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Neutral Functional Partial Differential Equations Driven by Fractional Brownian Motion with Non-Lipschitz Coefficients

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Abstract. Under a non-Lipschitz condition being considered as a generalized case of Lipschitz condition, the existence and uniqueness of mild solutions to neutral stochastic functional differential equations driven by fractional Brownian motion with Hurst parameter $1/2<H<1$ are investigated. Some known results are generalized and improved.

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1 Introduction

Recently, the theory for stochastic differential equations (without delay) driven by a fractional Brownian motion (fBm) has been studied intensively (see e.g. [1–6] and the references therein).

As for the stochastic functional differential equations driven by a fBm, even much less has been done, as far as we know, there exists only a few papers published in this field. In [7], the authors studied the existence and regularity of the density by using the Skorohod integral based on Malliavin calculus. In [8], Neuenkirch et al. studied the problem by using rough path analysis. In [9], Ferrante and Rovira studied the existence and convergence when the delay goes to zero by using the Riemann-Stieltjes integral. Using

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also the Riemann-Stieltjes integral, [10] proved the existence and uniqueness of mild solution in infinite dimensional space. In infinite dimensional space, [11] have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions by using Wiener integral. Very recently, [12] first investigated the following neutral stochastic functional differential equations driven by a fractional Brownian motion under the global Lipschitz and linear growth condition

$$\begin{cases} \mathrm{d}[x(t)+g(t,x(t-r(t)))]=[Ax(t)+f(t,x(t-\rho(t)))]\mathrm{d}t+\sigma(t)\mathrm{d}B^H(t), \quad 0\le t\le T, \\ x(t)=\varphi(t), \quad t\in[-\tau,0]. \end{cases} \tag{1.1}$$

Where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\ge 0}$, in a Hilbert space X, B^H is a Q-fractional Brownian motion on a real and separable Hilbert space Y, $r,\rho:[0,T]\to[0,\tau]$ $(\tau>0)$ are continuous, $f,g:[0,T]\times X\to X$, $\sigma:[0,T]\to\mathcal{L}_2^0(Y,X)$ are appropriate functions and $\varphi\in C([-\tau,0];L^2(\Omega,X))$. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see Section 2).

Unfortunately, for many practical situations, the nonlinear terms do not obey the global Lipschitz and linear growth condition, even the local Lipschitz condition. Motivated by the above papers, in this paper, we aim to extend the existence and uniqueness of mild solutions to cover a class of more general neutral stochastic functional differential equations driven by a fractional Brownian motion with Hurst parameter $1/2<H<1$ under a non-Lipschitz condition, with the Lipschitz condition being regarded as a special case, and a weakened linear growth condition.

The rest of this paper is organized as follows. In Section 2, we introduce some notations, concepts, and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and we recall some preliminary results about analytic semigroups and fractional power associated to its generator. In Section 3, the existence and uniqueness of mild solutions are proved.

2 Preliminaries

In this section we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian motion. In addition, we also recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators which will be used throughout the whole of this paper.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{\beta^H(t),\ t\in[0,T]\}$ be the one-dimensional fractional Brownian motion with Hurst parameter $H\in(1/2,1)$. This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(t,s)=\mathbb{E}(\beta_t^H\beta_s^H)=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}).$$

Moreover β^H has the following Wiener integral representation:

$$\beta^H(t)=\int_0^t K_H(t,s)\mathrm{d}\beta(s),$$

where $\beta=\{\beta(t):t\in[0,T]\}$ is a Wiener process, and $K_H(t,s)$ is the kernel given by

$$K_H(t,s)=c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\mathrm{d}u$$

for $t>s$, where $c_H=\sqrt{H(2H-1)/\beta(2-2H,H-\frac{1}{2})}$ and $\beta(\cdot,\cdot)$ denotes the Beta function. We put $K_H(t,s)=0$ if $t\leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of set of indicator functions $\{I_{[0,t]},\ t\in[0,T]\}$ with respect to the scalar product

$$\langle I_{[0,t]},I_{[0,s]}\rangle_{\mathcal{H}}=R_H(t,s).$$

The mapping $I_{[0,t]}\to\beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

