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Existence of Renormalized Solutions for Nonlinear Parabolic Equations

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Abstract. We give an existence result of a renormalized solution for a class of nonlinear parabolic equations

$$\frac{\partial b(x,u)}{\partial t}-\operatorname{div}\Big(a(x,t,u,\nabla u)\Big)+g(x,t,u,\nabla u)+H(x,t,\nabla u)=f,\quad \text{in } Q_T,$$

where the right side belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and where $b(x,u)$ is unbounded function of u and where $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u. The critical growth condition on g is with respect to ∇u and no growth condition with respect to u, while the function $H(x,t,\nabla u)$ grows as $|\nabla u|^{p-1}$.

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1 Introduction

In the present paper, we study a nonlinear parabolic problem of the type

$$\begin{cases}\dfrac{\partial b(x,u)}{\partial t}-\operatorname{div}\Big(a(x,t,u,\nabla u)\Big)+g(x,t,u,\nabla u)+H(x,t,\nabla u)=f, & \text{in } Q_T,\\ b(x,u)(t=0)=0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega\times(0,T),\end{cases}\tag{1.1}$$

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where Ω is a bounded open subset of $\mathbb{R}^N$, $N\geq 1$, $T>0$, $p>1$ and Q_T is the cylinder $\Omega\times(0,T)$. The operator $-\text{div}\big(a(x,t,u,\nabla u)\big)$ is a Leray-Lions operator which is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u, the function $b(x,u)$ is an unbounded on u. The functions g and H are two the Carathéodory functions with suitable assumptions (see Assumption (H_2)). Finally the data f is in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. We are interested in proving an existence result to (1.1). The difficulties connected to this problem are due to the data and the presence of the two terms g and H which induce a lack of coercivity.

For $b(x,u)=u$, the existence of a weak solution to Problem (1.1) (which belongs to $L^m(0,T;W_0^{1,m}(\Omega))$ with $p>2-1/(N+1)$ and $m<(p(N+1)-N)/N+1$ was proved in [1] (see also [2]) when $g=H=0$, and in [3] when $g=0$, and in [4–6] when $H=0$. In the present paper we prove the existence of renormalized solutions for a class of nonlinear parabolic problems (1.1). The notion of renormalized solution was introduced by Diperna and Lions [7] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (1.1) by Boccardo et al. [8] when the right hand side is in $W^{-1,p'}(\Omega)$, by Rakotoson [9] when the right hand side is in $L^1(\Omega)$, and finally by Dal Maso, Murat, Orsina and Prignet [10] for the case of right hand side is general measure data.

In the case where $H=0$ and where the function $g(x,t,u,\nabla u)\equiv g(u)$ is independent on the $(x,t,\nabla u)$ and g is continuous, the existence of a renormalized solution to Problem (1.1) is proved in [11]. The case $H=0$ is studied by Akdim et al. (see [12,13]). The case $H=0$ and where g depends on (x,t,u) is investigated in [14]. In [15] the authors prove the existence of a renormalized solution for the complete operator. The case $g(x,t,u,\nabla u)\equiv\text{div}(\phi(u))$ and $H=0$ is studied by Redwane in the classical Sobolev spaces $W^{1,p}(\Omega)$ and Orlicz spaces see [16,17], and where $b(x,u)=u$ (see [18]).

The aim of the present paper we prove an existence result for renormalized solutions to a class of problems (1.1) with the two lower order terms. It is worth noting that for the analogous elliptic equation with two lower order terms (see e.g. [19,20]). The plan of the article is as follows. In Section 2 we make precise all the assumptions on b, a, g, H, f and give the definition of a renormalized solution of (1.1). In Section 3 we establish the existence of such a solution (Theorem 3.1).

2 Basic assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true:

Assumption (H1)

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$, $T>0$ is given and we set $Q_T=\Omega\times(0,T)$, and

$$b:\Omega\times\mathbb{R}\to\mathbb{R} \text{ is a Carathéodory function,} \tag{2.1}$$

such that for every $x\in\Omega$, $b(x,.)$ is a strictly increasing C^1-function with $b(x,0)=0$.

Next, for any $k>0$, there exists $\lambda_k>0$ and functions $A_k\in L^\infty(\Omega)$ and $B_k\in L^p(\Omega)$ such that

$$\lambda_k\le\frac{\partial b(x,s)}{\partial s}\le A_k(x)\quad\text{and}\quad\Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big|\le B_k(x),\tag{2.2}$$

for almost every $x\in\Omega$, for every s such that $|s|\le k$, we denote by $\nabla_x\big(\frac{\partial b(x,s)}{\partial s}\big)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions. Also,

$$\begin{aligned}&a:Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N \text{ is a Carathéodory function,}\\&|a(x,t,s,\xi)|\le\beta[k(x,t)+|s|^{p-1}+|\xi|^{p-1}],\end{aligned}\tag{2.3}$$

for a.e. $(x,t)\in Q_T$, all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, some positive function $k(x,t)\in L^{p'}(Q_T)$ and $\beta>0$.

$$[a(x,t,s,\xi)-a(x,t,s,\eta)]\cdot(\xi-\eta)>0,\qquad\text{for all } (\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N,\text{ with } \xi\neq\eta,\tag{2.4}$$

$$a(x,t,s,\xi)\cdot\xi\ge\alpha|\xi|^p,\tag{2.5}$$

where α is a strictly positive constant.

$$f\in L^{p'}(0,T;W^{-1,p'}(\Omega)).\tag{2.6}$$

Assumption (H2)

Furthermore, let $g(x,t,s,\xi):\ Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $H(x,t,\xi):\ Q_T\times\mathbb{R}^N\to\mathbb{R}$ are two Carathéodory functions which satisfy, for almost every $(x,t)\in Q_T$ and for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$, the following conditions

$$|g(x,t,s,\xi)|\le L_1(|s|)(L_2(x,t)+|\xi|^p),\tag{2.7}$$

$$g(x,t,s,\xi)s\ge 0,\tag{2.8}$$

where $L_1:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous increasing function, while $L_2(x,t)$ is positive and belongs to $L^1(Q_T)$.

$$|H(x,t,\xi)|\le h(x,t)|\xi|^{p-1},\tag{2.9}$$

where $h(x,t)$ is positive and belongs to $L^r(Q_T)$ where $r>\max(N,p)$.

