JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **28**, No. 1, pp. 9-22

doi: 10.4208/jpde.v28.n1.2
March 2015

Wave Kernels with Bi-Inverse Square Potentials on Euclidean Plane

MOUSTAPHA Mohamed Vall Ould*

Université des Sciences, de Technologie et de la Médecine (USTM), Faculté des Sciences et Techniques.
Unité de recherche: Analyse EDP et Modélisation, Facult des Sciences et Techniques, Universit des Sciences de Technologie et de la médecine, Nouakchott BP: 5026 Mauritanie.

Received 1 June 2014; Accepted 17 December 2014

Abstract. The Cauchy problem for the wave equation with bi-inverse square potential on Euclidean plane is solved in terms of the two variables Appell F_2 hypergeometric functions. Our principal tools are the Hankel transforms and the special functions of mathematical physics.

AMS Subject Classifications: 35J05; 35J10

Chinese Library Classifications: O175.25

Key Words: Inverse square potential; wave equation; Hankel transform; Bessl functions; F_2-Appell hypergeometric function.

1 Introduction and statement of results

Consider the following Cauchy problem of the wave type on Euclidean plane

$$\begin{cases} L_{(\nu,\nu')}u(t,p)=\partial_t^2 u(t,p), \qquad (t,p)\in\mathbb{R}\times{\mathbb{R}_+^*}^2, \\ u(0,p)=0, \quad \partial_t u(0,p)=f(p)\in C_0^\infty({\mathbb{R}_+^*}^2), \end{cases} \tag{1.1}$$

where

$$L_{(\nu,\nu')}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1/4-\nu^2}{x^2}+\frac{1/4-\nu'^2}{y^2}, \tag{1.2}$$

*Corresponding author. *Email addresses:* `khames@ustm.mr` (M. V. O. Moustapha)

is called here the Schrödinger operator with bi-inverse square potential and ν and ν' are real parameters. For the classical Schrödinger operator with inverse square potential

$$L_\nu = \frac{\partial^2}{\partial x^2} + \frac{1/4-\nu^2}{x^2}, \tag{1.3}$$

on the half real line $\mathbb{R}_+^*$, the solution of the problem (1.1) is (Taylor [1], p.132-133):

$$u(t,x) = \int_0^\infty W_\nu(\sigma(t,x,x')f(x')\mathrm{d}x', \tag{1.4}$$

where

$$W_\nu(\sigma) = \begin{cases} 0, & \text{when } 1<\sigma, \\ \frac{1}{2}P_{\nu-\frac{1}{2}}(\sigma), & \text{when } -1<\sigma<1, \\ \frac{\cos\pi\nu}{\pi}Q_{\nu-\frac{1}{2}}(-\sigma), & \text{when } \sigma<-1, \end{cases} \tag{1.5}$$

where P_m and Q_m denote the Legendre functions of degree m of the first and second kind respectively:

$$P_m(\sigma) = F(-m,m+1,1;(1-\sigma)/2),$$
$$Q_m(\sigma) = B(1/2,m+1)\frac{1}{(2\sigma)^{m+1}}F((m+1)/2,(m+2)/2;m+3/2;1/\sigma^2),$$

and $\sigma(t,x,x') = (x^2+x'^2-t^2)/(2xx')$.

The Gauss hypergeometric function is defined by:

$$F(a,b,c;z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}z^n, \quad (|z|<1),$$

where as usual $(a)_n$ is the Pochhamer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{1.6}$$

The functions B and Γ are the classical Euler functions.

The inverse square potential arises in several contexts, one of them is the Schrödinger equation in non relativistic quantum mechanics (Reed and Simon [2]) . For example, the Hamiltonian for a spinzero particle in Coulomb field gives rise to a Schrödinger operator involving the inverse square potential (Case [3]).

The Cauchy problem for the wave equation with the inverse square potential in Euclidean space $\mathbb{R}^n$ is extensively studied (Burg et al [4]), (Cheeger and Taylor [5]), (Lamb [6]) and (Planchon et al [7]). The bi-inverse square potential has been considered by (Boyer [8]) in the case of the Schrödinger equation.