We recall that for $\psi,\varphi\in\mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle\psi,\varphi\rangle_{\mathcal{H}}=H(2H-1)\int_0^T\int_0^T\psi(s)\varphi(t)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0,T])$ defined by

$$(K_H^*)(s)=\int_s^T\varphi(r)\frac{\partial K}{\partial r}(r,s)\mathrm{d}r.$$

We refer to [13] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$. Moreover for any $\varphi\in\mathcal{H}$, we have

$$\beta^H(\varphi)=\int_0^T(K_H^*\varphi)(t)\mathrm{d}\beta(t).$$

It follows from [13] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|}:=\alpha_H\int_0^T\int_0^T|\psi(s)||\psi(t)||s-t|^{2H-2}\mathrm{d}s\mathrm{d}t<\infty,$$

where $\alpha_H=H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following conclusions [13].

Lemma 2.1.

$$L^2([0,T]) \subseteq L^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\psi \in L^2([0,T])$, *we have*

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1} \int_0^T |\psi(s)|^2 \mathrm{d}s.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in Y, X and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\mathrm{tr}Q = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \geq 0$ $(n=1,2\cdots)$ are non-negative real numbers and $\{e_n\}$ $(n=1,2\cdots)$ is a complete orthonormal basis in Y. We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),$$

where β^H_n are real, independent fBm. This process is a Y-valued Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t),x\rangle \langle B^H(s),y\rangle = R(s,t)\langle Q(x),y\rangle, \qquad \text{for all}\, x,y \in Y \text{ and}\, t,s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}^0_2 := \mathcal{L}^0_2(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi: Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}^0_2} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}^0_2$ equipped with the inner product $\langle \varphi,\psi\rangle_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Now, let $\phi(s), s \in [0,T]$ be a function with values in $\mathcal{L}^0_2(Y,X)$. The Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)\mathrm{d}B^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s)e_n \mathrm{d}\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} K^*_H(\phi e_n)(s)\mathrm{d}\beta_n(s), \tag{2.1}$$

where β_n is the standard Brownian motion used to present β^H_n.

Now we end this subsection by stating the following result in [12].

Lemma 2.2. *If* $\psi: [0,T] \to \mathcal{L}^0_2(Y,X)$ *satisfies* $\int_0^T \|\psi(s)\|^2_{\mathcal{L}^0_2} \mathrm{d}s < \infty$ *then the above sum in* (2.1) *is well defined as a X-valued random variable and we have*

$$\mathbb{E}\left\| \int_0^t \psi(s)\mathrm{d}B^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}^0_2} \mathrm{d}s.$$

Now we turn to state some notations and basic facts about the theory of semi-groups and fractional power operators. Let $A:D(A)\to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on X. For the theory of strongly continuous semigroup, we refer to Pazy [14]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $M\geq 1$ and $\lambda\in\mathbb{R}$ such that $\|S(t)\|\leq Me^{\lambda t}$ for every $t\geq 0$. If $(S(t))_{t\geq 0}$ is a uniformly bounded and analytic semigroup such that $0\in\rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^{\alpha}$ for $0<\alpha\leq 1$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, the subspace $D(-A)^{\alpha}$ is dense in X, and the expression

$$\|h\|_{\alpha}=\|(-A)^{\alpha}h\|$$

defines a norm in $D(-A)^{\alpha}$. If X_{α} represents the space $D(-A)^{\alpha}$ endowed with the norm $\|\cdot\|_{\alpha}$, then the following properties are well known (cf. Pazy [14, Theorem 6.13]).

Lemma 2.3. *Suppose that the preceding conditions are satisfied.*

(1) *Let* $0<\alpha\leq 1$. *Then* X_{α} *is a Banach space.*

(2) *If* $0<\beta\leq\alpha$ *then the injection* $X_{\alpha}\hookrightarrow X_{\beta}$ *is continuous.*

(3) *For every* $0<\beta\leq 1$ *there exists* $M_{\beta}>0$ *such that*

$$\|(-A)^{\beta}S(t)\|\leq M_{\beta}t^{-\beta}e^{-\lambda t},\qquad t>0,\ \ \lambda>0.$$

We also need the following Lemma 2.4.