We recall that, for $k>1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s)=\begin{cases}s, & \text{if } |s|\le k,\\ k\frac{s}{|s|}, & \text{if } |s|>k.\end{cases}$$

Definition 2.1. *A real-valued function u defined on Q_T is a renormalized solution of problem* (1.1) *if*

$$T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega)),\quad \text{for all } k\geq 0 \text{ and } b(x,u)\in L^\infty(0,T;L^1(\Omega)), \tag{2.10}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u\,\mathrm{d}x\mathrm{d}t\to 0,\qquad \text{as } m\to+\infty, \tag{2.11}$$

$$\begin{aligned}&\frac{\partial B_S(x,u)}{\partial t}-\mathrm{div}\Big(S'(u)a(x,t,u,\nabla u)\Big)+S''(u)a(x,t,u,\nabla u)\nabla u+g(x,t,u,\nabla u)S'(u)\\ &\quad+H(x,t,\nabla u)S'(u)=fS'(u),\qquad \text{in } \mathcal{D}'(Q_T),\end{aligned} \tag{2.12}$$

for all functions $S\in W^{2,\infty}(\mathbb{R})$ which are piecewise $\mathcal{C}^1$ and such that S' has a compact support in $\mathbb{R}$*, and*

$$B_S(x,u)(t=0)=0,\quad \text{in } \Omega, \tag{2.13}$$

where $B_S(x,z)=\int_0^z \frac{\partial b(x,r)}{\partial r}S'(r)\mathrm{d}r$.

Remark 2.1. Eq. (2.12) is formally obtained through pointwise multiplication of (1.1) by $S'(u)$. However, while $a(x,t,u,\nabla u)$, $g(x,t,u,\nabla u)$ and $H(x,t,\nabla u)$ does not in general make sense in $\mathcal{D}'(Q_T)$, all the terms in (2.12) have a meaning in $\mathcal{D}'(Q_T)$. Indeed, if M is such that $\mathrm{supp}S'\subset[-M,M]$, the following identifications are made in (2.12):

- $B_S(x,u)$ belongs to $L^\infty(Q_T)$ because $|B_S(x,u)|\leq\|A_M\|_{L^\infty(\Omega)}\|S\|_{L^\infty(\mathbb{R})}$.
- $S'(u)a(x,t,u,\nabla u)$ identifies with $S'(u)a(x,t,T_M(u),\nabla T_M(u))$ a.e. in Q_T. Since $|T_M(u)|\leq M$ a.e. in Q_T and $S'(u)\in L^\infty(Q_T)$, we obtain from (2.3) and (2.10) that
$$S'(u)a(x,t,T_M(u),\nabla T_M(u))\in(L^{p'}(Q_T))^N.$$
- $S''(u)a(x,t,u,\nabla u)\nabla u$ identifies with $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u)$ and
$$S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u)\in L^1(Q_T).$$
- $S'(u)\big(g(x,t,u,\nabla u)+H(x,t,\nabla u)\big)=S'(u)\big(g(x,t,T_M(u),\nabla T_M(u))+H(x,t,\nabla T_M(u))\big)$ a.e. in Q_T. Since $|T_M(u)|\leq M$ a.e. in Q_T and $S'(u)\in L^\infty(Q_T)$, we obtain from (2.3), (2.7) and (2.9) that $S'(u)\Big(g(x,t,T_M(u),\nabla T_M(u))+H(x,t,\nabla T_M(u))\Big)\in L^1(Q_T)$.
- In view of (2.6) and (2.10), we have $S'(u)f$ belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

The above considerations show that (2.12) holds in $\mathcal{D}'(Q_T)$ and that

$$\frac{\partial B_S(x,u)}{\partial t}\in L^{p'}(0,T;W^{-1,\,p'}(\Omega)).$$

Due to the properties of S, in view of (2.10) and (2.12), we have $\frac{\partial S(u)}{\partial t}\in L^{p'}(0,T;W^{-1,\,p'}(\Omega))$ and $S(u)\in L^p(0,T;W_0^{1,p}(\Omega))$, which implies that $S(u)\in C^0([0,T];L^1(\Omega))$ so that the initial condition (2.13) makes sense. Indeed, for every $S\in W^{1,\infty}(\mathbb{R})$, nondecreasing function such that supp$S'\subset[-M,M]$, in view of (2.2) we have

$$\lambda_M|S(r)-S(r')|\leq\Big|B_S(x,r)-B_S(x,r')\Big|\leq\|A_M\|_{L^\infty(\Omega)}|S(r)-S(r')|, \tag{2.14}$$

for almost every $x\in\Omega$ and for every r, $r'\in\mathbb{R}$.

Now we state the proposition is a slight modification of Gronwall's lemma (see [21]).

Proposition 2.1. *Given the function λ, γ, φ, ρ defined on $[a,+\infty[$, suppose that $a\geq 0$, $\lambda\geq 0$, $\gamma\geq 0$ and that $\lambda\gamma$, $\lambda\varphi$ and $\lambda\rho$ belong to $L^1([a,+\infty[)$. If for almost every $t\geq 0$ we have*

$$\varphi(t)\leq\rho(t)+\gamma(t)\int_t^{+\infty}\lambda(\tau)\varphi(\tau)\mathrm{d}\tau,$$

then

$$\varphi(t)\leq\rho(t)+\gamma(t)\int_t^{+\infty}\rho(\tau)\lambda(\tau)\left(\int_t^{\tau}\lambda(r)\gamma(r)\mathrm{d}r\right)\mathrm{d}\tau$$

for almost every $t\geq 0$.

3 Main results

In this section we establish the following existence theorem.

Theorem 3.1. *Assume that* (H1)–(H2) *hold true. Then, there exists a renormalized solution u of problem* (1.1) *in the sense of Definition* 2.1.

Proof. The proof of this theorem is done in five steps.

Step 1: Approximate problem and a priori estimates.