The aim of this paper is to provide an explicit formulas for the solutions of the Cauchy problem associated to the wave equations with bi-inverse square potential on Euclidian plane (1.1). The case considered most frequentely is obviously the one where $(\nu,\nu')=(\pm 1/2,\pm 1/2)$, the equation in (1.1) then turns into the classical wave equation on the Euclidean plane $\mathbb{R}^2$ and this equation appears in several branches of mathematics and physics, on the other hand it has the unique solution given by (Folland [9], p.171)

$$u(t,p)=\frac{1}{2\pi}\int_{|p-p'|<t}\left(t^2-|p-p'|^2\right)^{-1/2}f(p')\mathrm{d}p'. \tag{1.7}$$

Now we state the main result of this paper:

Theorem 1.1. *For* $(t,p,p')\in R_+\times\mathbb{R}^{*2}_+\times\mathbb{R}^{*2}_+$ *the functions:*

$$c^{\mu}_{\nu}z^{\beta}z'^{\beta'}a^{-1/2}F_2(1/2+\beta+\beta',\beta,\beta',2\beta,2\beta',z,z'),$$
$$c^{\mu}_{\nu}z^{1-\beta}z'^{\beta'}a^{-1/2}F_2(3/2-\beta+\beta',1-\beta,\beta',2-2\beta,2\beta',z,z'),$$
$$c^{\mu}_{\nu}z^{\beta}z'^{1-\beta'}a^{-1/2}F_2(3/2+\beta-\beta',\beta,1-\beta',2\beta,2-2\beta',z,z'),$$
$$c^{\mu}_{\nu}z^{1-\beta}z'^{1-\beta'}a^{-1/2}F_2(5/2-\beta-\beta',1-\beta,1-\beta',2-2\beta,2-2\beta',z,z'),$$

are independent solutions of the wave equation with bi-inverse square potential on the Euclidean plane in (1.1)*, where*

$$\beta=1/2+\nu,\quad \beta'=1/2+\nu',\quad z=-4xx'/(t^2-|p+p'|^2),\quad \text{and}\quad z'=-4yy'/(t^2-|p+p'|^2).$$

Theorem 1.2. *The Cauchy problem* (1.1) *for the wave equation with bi-inverse square potential on the Euclidean plane has the solutions given by:*

$$u(t,p)=\int_{|p+p'|<t}W_{(b,b')}(t,p,p')f(p')\mathrm{d}p', \tag{1.8}$$

where the kernel $W_{(b,b')}$ *is given by*

$$W_{(b,b')}(t,p,p')=\frac{C_2(xx')^b(yy')^{b'}}{(t^2-|p+p'|^2)^{1/2+b+b'}} \times F_2\left(1/2+b+b',b,b',2b,2b',\frac{-4xx'}{t^2-|p+p'|^2},\frac{-4yy'}{t^2-|p+p'|^2}\right), \tag{1.9}$$

and the constant C_2 *is given by*

$$C_2=\frac{(-1)^{b+b'}\Gamma(1/2+b+b')}{\sqrt{\pi}\Gamma(1/2+b)\Gamma(1/2+b')}, \tag{1.10}$$

with $b\in\{\beta,1-\beta\}$, $b'\in\{\beta',1-\beta'\}$, $dp'=dx'dy'$ *is the Lebesgue measure on* $\mathbb{R}^2$ *and* $F_2(\alpha,\beta,\beta',\gamma,\gamma';z,z')$ *is the two variables double series* F_2 *Appell hypergeometric function given by (Erdélyi et al* [10]*, p. 224)*

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',z,z')=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}z^m z'^n,\quad (|z|+|z'|<1). \tag{1.11}$$

Note that as applications of the results of this paper one can look for the estimates of Strichatz type for the wave and Schrödinger equations associated to the bi-inverse square potential (Burg et al [4]) and (Planchon et al [7]).

We can use Theorem 1.2 to give the fundamental solutions of generalized bi-axially symmetric Helmoltz equation (Anvar [11]).

The inverse square potential as a distinguished potential can be defined not only on the Euclidean space but also on the hyperbolic and spheric spaces and one can also consider the wave equation associated to the inverse square potential on these as well as the case of multidimensionnal inverse square potential. This will be treated later.