Lemma 2.4. (Caraballo [15], Lemma 1) *For* $u,v\in X$, *and* $0<c<1$,

$$\|u\|\leq\frac{1}{1-c}\|u-v\|^{2}+\frac{1}{c}\|v\|^{2}.$$

3 Existence and uniqueness

In this section we study the existence and uniqueness of mild solution for Eq. (1.1). For this equation we assume that the following conditions hold.

(H1) A is the infinitesimal generator of an analytic semigroup, $S(t)_{t\geq 0}$, of bounded linear operators on X. Further, to avoid unnecessary notations, we suppose that $0\in\rho(A)$, and that, see Lemma 2.3,

$$\|S(t)\|\leq M \text{ and } \|(-A)^{1-\beta}S(t)\|\leq\frac{M_{1-\beta}}{t^{1-\beta}},$$

for some constants M, M_{β} and every $t\in[0,T]$.

(H2) The function f satisfies the following non-Lipschitz condition: for any $x,y \in X$ and $t \geq 0$,

$$\|f(t,x)-f(t,y)\|^2 \leq \kappa(\|x-y\|^2),$$

where κ is a concave nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\kappa(0)=0$, $\kappa(u)>0$ and $\int_{0^+} \mathrm{d}u/\kappa(u)=\infty$, e.g., $\kappa \sim u^\alpha$, $1/2<\alpha<1$. We further assume that there is an $M'>0$ such that $\sup_{0\leq t\leq T}\|f(t,0)\| \leq M'$.

(H3) There exist constants $1/2<\beta\leq 1$, $K_1 \geq 0$ such that the function g is X_β-valued and satisfies for any $x,y \in X$ and $t \geq 0$,

$$\|(-A)^\beta g(t,x)-(-A)^\beta g(t,y)\| \leq K_1\|x-y\|,$$

and

$$\|(-A)^{-\beta}\|K_1<1.$$

We further assume that $g(t,0)\equiv 0$ for $t\geq 0$ and the function $(-A)^\beta$ is continuous in the quadratic mean sense:

$$\lim_{t\to s}\mathbb{E}\|(-A)^\beta g(t,x(t))-(-A)^\beta g(s,x(s))\|^2=0.$$

(H4) The function $\sigma:[0,+\infty)\to\mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0}\mathrm{d}s<\infty, \qquad \forall\ T>0.$$

Definition 3.1. *A X-valued process $x(t)$ is called a mild solution of (1.1) if*

$$x\in C([-\tau,T],\mathbb{L}^2(\Omega,X)),\ \text{for } t\in[-\tau,0], \quad x(t)=\varphi(t),\ \text{and for } t\in[0,T],$$

satisfies

$$\begin{aligned} x(t)=&S(t)(\varphi(0)+g(0,\varphi(-r(0))))-g(t,x(t-r(t)))\\ &-\int_0^t AS(t-s)g(s,x(s-r(s)))\mathrm{d}s+\int_0^t S(t-s)f(s,x(s-\rho(s)))\mathrm{d}s\\ &+\int_0^t S(t-s)\sigma(s)\mathrm{d}B^H(s)\ \mathbb{P}-\text{a.s.}\end{aligned}$$

Lemma 3.1. ([16, Theorem 1.8.2]) *Let $T>0$ and $c>0$. Let $\kappa:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous nondecreasing function such that $\kappa(t)>0$ for all $t>0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0,T]$, and let $v(\cdot)$ be a nonnegative integrable function on $[0,T]$. If*

$$u(t)\leq c+\int_0^t v(s)\kappa(u(s))\mathrm{d}s, \qquad \text{for all } 0\leq t\leq T.$$

Then

$$u(t)\leq J^{-1}\left(J(c)+\int_0^t v(s)\mathrm{d}s\right),$$

holds for all such $t\in[0,T]$ *that*

$$J(c)+\int_0^t v(s)\mathrm{d}s\in \mathrm{Dom}(J^{-1}),$$

where

$$J(r)=\int_0^r \mathrm{d}s/\kappa(s),\qquad \text{on } r>0,$$

and J^{-1} *is the inverse function of* J. *In particular, if, moreover,* $c=0$ *and* $\int_{0^+}\mathrm{d}s/\kappa(s)=\infty$, *then* $u(t)=0$ *for all* $t\in[0,T]$.