For $n>0$, let us define the following approximation of b, g and H. First, set

$$b_n(x,r)=b(x,T_n(r))+\frac{1}{n}r. \tag{3.1}$$

In view of (3.1), b_n is a Carathéodory function and satisfies (2.2), there exist $\lambda_n>0$ and functions $A_n\in L^\infty(\Omega)$ and $B_n\in L^p(\Omega)$ such that

$$\lambda_n\leq\frac{\partial b_n(x,s)}{\partial s}\leq A_n(x)\quad\text{and}\quad\left|\nabla_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\right|\leq B_n(x),$$

a.e. in Ω, $s\in\mathbb{R}$. Next, set

$$g_n(x,t,s,\xi)=\frac{g(x,t,s,\xi)}{1+\frac{1}{n}|g(x,t,s,\xi)|},\quad\text{and}\quad H_n(x,t,\xi)=\frac{H(x,t,\xi)}{1+\frac{1}{n}|H(x,t,\xi)|}.$$

Let us now consider the approximate problem

$$\left\{\begin{array}{l}\dfrac{\partial b_n(x,u_n)}{\partial t}-\operatorname{div}(a(x,t,u_n,\nabla u_n))+g_n(x,t,u_n,\nabla u_n)\\ \qquad +H_n(x,t,\nabla u_n)=f, \qquad \text{in } \mathcal{D}'(Q_T),\\ b_n(x,u_n)(t=0)=0, \qquad \text{in } \Omega,\\ b_n(x,u_n)=0, \qquad \text{on } \partial\Omega\times(0,T).\end{array}\right. \tag{3.2}$$

Note that $g_n(x,t,s,\xi)$ and $H_n(x,t,\xi)$ are satisfying the following conditions

$$|g_n(x,t,s,\xi)|\le \max\Big\{|g(x,t,s,\xi)|\ ;\ n\Big\} \quad \text{and} \quad |H_n(x,t,\xi)|\le \max\Big\{|H(x,t,\xi)|\ ;\ n\Big\}.$$

Moreover, since $f\in L^{p'}(0,T;W^{-1,\,p'}(\Omega))$, proving existence of a weak solution $u_n\in L^p(0,T;W_0^{1,\,p}(\Omega))$ of (3.2) is an easy task (see e.g. [22]). For $\varepsilon>0$ and $s\ge 0$, we define

$$\varphi_\varepsilon(r)=\left\{\begin{array}{ll}\operatorname{sign}(r), & \text{if } |r|>s+\varepsilon,\\ \dfrac{\operatorname{sign}(r)(|r|-s)}{\varepsilon}, & \text{if } s<|r|\le s+\varepsilon,\\ 0, & \text{otherwise.}\end{array}\right.$$

We choose $v=\varphi_\varepsilon(u_n)$ as test function in (3.2), we have

$$\begin{aligned}&\left[\int_\Omega B^n_{\varphi_\varepsilon}(x,u_n)\,\mathrm{d}x\right]_0^T+\int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot\nabla(\varphi_\varepsilon(u_n))\,\mathrm{d}x\mathrm{d}t\\ &\qquad +\int_{Q_T} g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\,\mathrm{d}x\mathrm{d}t+\int_{Q_T} H_n(x,t,\nabla u_n)\varphi_\varepsilon(u_n)\,\mathrm{d}x\mathrm{d}t\\ &=\int_0^T\langle f;\varphi_\varepsilon(u_n)\rangle\,\mathrm{d}t,\end{aligned}$$

where

$$B^n_{\varphi_\varepsilon}(x,r)=\int_0^r \frac{\partial b_n(x,s)}{\partial s}\varphi_\varepsilon(s)\,\mathrm{d}s.$$

Using

$$B^n_{\varphi_\varepsilon}(x,r)\ge 0, \quad g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\ge 0,$$

(2.9) and Hölder's inequality, we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\mathrm{d}t\\ &\le\left(\int_{\{s<|u_n|\le s+\varepsilon\}}|f|^{p'}\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_{\{s<|u_n|\le s+\varepsilon\}}\left(\frac{|\nabla u_n|}{\varepsilon}\right)^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\\ &\qquad+\int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Observe that,

$$\begin{aligned}&\int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}\mathrm{d}x\mathrm{d}t\\ \leq &\int_s^{+\infty}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}} h^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma. \end{aligned}\tag{3.3}$$

Because,

$$\begin{aligned}&\int_{\{s<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}\mathrm{d}x\mathrm{d}t\\ =&\int_s^{+\infty}\frac{-\mathrm{d}}{\mathrm{d}\sigma}\left(\int_{\{\sigma<|u_n|\}} h(x,t)|\nabla u_n|^{p-1}\mathrm{d}x\mathrm{d}t\right)\mathrm{d}\sigma\\ =&\int_s^{+\infty}\lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\leq\sigma+\delta\}} h(x,t)|\nabla u_n|^{p-1}\mathrm{d}x\mathrm{d}t\right)\mathrm{d}\sigma\\ \leq&\int_s^{+\infty}\lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{\{\sigma<|u_n|\leq\sigma+\delta\}} h^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\{\sigma<|u_n|\leq\sigma+\delta\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma\\ =&\int_s^{+\infty}\left(\lim_{\delta\to 0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\leq\sigma+\delta\}} h^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\lim_{\delta\to 0}\frac{1}{\delta}\int_{\{\sigma<|u_n|\leq\sigma+\delta\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma\\ =&\int_s^{+\infty}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}} h^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma. \end{aligned}$$

By (2.5) and (3.3), we deduce that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\{s<|u_n|\leq s+\varepsilon\}}\alpha|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\\ \leq&\left(\frac{1}{\varepsilon}\int_{\{s<|u_n|\leq s+\varepsilon\}}|f|^{p'}\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\left(\frac{1}{\varepsilon}\int_{\{s<|u_n|\leq s+\varepsilon\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\\ &\quad+\int_s^{+\infty}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}} h^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma. \end{aligned}\tag{3.4}$$

Letting ε go to zero, we obtain

$$\begin{aligned}&\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}\alpha|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\\ \leq&\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|f|^{p'}\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\\ &\quad+\int_s^{+\infty}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}} h^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma, \end{aligned}\tag{3.5}$$

where $\{s<|u_n|\}$ denotes the set $\{(x,t)\in Q_T, s<|u_n(x,t)|\}$ and $\mu(s)$ stands for the distribution function of u_n, that is $\mu(s)=|\{(x,t)\in Q_T,\ |u_n(x,t)|<s\}|$ for all $s\geq 0$.