The organization of the paper is as follows in Section 2 we give the proof of Theorems 1.1 and 1.2 respectively. Section 3 is devoted to some numerical results. In the appendis the Hankel transform and some of its properties are considered. The computation of a double integral involving Bessel functions is given.

2 Wave kernel for the bi-inverse square potential on Euclidean plane $\mathbb{R}^2$

Proof of Theorem 1.1. The Hankel transform (see Appendis) with respect to each of variables x, y transforms the problem (1) to

(a) $\left(-|\omega|^2-\frac{\partial^2}{\partial t^2}\right)[H_\nu U](t,\omega)=0;(t,\omega)\in\mathbb{R}^+\times\mathbb{R}^{+2}$,

(b) $[H_\nu U](0,\omega)=0$ and $\partial_t[H_\nu U](0,\omega)=[H_\nu f](\omega), f\in L^2(\mathbb{R}_+^{*2})$.

The Cauchy problem (a),(b) has the unique solution given by

$$[H_\nu U](t,\omega)=\frac{\sin t|\omega|}{|\omega|}[H_\nu f](\omega), \tag{2.1}$$

the Hankel inverse transform gives

$$\begin{aligned} U(t,p,p')=&\int_{\mathbb{R}_+^{*\,2}} f(p')\left(xx'yy\right)^{1/2}\mathrm{d}x'\mathrm{d}y'\int_0^\infty\int_0^\infty\frac{\sin t|\omega|}{|\omega|}J_{\nu_1}(x\omega_1)J_{\nu_1}(x'\omega_1)\\ &\times J_{\nu_2}(y\omega_2)J_{\nu_2}(y'\omega_2)\omega_1\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_2, \end{aligned} \tag{2.2}$$

from the formula (Magnus et al [12], p.73)

$$\sin z=\sqrt{\pi z/2}J_{1/2}(z), \tag{2.3}$$

we can write

$$\begin{aligned} U(t,p,p')=&\sqrt{\pi t/2}\int_{\mathbb{R}_+^{*\,2}} f(p')\left(xx'yy\right)^{1/2}\mathrm{d}x'\mathrm{d}y'\int_0^\infty\int_0^\infty|\omega|^{-1/2}J_{\frac{1}{2}}(t|\omega|)\\ &\times J_{\nu_1}(x\omega_1)J_{\nu_1}(x'\omega_1)J_{\nu_2}(y\omega_2)J_{\nu_2}(y'\omega_2)\omega_1\omega_2\mathrm{d}\omega_1\mathrm{d}\omega_2. \end{aligned} \tag{2.4}$$

For $\nu > -1$, $\nu' > -1$, $\mu > -1/2$, $p = (x,y)$ and $p' = (x',y')$, $\omega = (\omega_1,\omega_2)$ set

$$I^{\mu}_{(\nu,\nu')}(t,p,p') = \int_0^\infty \int_0^\infty |\omega|^{-\mu} J_\mu(t|\omega|) J_\nu(x\omega_1) J_\nu(x'\omega_1) \times J_{\nu'}(y\omega_2) J_{\nu'}(y'\omega_2)\omega_1\omega_2 \mathrm{d}\omega_1 \mathrm{d}\omega_2. \tag{2.5}$$

In the Appendis we prove at last formally the following formula:

$$I^{\mu}_{(\nu,\nu')}(t,x,x') = c^{\mu}_{(\nu,\nu')} (xx')^{\nu} (yy')^{\nu'} (t^2 - |p+p'|^2)^{\alpha} \times F_2\left(-\alpha,\beta,\beta',2\beta,2\beta',\frac{-4xx'}{t^2-|p+p'|^2},\frac{-4yy'}{t^2-|p+p'|^2}\right), \tag{2.6}$$

where $\alpha = \mu - \beta - \beta' - 1$.

$$c^{\mu}_{(\nu,\nu')} = \frac{4\sin\pi(\beta+\beta')\Gamma(1/2+\beta+\beta')}{\sqrt{2}\Gamma(\beta+1/2)\Gamma(\beta'+1/2)}. \tag{2.7}$$

In what follows we give a direct proof of Theorem 1.1.