To complete our main results, we need to prepare several lemmas which will be utilize in the sequel.

Note that $g(t,0)\equiv 0$ and

$$\|(-A)^\beta g(t,x)-(-A)^\beta g(t,y)\|\leq K_1\|x-y\|.$$

Then we easily get that $\|(-A)^\beta g(t,x)\|^2\leq K_1^2\|x\|^2$. Thus, by [12, Theorem 5], we can introduce the following successive approximating procedure: for each integer $n=1,2,3,\cdots$,

$$\begin{aligned} x^n(t)=&S(t)[\varphi(0)+g(0,\varphi(-r(0)))]-g(t,x^n(t-r(t)))\\ &-\int_0^t AS(t-s)g(s,x^n(s-r(s)))\mathrm{d}s+\int_0^t S(t-s)f(s,x^{n-1}(s-\rho(s)))\mathrm{d}s\\ &+\int_0^t S(t-s)\sigma(s)\mathrm{d}B^H(s) \end{aligned}\tag{3.1}$$

and for $n=0$,

$$x^0(t)=S(t)\varphi(0),\qquad t\in[0,T].$$

While for $n=1,2,\cdots$,

$$x^n(t)=\varphi(t),\qquad t\in[-\tau,0].$$

Lemma 3.2. *Let the hypothesis* (H1)-(H4) *hold. Then there is a positive constant* C_1, *which is independent of* $n\geq 1$, *such that for any* $t\in[0,T]$

$$\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)\|^2\leq C_1.\tag{3.2}$$

Proof. For $0\le t\le T$, it follows easily from (3.1) that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|x^n(t)+g(t,x^n(t-r(t)))\|^2\\
\le&4\mathbb{E}\sup_{0\le t\le T}\|S(t)[\varphi(0)+g(0,\varphi(-r(0)))]\|^2\\
&+4\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t AS(t-s)g(s,x^n(s-r(s)))\mathrm{d}s\right\|^2\\
&+4\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t S(t-s)f(s,x^{n-1}(s-\rho(s)))\mathrm{d}s\right\|^2\\
&+4\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t S(t-s)\sigma(s)\mathrm{d}B^H(s)\right\|^2\\
=:&4(I_1+I_2+I_3+I_4).
\end{aligned}\tag{3.3}$$

Note from [14] that $(-A)^{-\beta}$ for $0<\beta\le 1$ is a bounded operator. Employing the assumption (H3), it follows that

$$\begin{aligned}
I_1\le&2\left[\mathbb{E}\sup_{0\le t\le T}\|S(t)\varphi(0)\|^2+\mathbb{E}\sup_{0\le t\le T}\|S(t)(-A)^{-\beta}(-A)^{\beta}g(0,\varphi(-r(0)))\|^2\right]\\
\le&2(1+K_1^2\|(-A)^{-\beta}\|^2)\|\varphi\|_C^2.
\end{aligned}\tag{3.4}$$

Applying the Hölder's inequality and taking into account Lemma 2.3 as well as (H3), and the fact that $1/2<\beta<1$, we obtain

$$\begin{aligned}
I_2=&\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t(-A)^{1-\beta}S(t-s)(-A)^{\beta}g(s,x^n(s-r(s)))\mathrm{d}s\right\|^2\\
\le&\frac{T^{2\beta-1}}{2\beta-1}M_{1-\beta}^2\mathbb{E}\sup_{0\le t\le T}\int_0^t\|(-A)^{\beta}g(s,x^n(s-r(s)))\|^2\mathrm{d}s\\
\le&\frac{T^{2\beta-1}}{2\beta-1}M_{1-\beta}^2K_1^2\mathbb{E}\sup_{0\le t\le T}\int_0^t\|x^n(s-r(s))\|^2\mathrm{d}s.
\end{aligned}\tag{3.5}$$