Now, we recall the following inequality (see for example [23]), we have for almost every $s>0$

$$1\leq \left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(-\frac{\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}. \tag{3.6}$$

Using (3.6), we have

$$\begin{aligned}
&\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}\alpha|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\\
=&\alpha\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\\
\leq&\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|f|^{p'}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad\times\int_s^{+\infty}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}h^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma,
\end{aligned} \tag{3.7}$$

which implies that,

$$\begin{aligned}
&\alpha\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\\
\leq&\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|f|^{p'}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\\
&\quad\times\int_s^{+\infty}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}h^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{-\mathrm{d}}{\mathrm{d}\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\sigma.
\end{aligned} \tag{3.8}$$

Now, we consider two functions $B(s)$ and $F(s)$ (see [24, Lemma 2.2]) defined by

$$\int_{\{s<|u_n|\}}h^p(x,t)\mathrm{d}x\mathrm{d}t=\int_0^{\mu(s)}B^p(\sigma)\mathrm{d}\sigma, \tag{3.9}$$

$$\int_{\{s<|u_n|\}}|f|^{p'}\mathrm{d}x\mathrm{d}t=\int_0^{\mu(s)}F^{p'}(\sigma)\mathrm{d}\sigma, \tag{3.10}$$

$$||B||_{L^p(0,T;W_0^{1,p}(\Omega))}\leq||h||_{L^p(0,T;W_0^{1,p}(\Omega))},\quad ||F||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\leq||f||_{L^{p'}(0,T;W^{-1,p'}(\Omega))}. \tag{3.11}$$

From (3.8), (3.9) and (3.10) becomes

$$\begin{aligned}&\alpha\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\\ \leq& F(\mu(s))(-\mu'(s))^{\frac{1}{p'}}+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\\ &\times\int_s^{+\infty}B(\mu(\nu))(-\mu'(\nu))^{\frac{1}{p}}\left(-\frac{\mathrm{d}}{\mathrm{d}\nu}\int_{\{\nu<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\mathrm{d}\nu.\end{aligned}$$

From Proposition 2.1, we obtain

$$\begin{aligned}&\alpha\left(\frac{-\mathrm{d}}{\mathrm{d}s}\int_{\{s<|u_n|\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p'}}\\ \leq& F(\mu(s))(-\mu'(s))^{\frac{1}{p'}}+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\\ &\times\int_s^{+\infty}F(\mu(\sigma))B(\mu(\sigma))(-\mu'(\sigma))\exp\left(\int_s^{\sigma}(NC_N^{\frac{1}{N}})^{-1})B(\mu(r))(\mu(r))^{\frac{1}{N}-1}(-\mu'(r))\mathrm{d}r\right)\mathrm{d}\sigma.\end{aligned}$$

Raising to the power p', integrating between 0 and $+\infty$ and by a variable change we have

$$\begin{aligned}&\alpha^{p'}\int_{Q_T}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\leq c_0\int_0^{|Q_T|}F^{p'}(\lambda)\mathrm{d}\lambda\\ &\quad+c_0\int_0^{|Q_T|}\lambda^{(\frac{1}{N}-1)p'}\left[\int_0^{\lambda}F(z)B(z)\exp\left(\int_z^{\lambda}(NC_N^{\frac{1}{N}})^{-1}B(v)v^{\frac{1}{N}-1}\mathrm{d}v\right)\mathrm{d}z\right]^{p'}\mathrm{d}\lambda.\end{aligned}$$

Using Hölder's inequality and (3.11), then we get

$$||u_n||_{L^p(0,T;W_0^{1,p}(\Omega))}\leq c_1, \tag{3.12}$$

where c_1 is a positive constant independent of n. Then there exists $u\in L^p(0,T;W_0^{1,p}(\Omega))$ such that, for some subsequence

$$u_n\rightharpoonup u \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.13}$$

we conclude that

$$||T_k(u_n)||^p_{L^p(0,T;W_0^{1,p}(\Omega))}\leq c_2k. \tag{3.14}$$

We deduce from the above inequality, (2.2) and (3.14), that

$$\int_\Omega B^n_{T_k}(x,u_n)\mathrm{d}x\leq Ck, \tag{3.15}$$

where

$$B^n_{T_k}(x,z)=\int_0^z\frac{\partial b_n(x,s)}{\partial s}T_k(s)\mathrm{d}s.$$

Now, we turn to prove the almost every convergence of u_n and $b_n(x,u_n)$. Consider now a function non decreasing $g_k \in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\le k/2$ and $g_k(s)=k$ for $|s|\ge k$. Multiplying the approximate equation by $g_k'(u_n)$, we obtain

$$\begin{aligned}&\frac{\partial B^n_{g_k'}(x,u_n)}{\partial t}-\operatorname{div}(a(x,t,u_n,\nabla u_n)g_k'(u_n))+a(x,t,u_n,\nabla u_n)g_k''(u_n)\nabla u_n\\&\quad+(g_n(x,t,u_n,\nabla u_n)+H_n(x,t,\nabla u_n))g_k'(u_n)=fg_k'(u_n),\quad \text{in } \mathcal{D}'(Q_T),\end{aligned}\tag{3.16}$$

where

$$B^n_{g_k'}(x,z)=\int_0^z\frac{\partial b_n(x,s)}{\partial s}g_k'(s)\mathrm{d}s.$$

As a consequence of (3.14), we deduce that $g_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and $\partial B^n_{g_k'}(x,u_n)/\partial t$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Due to the properties of g_k and (2.2), we conclude that $\partial g_k(u_n)/\partial t$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, which implies that $g_k(u_n)$ is compact in $L^1(Q_T)$.

Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q_T, which implies that u_n converges almost everywhere to some measurable function u in Q_T. Thus by using the same argument as in [11, 25] and [26], we can show

$$u_n\to u,\ \text{ a.e. in } Q_T,\tag{3.17}$$

$$b_n(x,u_n)\to b(x,u),\ \text{ a.e. in } Q_T.\tag{3.18}$$

We can deduce from (3.14) that

$$T_k(u_n)\rightharpoonup T_k(u),\ \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)).\tag{3.19}$$

Which implies, by using (2.3), for all $k>0$ that there exists a function $\overline{a}\in(L^{p'}(Q_T))^N$, such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\overline{a},\ \text{ weakly in } (L^{p'}(Q_T))^N.\tag{3.20}$$

We now establish that $b(.,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Using (3.17) and passing to the limit inf in (3.15) as n tends to $+\infty$, we obtain that

$$\frac{1}{k}\int_\Omega B_{T_k}(x,u)(\tau)\mathrm{d}x\le C,$$

for almost any τ in $(0,T)$. Due to the definition of $B_{T_k}(x,s)$ and the fact that $\frac{1}{k}B_{T_k}(x,u)$ converges pointwise to $b(x,u)$, as k tends to $+\infty$, shows that $b(x,u)$ belong to $L^\infty(0,T;L^1(\Omega))$.