Let $a = t^2 - |p+p'|^2$, $t \in \mathbb{R}$, $p,p' \in \mathbb{R}^{*2}$ set:

$$\Omega\varphi(t,p) = (xx')^{-\beta} (yy')^{-\beta'} a^{-\alpha} \times \left[\Delta - \frac{\beta(\beta-1)}{x^2} - \frac{\beta'(\beta'-1)}{y^2} - \frac{\partial^2}{\partial t^2}\right] (xx')^{\beta} (yy')^{\beta'} a^{\alpha} \varphi(t,x), \tag{2.8}$$

then we have

$$\Omega\varphi(t,p) = \left[\Delta - \frac{\partial^2}{\partial t^2}\right] + \left[\frac{2\beta}{x} - \frac{4\alpha(x+x')}{a}\right]\frac{\partial}{\partial x} + \left[\frac{2\beta'}{y} - \frac{4\alpha(y+y')}{a}\right]\frac{\partial}{\partial y} - \frac{4\alpha t}{a}\frac{\partial}{\partial t} - \frac{4\alpha}{a}\left[\frac{\beta x'}{x} + \frac{\beta' y'}{y}\right] - \frac{4\alpha}{a}\left[\alpha + \frac{1}{2} + \beta + \beta'\right]\varphi. \tag{2.9}$$

Now set

$$z = \frac{-4xx'}{t^2-|p+p'|^2}, \qquad z' = \frac{-4yy'}{t^2-|p+p'|^2}, \tag{2.10}$$

we have:

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial y} = \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial z'}{\partial y}\frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial z'}{\partial t}\frac{\partial}{\partial z'}. \tag{2.11}$$

We have

$$\Omega\varphi(t,p) = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 - \left(\frac{\partial z}{\partial t}\right)^2\right]\frac{\partial^2}{\partial z^2} + \left[\left(\frac{\partial z'}{\partial x}\right)^2 + \left(\frac{\partial z'}{\partial y}\right)^2 - \left(\frac{\partial z'}{\partial t}\right)^2\right]\frac{\partial^2}{\partial z'^2}$$

$$
\begin{aligned}
&+2\left[\frac{\partial z}{\partial x}\frac{\partial z'}{\partial x}+\frac{\partial z}{\partial y}\frac{\partial z'}{\partial y}-\frac{\partial z}{\partial t}\frac{\partial z'}{\partial t}\right]\frac{\partial^2}{\partial z\partial z'}+\left[\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial t^2}\right]\frac{\partial}{\partial z}\\
&+\left[\frac{\partial^2 z'}{\partial x^2}+\frac{\partial^2 z'}{\partial y^2}-\frac{\partial^2 z'}{\partial t^2}\right]\frac{\partial}{\partial z'}+\left[A_x\frac{\partial z}{\partial x}+A_y\frac{\partial z}{\partial y}-A_t\frac{\partial z}{\partial t}\right]\frac{\partial}{\partial z}\\
&+\left[A_x\frac{\partial z'}{\partial x}+A_y\frac{\partial z'}{\partial y}-A_t\frac{\partial z'}{\partial t}\right]\frac{\partial}{\partial z'}+\frac{4\alpha}{a}\left[\frac{\beta x'}{x}+\frac{\beta' y'}{y}\right]\\
&-\frac{4\alpha}{a}\left[\alpha+\frac{1}{2}+\beta+\beta'\right]\varphi(z,z'),
\end{aligned}
\tag{2.12}
$$

where

$$
A_x=\frac{2\beta}{x}-\frac{4\alpha(x+x')}{a},\qquad A_y=\frac{2\beta'}{y}-\frac{4\alpha(y+y')}{a},\qquad A_t=\frac{4\alpha t}{a}. \tag{2.13}
$$