On the other hand, in view of (H2), we obtain that

$$\begin{aligned}
I_3\le&T\mathbb{E}\sup_{0\le t\le T}\int_0^t\|S(t-s)[f(s,x^{n-1}(s-\rho(s)))-f(s,0)+f(s,0)]\|^2\mathrm{d}s\\
\le&2TM^2\left[M'^2T+\mathbb{E}\sup_{0\le t\le T}\int_0^t\kappa(\|x^{n-1}(s-\rho(s))\|^2)\right].
\end{aligned}\tag{3.6}$$

Next, by Lemma 2.2, we have

$$I_4\le 2M^2HT^{2H-1}\int_0^T\|\sigma(s)\|_{\mathcal{L}_2^0}^2\mathrm{d}s<\infty.\tag{3.7}$$

Since $\kappa(u)$ is concave on $u\geq 0$, there is a pair of positive constants a, b such that

$$\kappa(u)\leq a+bu.$$

Putting (3.4)-(3.7) into (3.3) yields that, for some positive constants C_2 and C_3,

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)+g(t,x^n(t-r(t)))\|^2\leq & C_2+C_3\mathbb{E}\sup_{0\leq t\leq T}\int_0^t\|x^n(s-r(s))\|^2\mathrm{d}s\\ &+C_3\mathbb{E}\sup_{0\leq t\leq T}\int_0^t\|x^{n-1}(s-r(s))\|^2\mathrm{d}s. \qquad (3.8)\end{aligned}$$

While, for $\|(-A)^{-\beta}\|K_1<1$, by Lemma 2.4,

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)\|^2\leq & \frac{1}{1-K_1\|(-A)^{-\beta}\|}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)+g(t,x^n(t-r(t)))\|^2\\ &+\frac{1}{K_1\|(-A)^{-\beta}\|}\mathbb{E}\sup_{0\leq t\leq T}\|g(t,x^n(t-r(t)))\|^2\\ \leq & \frac{1}{1-K_1\|(-A)^{-\beta}\|}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)+g(t,x^n(t-r(t)))\|^2\\ &+K_1\|(-A)^{-\beta}\|\mathbb{E}\|\varphi\|_C^2+K_1\|(-A)^{-\beta}\|\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)\|^2,\end{aligned}$$

which further implies that

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)\|^2\leq & \frac{1}{(1-K_1\|(-A)^{-\beta}\|)^2}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)+g(t,x^n(t-r(t)))\|^2\\ &+\frac{K_1\|(-A)^{-\beta}\|}{1-K_1\|(-A)^{-\beta}\|}\mathbb{E}\|\varphi\|_C^2.\end{aligned}$$

Thus, by (3.8) we have

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)\|^2\leq & \left[\frac{K_1\|(-A)^{-\beta}\|}{1-K_1\|(-A)^{-\beta}\|}+\frac{2C_3r}{(1-K_1\|(-A)^{-\beta}\|)^2}\right]\mathbb{E}\|\varphi\|_C^2\\ &+\frac{C_3}{(1-K_1\|(-A)^{-\beta}\|)^2}\left[\int_0^T\mathbb{E}\sup_{0\leq r\leq s}\|x^{n-1}(r)\|\mathrm{d}s\right.\\ &\left.+\int_0^T\mathbb{E}\sup_{0\leq r\leq s}\|x^n(r)\|\mathrm{d}s\right]+\frac{C_2}{(1-K_1\|(-A)^{-\beta}\|)^2}.\end{aligned}$$

Observing that

$$\max_{1\leq n\leq k}\mathbb{E}\sup_{0\leq t\leq T}\|x^{n-1}(t)\|^2\leq\mathbb{E}\|\varphi\|_C^2+\max_{1\leq n\leq k}\mathbb{E}\sup_{0\leq t\leq T}\|x^n(t)\|^2,$$

we then derive that, for some positive constants C_4, C_5,

$$\max_{1\le n\le k}\mathbb{E}\sup_{0\le t\le T}\|x^n(t)\|^2\le C_4+C_5\mathbb{E}\int_0^T\max_{1\le n\le k}\mathbb{E}\sup_{0\le r\le s}\|x^n(s)\|^2\mathrm{d}s.$$