Lemma 3.1. *Let u_n be a solution of the approximate problem* (3.2). *Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\mathrm{d}x\mathrm{d}t=0.\tag{3.21}$$

Proof. Considering the function $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ in (3.2) this function is admissible since $\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned}&\int_0^T \Big\langle \frac{\partial b_n(x,u_n)}{\partial t}\ ;\ \alpha_m(u_n)\Big\rangle \mathrm{d}t + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot \nabla u_n \alpha_m'(u_n)\mathrm{d}x\mathrm{d}t\\ &\quad + \int_{Q_T}\Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big)\alpha_m(u_n)\mathrm{d}x\mathrm{d}t\\ &\leq \|\nabla u_n\|_{L^p(Q_T)}\Big(\int_{\{m\leq u_n\}}|f|^{p'}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p'}}.\end{aligned}$$

Which, by setting

$$B^n_{\alpha_m}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\alpha_m(s)\mathrm{d}s,$$

(2.8) and (2.9) gives

$$\begin{aligned}&\int_\Omega B^n_{\alpha_m}(x,u_n)(T)\mathrm{d}x + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \mathrm{d}x\mathrm{d}t\\ &\leq \|\nabla u_n\|_{L^p(Q_T)}\Big(\int_{\{m\leq u_n\}}|f|^{p'}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p'}} + \int_{Q_T} h(x,t)|\nabla u_n|^{p-1}\alpha_m(u_n)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using this Hölder's inequality and (3.12), we deduce

$$\begin{aligned}&\int_\Omega B^n_{\alpha_m}(x,u_n)(T)\mathrm{d}x + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \mathrm{d}x\mathrm{d}t\\ &\leq c_1\Big(\int_{\{m\leq u_n\}}|f|^{p'}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p'}} + c_1\Big(\int_{\{m\leq u_n\}}|h(x,t)|^{p}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p'}}.\end{aligned}$$

Since $B^n_{\alpha_m}(x,u_n)(T)\geq 0$ and by Lebesgue's theorem, we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\Big(\int_{\{m\leq u_n\}}|f|^{p'}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p'}} = 0. \tag{3.22}$$

Similarly, since $b\in L^r(Q_T)$ (with $r\geq p$), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\Big(\int_{\{m\leq u_n\}}|h(x,t)|^{p}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p'}} = 0. \tag{3.23}$$

We conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\mathrm{d}x\mathrm{d}t = 0. \tag{3.24}$$

On the other hand, let $\varphi = T_1(u_n - T_m(u_n))^-$ as test function in (3.2) and reasoning as in the proof of (3.24) we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \,\mathrm{d}x\mathrm{d}t = 0. \tag{3.25}$$

Thus (3.21) follows from (3.24) and (3.25). □

Step 2: Almost everywhere convergence of the gradients.

This step is devoted to introduce for $k\ge 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes (see [27, Lemma 6, proposition 3 and proposition 4]). For $k>0$ fixed, and let $\varphi(t)=te^{\gamma t^2}$, $\gamma>0$. It is will known that when $\gamma > (L_1(k)/2\alpha)^2$, one has

$$\varphi'(s) - \left(\frac{L_1(k)}{\alpha}\right)|\varphi(s)| \ge \frac{1}{2}, \quad \text{for all } s\in\mathbb{R}. \tag{3.26}$$

Let $\psi_i \in \mathcal{D}(\Omega)$ be a sequence which converge strongly to u_0 in $L^1(\Omega)$.

Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad |w^i_\mu|\le k, \tag{3.27}$$

$$w^i_\mu \to T_k(u), \quad \text{strongly in } L^p(0,T;W^{1,p}_0(\Omega)), \text{ as } \mu\to\infty. \tag{3.28}$$

We introduce the following function of one real:

$$h_m(s) = \begin{cases} 1, & \text{if } |s|\le m,\\ 0, & \text{if } |s|\ge m+1,\\ m+1-|s|, & \text{if } m\le |s|\le m+1,\end{cases}$$

where $m>k$. Let $\theta^{\mu,i}_n = T_k(u_n) - w^i_\mu$ and $z^{\mu,i}_{n,m} = \varphi(\theta^{\mu,i}_n)h_m(u_n)$.

Using in (3.2) the test function $z^{\mu,i}_{n,m}$, we obtain

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t}\,;\,\varphi(T_k(u_n)-w^i_\mu)h_m(u_n)\right\rangle \mathrm{d}t\\
&\quad + \int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot[\nabla T_k(u_n) - \nabla w^i_\mu]\varphi'(\theta^{\mu,i}_n)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,\varphi(\theta^{\mu,i}_n)h'_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_{Q_T}\Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big)z^{\mu,i}_{n,m}\,\mathrm{d}x\mathrm{d}t\\
&= \int_0^T \langle f\,;\,z^{\mu,i}_{n,m}\rangle\,\mathrm{d}t,
\end{aligned}$$

which implies since $g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\geq 0$ on $\{|u_n|>k\}$:

$$\begin{aligned}&\int_0^T\Big\langle\frac{\partial b_n(x,u_n)}{\partial t}\,;\,\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\Big\rangle\mathrm{d}t\\&\quad+\int_{Q_T}a(x,t,u_n,\nabla u_n)\cdot[\nabla T_k(u_n)-\nabla w_\mu^i]\varphi'(\theta_n^{\mu,i})h_m(u_n)\mathrm{d}x\mathrm{d}t\\&\quad+\int_{Q_T}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\varphi(\theta_n^{\mu,i})h_m'(u_n)\mathrm{d}x\mathrm{d}t\\&\quad+\int_{\{|u_n|\leq k\}}g_n(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\mathrm{d}x\mathrm{d}t\\&\leq\int_0^T\langle f\,;\,z_{n,m}^{\mu,i}\rangle\mathrm{d}t+\int_{Q_T}|H_n(x,t,\nabla u_n)z_{n,m}^{\mu,i}|\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.29}$$

In the sequel and throughout the paper, we will omit for simplicity the denote $\varepsilon(n,\mu,i,m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{n\to\infty}\varepsilon(n,\mu,i,m)=0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then μ,i and finally m. Similarly we will write only $\varepsilon(n)$, or $\varepsilon(n,\mu),\cdots$ to mean that the limits are made only on the specified parameters.