We have:

$$
\frac{\partial z}{\partial x}=\frac{-4x'a-8(x+x')xx'}{a^2};\qquad \frac{\partial z'}{\partial x}=\frac{-8yy'(x+x')}{a^2}, \tag{2.14}
$$

$$
\frac{\partial z'}{\partial y}=\frac{-4y'a-8(y+y')yy'}{a^2};\qquad \frac{\partial z}{\partial y}=\frac{-8xx'(y+y')}{a^2}, \tag{2.15}
$$

$$
\frac{\partial z}{\partial t}=\frac{8xx't}{a^2};\qquad \frac{\partial z'}{\partial t}=\frac{8yy't}{a^2}, \tag{2.16}
$$

$$
\frac{\partial^2 z}{\partial x^2}=\frac{-24xx'a^2-16x'^2a^2-32(x+x')^2xx'a}{a^4}, \tag{2.17}
$$

$$
\frac{\partial^2 z}{\partial y^2}=\frac{-8yy'a^2-32xx'(y+y')^2a}{a^4}, \tag{2.18}
$$

$$
\frac{\partial^2 z}{\partial t^2}=\frac{8xx'a^2-32axx't^2}{a^4}, \tag{2.19}
$$

$$
\frac{\partial^2 z'}{\partial x^2}=\frac{-8xx'a^2-32yy'(x+x')^2a}{a^4}, \tag{2.20}
$$

$$
\frac{\partial^2 z'}{\partial y^2}=\frac{-24yy'a^2-16y'^2a^2-32(y+y')^2yy'a}{a^4}, \tag{2.21}
$$

$$
\frac{\partial^2 z'}{\partial t^2}=\frac{8yy'a^2-32ayy't^2}{a^4}, \tag{2.22}
$$

from (2.14)-(2.16) we have

$$
\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2-\left(\frac{\partial z}{\partial t}\right)^2=\frac{z^2}{x^2}(1-z), \tag{2.23}
$$

$$
\left(\frac{\partial z'}{\partial x}\right)^2+\left(\frac{\partial z'}{\partial y}\right)^2-\left(\frac{\partial z'}{\partial t}\right)^2=\frac{z'^2}{y^2}(1-z'), \tag{2.24}
$$

$$2\left[\frac{\partial z}{\partial x}\frac{\partial z'}{\partial x}+\frac{\partial z}{\partial y}\frac{\partial z'}{\partial y}-\frac{\partial z}{\partial t}\frac{\partial z'}{\partial t}\right]=-\frac{z^2}{x^2}z'-\frac{z'^2}{y^2}z, \tag{2.25}$$

from (2.17)-(2.19) we have

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial t^2}=\frac{-z^2}{x^2}+\frac{2}{a}z, \tag{2.26}$$

from (2.20)-(2.22) we have

$$\frac{\partial^2 z'}{\partial x^2}+\frac{\partial^2 z'}{\partial y^2}-\frac{\partial^2 z'}{\partial t^2}=\frac{-z'^2}{y^2}+\frac{2}{a}z', \tag{2.27}$$

from (2.13)-(2.16) we have

$$A_x\frac{\partial z}{\partial x}+A_y\frac{\partial z}{\partial y}-A_t\frac{\partial z}{\partial t}=\frac{4\alpha+4(\beta+\beta')}{a}z+2\beta\frac{z}{x^2}+\alpha\frac{z^2}{x^2}-\beta\frac{z^2}{x^2}+\frac{\beta' z' z}{y^2}, \tag{2.28}$$

$$A_x\frac{\partial z'}{\partial x}+A_y\frac{\partial z'}{\partial y}-A_t\frac{\partial z'}{\partial t}=\frac{4\alpha+4(\beta+\beta')}{a}z'+2\beta'\frac{z'}{y^2}+\alpha\frac{z'^2}{y^2}-\beta'\frac{z'^2}{y^2}+\frac{\beta z z'}{x^2}. \tag{2.29}$$

To replace in the formula (2.12) using the formulas (2.23)-(2.29) we get:

$$\begin{aligned}\Omega\varphi=&zx^{-2}\left[z(1-z)\frac{\partial^2}{\partial z^2}-zz'\frac{\partial^2}{\partial z'\partial z}+[2\beta+(-\alpha+\beta+1)z]\frac{\partial}{\partial z}+-\beta z'\frac{\partial}{\partial z'}+\alpha\beta\right]\varphi\\&+z'y^{-2}\left[z'(1-z')\frac{\partial^2}{\partial z'^2}-z'z\frac{\partial^2}{\partial z\partial z'}+\left[2\beta'+(-\alpha+\beta'+1)z'\right]\frac{\partial}{\partial z'}-\beta' z\frac{\partial}{\partial z}+\alpha\beta'\right]\varphi\\&+\frac{4}{a}(\alpha+\beta+\beta'+1/2)\left[\left(z\frac{\partial}{\partial z}+z'\frac{\partial}{\partial z'}\right)\varphi(z,z')-\varphi(z,z')\right]=0.\end{aligned} \tag{2.30}$$