Now, an application of the well-known Gronwall's inequality yields that

$$\max_{1\le n\le k}\mathbb{E}\sup_{0\le t\le T}\|x^n(t)\|^2\le C_4+e^{C_5T}.$$

The required assertion (3.2) is obtained since k is arbitrary. $\square$

Lemma 3.3. *Let the condition* (H1)-(H4) *be satisfied. For* $1/2<\beta\le 1$, *we further assume that*

$$\frac{2K_1^2M_{1-\beta}^2\gamma^{-2\beta}\Gamma(2\beta-1)}{1-K_1\|(-A)^{-\beta}\|}+K_1\|(-A)^{-\beta}\|<1, \tag{3.9}$$

where $\Gamma(\cdot)$ *is the Gamma function and* $M_{1-\beta}$ *is a constant in Lemma* 2.3. *Then there exists a positive constant* $\overline{C}$ *such that, for all* $0\le t\le T$ *and* n, $m\ge 1$,

$$\mathbb{E}\sup_{0\le s\le t}\|x^{n+m}(s)-x^n(s)\|^2\le\overline{C}\int_0^t\kappa\left(\mathbb{E}\sup_{0\le u\le s}\|x^{n+m-1}(u)-x^{n-1}(u)\|^2\right)\mathrm{d}s. \tag{3.10}$$

Proof. From (3.1), it is easy to see that for any $0\le t\le T$

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\|x^{n+m}(s)-x^n(s)+g(s,x^{n+m}(s)-g(s,x^n(s))\|^2\\
\le&2\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s AS(s-u)[g(u,x^{m+n}(u-r(u)))-g(u,x^n(u-r(u)))]\mathrm{d}u\right\|^2\\
&\quad+2\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S(s-u)[f(u,x^{m+n-1}(u-\rho(u)))-f(u,x^{n-1}(u-\rho(u)))]\mathrm{d}u\right\|^2.
\end{aligned}$$

Following from the proof of Lemma 3.2, there exists a positive C_6 satisfying

$$\begin{aligned}
&2\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s S(s-u)[f(u,x^{m+n-1}(u-\rho(u)))-f(u,x^{n-1}(u-\rho(u)))]\mathrm{d}u\right\|^2\\
\le&C_6\int_0^t\kappa\left(\sup_{0\le u\le s}\|x^{m+n-1}(u)-x^{n-1}(u)\|^2\right)\mathrm{d}s,
\end{aligned}$$

the last inequality holds from the Jensen's inequality. Now, by the condition (H3), Lemma

2.3 and Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\left\|\int_0^s AS(s-u)[g(u,x^{m+n}(u-r(u)))-g(u,x^n(u-r(u)))]\mathrm{d}u\right\|^2\\
\le&\mathbb{E}\sup_{0\le s\le t}(\int_0^s\|(-A)^{1-\beta}S(s-u)[(-A)^\beta g(u,x^{m+n}(u-r(u)))\\
&\quad-(-A)^\beta g(u,x^n(u-r(u)))]\|\mathrm{d}u)^2\\
\le&\mathbb{E}\sup_{0\le s\le t}\left(\int_0^s K_1\frac{M_{1-\beta}e^{-\gamma(s-u)}}{(s-u)^{1-\beta}}\|x^{m+n}(u-r(u))-x^n(u-r(u))\|\right)^2\\
\le&\mathbb{E}[\sup_{0\le s\le t}\int_0^s K_1^2\frac{M_{1-\beta}^2e^{-\gamma(s-u)}}{(s-u)^{1-\beta}}\mathrm{d}u\int_0^s e^{-\gamma(s-u)}\|x^{m+n}(u-r(u))\\
&\quad-x^n(u-r(u))\|^2\mathrm{d}u]\\
\le&K_1^2M_{1-\beta}^2\gamma^{1-2\beta}\Gamma(2\beta-1)\mathbb{E}\sup_{0\le s\le t}\int_0^s e^{-\gamma(s-u)}\|x^{m+n}(u-r(u))-x^n(u-r(u))\|^2\mathrm{d}u\\
\le&K_1^2M_{1-\beta}^2\gamma^{-2\beta}\Gamma(2\beta-1)\mathbb{E}\sup_{0\le s\le t}\|x^{n+m}(s)-x^n(s)\|^2.
\end{aligned}$$