We will deal with each term of (3.29). First of all, observe that

$$\int_0^T\langle f\,;\,z_{n,m}^{\mu,i}\rangle\mathrm{d}t+\int_{Q_T}|H_n(x,t,\nabla u_n)z_{n,m}^{\mu,i}|\mathrm{d}x\mathrm{d}t=\varepsilon(n,\mu),\tag{3.30}$$

since $\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)$ converges to $\varphi(T_k(u)-(T_k(u))_\mu+e^{-\mu t}T_k(\psi_i))h_m(u)$ strongly in $L^p(Q_T)$ and weakly$-*$ in $L^\infty(Q_T)$ as $n\to\infty$ and finally $\varphi(T_k(u)-(T_k(u))_\mu+e^{-\mu t}T_k(\psi_i))\times h_m(u)$ converges to 0 strongly in $L^p(Q_T)$ and weakly$-*$ in $L^\infty(Q_T)$ as $\mu\to\infty$.

On the one hand. The definition of the sequence w_μ^i makes it possible to establish the following Lemma 3.2.

Lemma 3.2. *For $k\geq 0$ we have*

$$\int_0^T\Big\langle\frac{\partial b_n(x,u_n)}{\partial t}\,;\,\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\Big\rangle\mathrm{d}t\geq\varepsilon(n,m,\mu,i).\tag{3.31}$$

Proof. (see Blanchard and Redwane [28]). □

On the other hand, the second term of the left hand side of (3.29) can be written

$$\begin{aligned}
&\int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
=&\int_{\{|u_n|\le k\}} a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
=&\int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

since $m>k$ and $h_m(u_n)=1$ on $\{|u_n|\le k\}$, we deduce that

$$\begin{aligned}
&\int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
=&\int_{Q_T}\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\
&\qquad\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_{Q_T} a(x,t,u_n,\nabla u_n)\cdot\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
=&K_1+K_2+K_3+K_4.
\end{aligned}$$

Using (2.3), (3.20) and Lebesgue theorem we have $a(x,t,T_k(u_n),\nabla T_k(u))$ converges to $a(x,t,T_k(u),\nabla T_k(u))$ strongly in $(L^{p'}(Q_T))^N$ and $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $(L^p(Q_T))^N$, then $K_2=\varepsilon(n)$. Using (3.20) and (3.28) we have

$$K_3=\int_{Q_T}\overline{a}\cdot\nabla T_k(u)\,\mathrm{d}x\mathrm{d}t+\varepsilon(n,\mu).$$

For what concerns K_4 can be written, since $h_m(u_n)=0$ on $\{|u_n|>m+1\}$

$$\begin{aligned}
K_4=&-\int_{Q_T} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
=&-\int_{\{|u_n|\le k\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_{\{k<|u_n|\le m+1\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w_\mu^i\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

and, as above, by letting $n\to\infty$

$$\begin{aligned}K_4=&-\int_{\{|u|\le k\}}\overline{a}\cdot\nabla w_\mu^i\varphi'(T_k(u)-w_\mu^i)\mathrm{d}x\mathrm{d}t\\&-\int_{\{k<|u|\le m+1\}}\overline{a}\cdot\nabla w_\mu^i\varphi'(T_k(u)-w_\mu^i)h_m(u)\mathrm{d}x\mathrm{d}t+\varepsilon(n),\end{aligned}$$

so that, by letting $\mu\to\infty$

$$K_4=-\int_{Q_T}\overline{a}\cdot\nabla T_k(u)\mathrm{d}x\mathrm{d}t+\varepsilon(n,\mu).$$

We conclude then that

$$\begin{aligned}&\int_{Q_T}a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla w_\mu^i)\varphi'(T_k(u_n)-w_\mu^i)h_m(u_n)\mathrm{d}x\mathrm{d}t\\=&\int_{Q_T}\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\&\quad\times(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(T_k(u_n)-w_\mu^i)\mathrm{d}x\mathrm{d}t+\varepsilon(n,\mu).\end{aligned}\tag{3.32}$$

To deal with the third term of the left hand side of (3.29), observe that

$$\begin{aligned}&\left|\int_{Q_T}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\varphi(\theta_n^{\mu,i})h_m'(u_n)\mathrm{d}x\mathrm{d}t\right|\\\le&\varphi(2k)\int_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Thanks to (3.21), we obtain

$$\left|\int_{Q_T}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\varphi(\theta_n^{\mu,i})h_m'(u_n)\mathrm{d}x\mathrm{d}t\right|\le\varepsilon(n,m).\tag{3.33}$$

We now turn to fourth term of the left hand side of (3.29), can be written

$$\begin{aligned}&\left|\int_{\{|u_n|\le k\}}g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\mathrm{d}x\mathrm{d}t\right|\\\le&\int_{\{|u_n|\le k\}}L_1(k)(L_2(x,t)+|\nabla T_k(u_n)|^p|\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\mathrm{d}x\mathrm{d}t\\\le&L_1(k)\int_{Q_T}L_2(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t\\&\quad+\frac{L_1(k)}{\alpha}\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.34}$$

since $L_2(x,t)$ belong to $L^1(Q_T)$ it is easy to see that

$$L_1(k)\int_{Q_T}L_2(x,t)|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t=\varepsilon(n,\mu).$$