Take $\alpha=-1/2-\beta-\beta'$ we get:

$$\begin{cases}\left[z(1-z)\frac{\partial^2}{\partial z^2}-zz'\frac{\partial^2}{\partial z'\partial z}+[2\beta+(-\alpha+\beta+1)z]\frac{\partial}{\partial z}-\beta z'\frac{\partial}{\partial z'}+\alpha\beta\right]\varphi(z,z')=0,\\\left[z'(1-z')\frac{\partial^2}{\partial z'^2}-z'z\frac{\partial^2}{\partial z\partial z'}+[2\beta'+(-\alpha+\beta'+1)z']\frac{\partial}{\partial z'}-\beta' z\frac{\partial}{\partial z}+\alpha\beta'\right]\varphi(z,z')=0.\end{cases}$$

This is an F_2 Appell hypergeometric system and for $2\beta\neq 1$ and $2\beta'\neq 1$ the system has four independent solutions of the form:

- $F_2(-\alpha,\beta,\beta',2\beta,2\beta',z,z')$,
- $z^{1-2\beta}F_2(-\alpha+1-2\beta,1-\beta,\beta',2-2\beta,2\beta',z,z')$,
- $z'^{1-2\beta'}F_2(-\alpha+1-2\beta',\beta,1-\beta',2\beta,2-2\beta',z,z')$,
- $z^{1-2\beta}z'^{1-2\beta'}F_2(-\alpha+2-2\beta-2\beta',1-\beta,1-\beta',2-2\beta,2-2\beta',z,z')$.

And the proof of Theorem 1.1 is finished. □

Proof of Theorem 1.2. For the proof of Theorem 1.2 we need the following lemma.

Lemma 2.1. *Let F_2 be the Appell hypergeometric function with $(h,k)\in R^2$ and $a\in R^*$ then we have:*

i)

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[a^{\alpha}F_2\left(-\alpha,\beta,\beta',2\beta,2\beta',h/a,k/a\right)\right] =-\alpha a^{\alpha-1}F_2\left(-\alpha+1,\beta,\beta',2\beta,2\beta',h/a,k/a\right), \tag{2.31}$$

ii)

$$a^{\alpha}\Gamma(-\alpha)F_2\left(-\alpha,\beta,\beta',2\beta,2\beta',h/a,k/a\right) =\int_0^{\infty}e^{-(a-h/2-k/2)t}t^{-\alpha-1}\Gamma(\beta+1/2)\Gamma(\beta'+1/2)\left(\frac{th}{4}\right)^{1/2-\beta}\left(\frac{tk}{4}\right)^{1/2-\beta'} \times I_{\beta-1/2}(th/2)I_{\beta'-1/2}(tk/2)\mathrm{d}t, \tag{2.32}$$

iii)

$$a^{\alpha}\Gamma(-\alpha)F_2\left(-\alpha,\beta,\beta',\gamma,\gamma',h/a,k/a\right) \sim\Gamma(-|\beta|-\alpha)\frac{\Gamma(2\beta)}{\Gamma(\beta)}\frac{\Gamma(2\beta')}{\Gamma(\beta')}h^{-\beta}k^{-\beta'}(a-h-k)_+^{\alpha+|\beta|}, \qquad \text{as } a\to 0. \tag{2.33}$$

.