On the other hand, Lemma 2.4 and (H3) give that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\|x^{n+m}(s)-x^n(s)\|^2\\
\le&\frac{1}{1-K_1\|(-A)^{-\beta}\|}\mathbb{E}\sup_{0\le s\le t}\|x^{n+m}(s)-x^n(s)+g(s,x^{n+m}(s))-g(s,x^n(s))\|^2\\
&\quad+K_1\|(-A)^{-\beta}\|\mathbb{E}\sup_{0\le s\le t}\|x^{n+m}(s)-x^n(s)\|^2. \qquad (3.11)
\end{aligned}$$

So the desired assertion (3.10) follows from (3.11). □

We can now state the main result of this paper.

Theorem 3.1. *Under the conditions of Lemma* 3.3*, then Eq.* (1.1) *admits a unique mild solution.*

Proof. Uniqueness: Denote by $x(t)$ and $\overline{x}(t)$ the mild solutions to (1.1). In the same way as Lemma 3.3 was done, we can show that for some $\overline{K}>0$

$$\mathbb{E}\sup_{0\le s\le t}\|x(s)-\overline{x}(s)\|^2\le\overline{K}\int_0^t\kappa\left(\mathbb{E}\sup_{0\le u\le s}\|x(r)-\overline{x}(r)\|\right)\mathrm{d}s.$$

This, together with Lemma 3.1, leads to

$$\mathbb{E}\sup_{0\le s\le t}\|x(s)-\overline{x}(s)\|^2=0,$$

which further implies $x(t)=\overline{x}(t)$ almost surely for any $0\leq t\leq T$.

Existence: Following the proof of Lemma 3.3, there exists a positive $\overline{C}$ such that, for all $0\leq t\leq T$ and n, $m\geq 1$,

$$\mathbb{E}\sup_{0\leq s\leq t}\|x^{n+1}(s)-x^{m+1}(s)\|^2\leq\overline{C}\int_0^t\kappa\left(\mathbb{E}\sup_{0\leq u\leq s}\|x^n(u)-x^m(u)\|^2\right)\mathrm{d}s.$$

Integrating both sides and applying Jensen's inequality gives that

$$\begin{aligned}&\int_0^t\mathbb{E}\sup_{0\leq l\leq s}\|x^{n+1}(l)-x^{m+1}(l)\|^2\mathrm{d}s\\ \leq&\overline{C}\int_0^t\int_0^s\kappa\left(\mathbb{E}\sup_{0\leq u\leq l}\|x^n(u)-x^m(u)\|^2\right)\mathrm{d}l\mathrm{d}s\\ =&\overline{C}\int_0^t s\int_0^s\kappa\left(\mathbb{E}\sup_{0\leq u\leq l}\|x^n(u)-x^m(u)\|^2\right)\frac{1}{s}\mathrm{d}l\mathrm{d}s\\ \leq&\overline{C}t\int_0^t\kappa\left(\int_0^s\mathbb{E}\sup_{0\leq u\leq l}\|x^n(u)-x^m(u)\|^2\frac{1}{s}dl\right)\mathrm{d}s.\end{aligned}$$

Then

$$h_{n+1,m+1}(t)\leq\overline{C}\int_0^t\kappa(h_{n,m}(s))\mathrm{d}s,$$

where

$$h_{n,m}(t)=\frac{\int_0^t\mathbb{E}\sup_{0\leq l\leq s}\|x^{n+1}(l)-x^{m+1}(l)\|^2\mathrm{d}s}{t}.$$