On the other hand, the second term of the right hand side of (3.34), write as

$$
\begin{aligned}
&\frac{L_1(k)}{\alpha}\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t\\
=&\frac{L_1(k)}{\alpha}\int_{Q_T}\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\
&\quad\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{L_1(k)}{\alpha}\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u))\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{L_1(k)}{\alpha}\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u))\cdot\nabla T_k(u)|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

and, as above, by letting first n then finally μ go to infinity, we can easily see, that each one of last two integrals is of the form $\varepsilon(n,\mu)$. This implies that

$$
\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g(x,t,u_n,\nabla u_n)\varphi(T_k(u_n)-w_\mu^i)h_m(u_n)\mathrm{d}x\mathrm{d}t\right|\\
\le&\frac{L_1(k)}{\alpha}\int_{Q_T}\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\
&\quad\cdot(\nabla T_k(u_n)-\nabla T_k(u))|\varphi(T_k(u_n)-w_\mu^i)|\mathrm{d}x\mathrm{d}t+\varepsilon(n,\mu).
\end{aligned}
\tag{3.35}
$$

Combining (3.29), (3.31), (3.32), (3.33) and (3.35), we get

$$
\begin{aligned}
&\int_{Q_T}\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\\
&\quad\cdot(\nabla T_k(u_n)-\nabla T_k(u))\left(\varphi'(T_k(u)-w_\mu^i)-\frac{L_1(k)}{\alpha}|\varphi(T_k(u_n)-w_\mu^i)|\right)\mathrm{d}x\mathrm{d}t\le\varepsilon(n,\mu,i,m),
\end{aligned}
$$

and so, thanks to (3.26), we have

$$
\begin{aligned}
&\int_{Q_T}\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\Big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\mathrm{d}t\\
\le&\varepsilon(n).
\end{aligned}
\tag{3.36}
$$

Hence by passing to the limit sup over n, we get

$$
\limsup_{n\to\infty}\int_{Q_T}\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\cdot(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\mathrm{d}t=0.
$$

This implies that

$$
T_k(u_n)\to T_k(u),\quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ for all } k. \tag{3.37}
$$

Now, observe that for every $\sigma>0$,

$$\begin{aligned}&\text{meas}\Big\{(x,t)\in Q_T:\ |\nabla u_n-\nabla u|>\sigma\Big\}\\ \le&\text{meas}\Big\{(x,t)\in Q_T:\ |\nabla u_n|>k\Big\}+\text{meas}\Big\{(x,t)\in Q_T:\ |u|>k\Big\}\\ &+\text{meas}\Big\{(x,t)\in Q_T:\ \big|\nabla T_k(u_n)-\nabla T_k(u)\big|>\sigma\Big\},\end{aligned}$$

then as a consequence of (3.37) we have that ∇u_n converges to ∇u in measure and therefore, always reasoning for a subsequence,

$$\nabla u_n\to\nabla u,\quad \text{a.e. in } Q_T, \tag{3.38}$$

which implies

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup a(x,t,T_k(u),\nabla T_k(u)),\quad \text{weakly in } (L^{p'}(Q_T))^N. \tag{3.39}$$

Step 3: Equi-integrability of H_n and g_n.

We shall now prove that $H_n(x,t,\nabla u_n)$ converges to $H(x,t,\nabla u)$ and $g_n(x,t,u_n,\nabla u_n)$ converges to $g(x,t,u,\nabla u)$ strongly in $L^1(Q_T)$ by using Vitali's theorem.

Since $H_n(x,t,\nabla u_n)\to H(x,t,\nabla u)$ a.e. Q_T and $g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u)$ a.e. Q_T, thanks to (2.7) and (2.9), it suffices to prove that $H_n(x,t,\nabla u_n)$ and $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q_T. We will now prove that $H_n(x,\nabla u_n)$ is uniformly equi-integrable, we use Hölder's inequality and (3.12), we have

$$\int_E|H_n(x,\nabla u_n)|\le\Big(\int_E h^p(x,t)\,\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{Q_T}|\nabla u_n|^p\Big)^{\frac{1}{p'}}\le c_1\Big(\int_E h^p(x,t)\,\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p}}, \tag{3.40}$$

which is small uniformly in n when the measure of E is small.

To prove the uniform equi-integrability of $g_n(x,t,u_n,\nabla u_n)$. For any measurable subset $E\subset Q_T$ and $m\ge 0$,

$$\begin{aligned}&\int_E|g(x,t,u_n,\nabla u_n)|\,\mathrm{d}x\mathrm{d}t\\ =&\int_{E\cap\{|u_n|\le m\}}|g(x,t,u_n,\nabla u_n)|\,\mathrm{d}x\mathrm{d}t+\int_{E\cap\{|u_n|>m\}}|g(x,t,u_n,\nabla u_n)|\,\mathrm{d}x\mathrm{d}t\\ \le&L_1(m)\int_{E\cap\{|u_n|\le m\}}[L_2(x,t)+|\nabla u_n|^p]\,\mathrm{d}x\mathrm{d}t+\int_{E\cap\{|u_n|>m\}}|g(x,t,u_n,\nabla u_n)|\,\mathrm{d}x\mathrm{d}t\\ \le&L_1(m)\int_{E\cap\{|u_n|\le m\}}[L_2(x,t)+|\nabla T_m(u_n)|^p]\,\mathrm{d}x\mathrm{d}t+\int_{E\cap\{|u_n|>m\}}|g(x,t,u_n,\nabla u_n)|\,\mathrm{d}x\mathrm{d}t\\ =&K_1+K_2.\end{aligned} \tag{3.41}$$

For fixed m, we get

$$K_1\le L_1(m)\int_E[L_2(x,t)+|\nabla T_m(u_n)|^p]\,\mathrm{d}x\mathrm{d}t,$$

which is thus small uniformly in n for m fixed when the measure of E is small (recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$). We now discuss the behavior of the second integral of the right hand side of (3.41), let ψ_m be a function such that

$$\begin{cases} \psi_m(s)=0, & \text{if} \quad |s|\le m-1, \\ \psi_m(s)=\text{sign}(s), & \text{if} \quad |s|\ge m, \\ \psi'_m(s)=1, & \text{if} \quad m-1<|s|<m. \end{cases} \tag{3.42}$$