Proof. i) is a consequence of the formulas

$$a^{\alpha}F_2\left(-\alpha,\beta,\beta',\gamma,\gamma',k/a\right)=\sum_{m,n\geq 0}\frac{(-\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m m!(\gamma')_n n!}h^m k^n a^{-m-n+\alpha}, \tag{2.34}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}[a^{\alpha}F_2\left(-\alpha,\beta,\beta',\gamma,\gamma',h/a,k/a\right)]=\sum_{m\geq 0}\frac{(-\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m m!(\gamma')_n n!} \times(\alpha-m-n)h^m k^n a^{-m-n+\alpha-1}, \tag{2.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}[a^{\alpha}F_2]\left(-\alpha,\beta,\beta',\gamma,\gamma',h/a,k/a\right)]=-\alpha a^{\alpha-1}\sum_{m\geq 0}\frac{(-\alpha+1)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!} \times h^m k^n a^{-m-n}, \tag{2.36}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}[a^{\alpha}F_2]\left(-\alpha,\beta,\beta',\gamma,\gamma',h/a,k/a\right)]=-\alpha a^{\alpha-1}F_2\left(-\alpha+1,\beta,\beta',\gamma,\gamma',h/a,k/a\right). \tag{2.37}$$

ii) is a consequence of the integral representation (Srivastava and Morgautown [13], p.69):

$$F_2\left(\alpha,\beta,\beta',\gamma,\gamma',k_1/a,k_2/a\right)=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\phi(\beta,\gamma,ht/a)\phi(\beta',\gamma',kt/a)\mathrm{d}t, \quad (2.38)$$

and (Lebedev [14], p.274):

$$\Phi(\nu+1/2,2\nu+1,2z)=\Gamma(\nu+1)e^z(z/2)^{-\nu}I_\nu(z). \quad (2.39)$$

where $\phi(a,c,z)$ and I_ν are respectively the first kind Kummer confluent hypergeometric function and the modified Bessel function.

iii) is a result of ii) and the asymptotic formula (Bell [15], p. 130):

$$I_\nu(x)\sim e^x(2\pi x)^{-1/2} \qquad x\longrightarrow\infty. \quad (2.40)$$

□

To finish the proof of Theorem 1.2, we prove the limit conditions in (1): from iii) of Lemma 2.1 and the Legendre duplication formula for the Γ-Euler function (Bell [15], p.29)

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2)=\sqrt{\pi}\Gamma(2z), \quad (2.41)$$

we have for $t\longrightarrow 0$ and $p,p'\in {\mathbb{R}_+^*}^2$

$$W^{\mathbb{R}^2}_{(a,b)}(t,p,p')\sim\frac{1}{\pi}(t^2-|p-p'|^2)_+^{-1/2}. \quad (2.42)$$

The polar coordinates $p'=p+r(\cos\theta,\sin\theta)$ we have for $t\longrightarrow 0$

$$u(t,p)\sim\frac{1}{\pi}\int_{\mathbb{R}^{*2}}(t^2-|p-p'|^2)_+^{-1/2}f(p')\mathrm{d}p', \quad (2.43)$$

$$u(t,p)\sim\frac{t}{\pi}\int_0^1 f_p^{\#}(ts)(1-s^2)s\mathrm{d}s, \quad (2.44)$$

with

$$f_p^{\#}(r)=\int_0^\pi f(x+r(\cos\theta,\sin\theta)\mathrm{d}\theta, \quad (2.45)$$

and from (2.44) and (2.45) we see that the limit conditions in (1.1) hold. The proof of Theorem 1.1 is finished. □

3 Numerical examples

For the special values of parameters β and β' we can express the solutions of the problem (1.1) in terms of elementary functions for example we have (Murley and Saad [16], p.29) :

$$F_2((3-n)/2,-n/2,1,-n,2,x,y)=\Big(\frac{\sqrt{1-y}+\sqrt{(1-x-y)})^{n+1}-(\sqrt{1-y}-\sqrt{1-x-y})^{n+1}}{\sqrt{1-x-y}\sqrt{1-y}} \\ -\frac{(1+\sqrt{1-x})^{n+1}-(1-\sqrt{(1-x)})^{n+1}}{\sqrt{1-x}}. \tag{3.1}$$

