. □

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