While by Lemma 3.2, it is easy to see that

$$\sup_{n,m}h_{n,m}(t)<\infty.$$

So letting $h(t):=\limsup_{n,m\to\infty}h_{n,m}(t)$ and taking into account the Fatou's lemma, we yield that

$$h(t)\leq\overline{C}\int_0^t\kappa(h(s)).$$

Now, applying the Lemma 3.1 immediately reveals $h(t)=0$ for any $t\in[0,T]$. This further means $\{x^n(t),n\in\mathbb{N}$ is a Cauchy sequence in L^2. So there is a $x\in L^2$ such that

$$\lim_{n\to\infty}\int_0^T\mathbb{E}\sup_{0\leq s\leq t}\|x^n(s)-x(s)\|^2\mathrm{d}s=0.$$

In addition, by Lemma 3.2, it is easy to follow that $\mathbb{E}\|x(t)\|^2 \le C_1$. In what follows, we claim that $x(t)$ is a mild solution to (1.1). On one hand, by (H3),

$$\begin{aligned}
&\mathbb{E}\|g(t,x^n(t-r(t)))-g(t,x(t-r(t)))\|^2\\
=&\mathbb{E}\|(-A)^{-\beta}[(-A)^{\beta}g(t,x^n(t-r(t)))-(-A)^{\beta}g(t,x(t-r(t)))]\|^2\\
\le&\|(-A)^{-\beta}\|^2K_1^2\mathbb{E}\sup_{0\le s\le t}\|x^n(s)-x(s)\|^2\to 0,
\end{aligned}$$

whenever $n\to\infty$. On the other hand, by $(H3)$ and Lemma 2.3, compute for $t\in[0,T]$

$$\begin{aligned}
&\mathbb{E}\|\int_0^t AS(t-s)[g(t,x^n(t-r(t)))-g(t,x(t-r(t)))]\mathrm{d}s\|^2\\
=&\mathbb{E}\int_0^t\|(-A)^{1-\beta}S(t-s)[(-A)^{\beta}g(t,x^n(t-r(t)))-(-A)^{\beta}g(t,x(t-r(t)))]\mathrm{d}s\|^2\\
\le&\frac{T^{2\beta-1}}{2\beta-1}M_{1-\beta}^2\int_0^T\mathbb{E}\sup_{0\le u\le s}\|x^n(u)-x(u)\|^2\mathrm{d}s\\
\to&0,\qquad \text{as } n\to\infty.
\end{aligned}$$

While, applying (H2), the Hölder's inequality and [17, Theorem 1.2.6] and letting $n\to\infty$, we can also claim that for $t\in[0,T]$

$$\mathbb{E}\left\|\int_0^t S(t-s)[f(t,x^n(t-\rho(t)))-f(t,x(t-\rho(t)))]\mathrm{d}s\right\|^2\to 0.$$

Hence, taking limits on both sides of (3.1),

$$\begin{aligned}
x(t)=&S(t)[\varphi(0)+g(0,\varphi(-r(0)))]-g(t,x(t-r(t)))\\
&-\int_0^t AS(t-s)g(s,x(s-r(s)))\mathrm{d}s+\int_0^t S(t-s)f(s,x(s-\rho(s)))\mathrm{d}s\\
&+\int_0^t S(t-s)\sigma(s)\mathrm{d}B^H(s).
\end{aligned}$$

This certainly demonstrates by the Definition 3.1 that $x(t)$ is a mild solution to (1.1) on the interval $[0,T]$. □

Remark 3.1. If $H=1/2$, then $B_Q^H(t)$ is standard Q-Bm. Consequently, our results can be reduced to some results in [18]. In other words, in this special case, we generalize [18].

Remark 3.2. In this work, we consider the existence and uniqueness of mild solutions to SNFDEs driven by a fractional Brownian motion under a non-Lipschitz condition with the Lipschitz condition being regarded as special case and a weakened linear growth assumption. Therefore, some of the results in [12] are improved to cover a class of more general SNFDEs driven by a fractional Brownian motion.

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