We chooses $\psi_m(u_n)$ as a test function for $m>1$ in (3.2), we obtain

$$\begin{aligned} &\Big[\int_\Omega B_m^n(x,u_n)\mathrm{d}x\Big]_0^T+\int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n\psi'_m(u_n)\mathrm{d}x\mathrm{d}t \\ &\qquad +\int_{Q_T} g_n(x,t,u_n,\nabla u_n)\psi_m(u_n)\mathrm{d}x\mathrm{d}t+\int_{Q_T} H_n(x,t,\nabla u_n)\psi_m(u_n)\mathrm{d}x\mathrm{d}t \\ &=\int_0^T \langle f\,;\,\psi_m(u_n)\rangle\mathrm{d}t, \end{aligned}$$

where

$$B_m^n(x,r)=\int_0^r \frac{\partial b_n(x,s)}{\partial s}\psi_m(s)\mathrm{d}s,$$

which implies, since $B_m^n(x,r)\ge 0$ and using (2.5), Hölder's inequality

$$\begin{aligned} &\int_{\{m-1\le|u_n|\}} |g_n(x,t,u_n,\nabla u_n)|\mathrm{d}x\mathrm{d}t \\ &\le \int_E |H_n(x,t,\nabla u_n)|\mathrm{d}x\mathrm{d}t+\|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\Big(\int_{\{m-1\le|u_n|\le m\}}|\nabla u_n|^p\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p}}. \end{aligned}$$

By (3.12), we have

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>m-1\}}|g_n(x,t,u_n,\nabla u_n)|\mathrm{d}x\mathrm{d}t=0.$$

Thus we proved that the second term of the right hand side of (3.41) is also small, uniformly in n and in E when m is sufficiently large. Which shows that $g_n(x,t,u_n,\nabla u_n)$ and $H_n(x,t,\nabla u_n)$ are uniformly equi-integrable in Q_T as required, we conclude that

$$\begin{cases} H_n(x,t,\nabla u_n)\to H(x,t,\nabla u), & \text{strongly in } L^1(Q_T), \\ g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u), & \text{strongly in } L^1(Q_T). \end{cases} \tag{3.43}$$

Step 4: In this step we prove that u satisfies (2.11).

Lemma 3.3. *The limit u of the approximate solution u_n of* (3.2) *satisfies*

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\cdot\nabla u\mathrm{d}x\mathrm{d}t=0.$$

Proof. Note that for any fixed $m\geq 0$, one has

$$\begin{aligned}&\int_{\{m\leq|u_n|\leq m+1\}}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\mathrm{d}t\\=&\int_{Q_T}a(x,t,u_n,\nabla u_n)\cdot(\nabla T_{m+1}(u_n)-\nabla T_m(u_n))\mathrm{d}x\mathrm{d}t\\=&\int_{Q_T}a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla T_{m+1}(u_n)\mathrm{d}x\mathrm{d}t\\&\quad-\int_{Q_T}a(x,t,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

According to (3.37) and (3.39), one can pass to the limit as $n\to+\infty$ for fixed $m\geq 0$, to obtain

$$\begin{aligned}&\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,\mathrm{d}x\mathrm{d}t\\=&\int_{Q_T}a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\cdot\nabla T_{m+1}(u)\mathrm{d}x\mathrm{d}t\\&\quad-\int_{Q_T}a(x,t,T_m(u),\nabla T_m(u))\cdot\nabla T_m(u_n)\mathrm{d}x\mathrm{d}t\\=&\int_{\{m\leq|u_n|\leq m+1\}}a(x,t,u,\nabla u)\cdot\nabla u\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.44}$$

Taking the limit as $m\to+\infty$ in (3.44) and using the estimate (3.21), we show that u satisfies (2.11) and the proof is complete. ☐

Step 5: In this step we prove that u satisfies (2.12) and (2.13).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let M be a positive real number such that support of S' is a subset of $[-M,M]$. Pointwise multiplication of the approximate equation (3.2) by $S'(u_n)$ leads to

$$\begin{aligned}&\frac{\partial B_S^n(x,u_n)}{\partial t}-\mathrm{div}\Big(S'(u_n)a(x,t,u_n,\nabla u_n)\Big)+S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n\\&\quad+S'(u_n)\Big(g_n(x,t,u_n,\nabla u_n)+H_n(x,t,\nabla u_n)\Big)=fS'(u_n),\quad\text{in }\mathcal{D}'(Q_T).\end{aligned}\tag{3.45}$$

Passing to the limit, as n tends to $+\infty$, we have

• Since S is bounded and continuous, $u_n\to u$ a.e. in Q_T implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e. in Q_T and L^∞ weak*. Then $\partial B_S^n(x,u_n)/\partial t$ converges to $\partial B_S(x,u)/\partial t$ in $\mathcal{D}'(Q_T)$ as n tends to $+\infty$.

• Since $\mathrm{supp}(S')\subset[-M,M]$, we have for $n\geq M$,

$$S'(u_n)a_n(x,t,u_n,\nabla u_n)=S'(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n)),\ \text{ a.e. in }\ Q_T.$$

The pointwise convergence of u_n to u and (3.39) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(x,t,u_n,\nabla u_n) \rightharpoonup S'(u)a(x,t,T_M(u),\nabla T_M(u)), \quad \text{in } (L^{p'}(Q_T))^N, \tag{3.46}$$

as n tends to $+\infty$. $S'(u)a(x,t,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,t,u,\nabla u)$ in Eq. (2.12).

• Regarding the 'energy' term, we have

$$S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a(x,t,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n), \quad \text{a.e. in } Q_T.$$

The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (3.39) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that $S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n$ converges to $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u)$ weakly in $L^1(Q_T)$. Recall that

$$S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u) = S''(u)a(x,t,u,\nabla u)\nabla u, \quad \text{a.e. in } Q_T.$$

• Since $\text{supp}(S') \subset [-M,M]$, by (3.43), we have

$$S'(u_n)\Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big) \rightarrow S'(u)\Big(g(x,t,u,\nabla u) + H(x,t,\nabla u)\Big)$$

strongly in $L^1(Q_T)$, as n tends to $+\infty$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (3.45) and to conclude that u satisfies (2.12).

It remains to show that $B_S(x,u)$ satisfies the initial condition (2.13). To this end, firstly remark that, S being bounded, $B_S^n(x,u_n)$ is bounded in $L^\infty(Q_T)$. Secondly, (3.45) and the above considerations on the behavior of the terms of this equation show that $\partial B_S^n(x,u_n)/\partial t$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type lemma (see, e.g, [29]) implies that $B_S^n(x,u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. It follows that on the one hand, $B_S^n(x,u_n)(t=0) = B_S^n(x,0) = 0$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that $B_S(x,u)(t=0) = 0$ in Ω.

As a conclusion, steps 1–5 complete the proof of Theorem 3.1. □

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