We need the following formulas (Saad and Hall [17], Lemma 3 and Lemma 5)

$$F_2(d,a,a';a,a';x;y)=(1-x-y)^{-d}, \tag{3.2}$$
$$F_2(d,a,a';d,a';x;y)=(1-y)^{a-d}(1-x-y)^{-a}, \tag{3.3}$$
$$F_2(d,a,a';a,d;x;y)=(1-x)^{a'-d}(1-x-y)^{-a'}, \tag{3.4}$$
$$F_2(d,a,a';d,c';x;y)=(1-x)^{-a}F_1(a',a,d-a,c',y/(1-x),y), \tag{3.5}$$
$$F_2(d,a,a';c,d;x;y)=(1-y)^{-a'}F_1(a,d-a',a',c,x,x/(1-y)), \tag{3.6}$$

we have

$$F_2(2\beta,\beta,\beta-1/2,2\beta,2\beta-1,x,y)=(1-x)^{-\beta}F_1(\beta-1/2,\beta,\beta,2\beta-1,y/(1-x),y), \tag{3.7}$$
$$F_2(2\beta',\beta'-1/2,\beta',2\beta'-1,2\beta',x,y)=(1-y)^{-\beta'}F_1(\beta'-1/2,\beta',\beta',2\beta'-1,x,x/(1-y)). \tag{3.8}$$

From the formulas (3.2)-(3.8) we have the following particular cases

$$W_{0,0}(t,p,p')=c\left(t^2-(x-x')^2-(y-y')^2\right)^{-1/2}, \tag{3.9}$$
$$W_{1/2,0}(t,p,p')=c\frac{(xx')^{1/2}}{2\pi}\left(t^2-(x+x')^2-(y-y')^2\right)^{-1/2} \\ \times\left(t^2-(x-x')^2-(y-y')^2\right)^{-1/2}, \tag{3.10}$$
$$W_{0,1/2}(t,p,p')=c\frac{(yy')^{1/2}}{2\pi}\left(t^2-(x-x')^2-(y+y')^2\right)^{-1/2} \\ \times\left(t^2-(x-x')^2-(y-y')^2\right)^{-1/2}, \tag{3.11}$$
$$W_{\beta,\beta-1/2}(t,p,p')=c(xx')^{\beta}(yy')^{\beta-1/2}(A^{++}A^{-+})^{-2\beta} \\ \times F_1\left(\beta-1/2,\beta,\beta,2\beta-1,\frac{-4yy'}{A^{++2}},\frac{-4yy'}{A^{+-2}}\right), \tag{3.12}$$
$$W_{\beta,\beta+1/2}(t,p,p')=c(xx')^{\beta}(yy')^{\beta+1/2}(A^{++}A^{+-})^{-2\beta-1} \\ \times F_1\left(\beta,\beta+1/2,\beta+1/2,2\beta,\frac{-4xx'}{A^{++2}},\frac{-4xx'}{A^{-+2}}\right), \tag{3.13}$$

where we have set

$$A^{++}=\sqrt{t^2-(x+x')^2-(y+y')^2}, \qquad A^{+-}=\sqrt{t^2-(x+x')^2-(y-y')^2}, \tag{3.14}$$

$$A^{-+}=\sqrt{t^2-(x-x')^2-(y+y')^2}, \qquad A^{--}=\sqrt{t^2-(x-x')^2-(y-y')^2}. \tag{3.15}$$

To express the wave kernel $W_{\beta,\beta-1/2}$ and $W_{\beta,\beta+1/2}$ in terms of elementary functions, I can use the formula (11) in (Mourad and Pitman [18], p.3)for $|x|<1$ and $|y|<1$:

$$F_1(\alpha,\alpha+1/2,\alpha+1/2,2\alpha,x,y)=\frac{1}{2}\left(1+\frac{1}{\sqrt{1-x}\sqrt{1-y}}\right)\left(\frac{2}{\sqrt{1-x}+\sqrt{1-y}}\right)^{2\alpha}, \tag{3.16}$$

but it seems that this formula is not correct at least for $\alpha=-1$, compare it with the following formula given by Mathematica:

$$F_1(-1,-1/2,-1/2,-2,x,y)=\frac{1}{4}(4-x-y), \tag{3.17}$$

$$W_{-1/2,-1}(t,p,p')=c(xx')^{-1/2}(yy')^{-1}A^{++}A^{-+}\left(1+\frac{4yy'}{A^{-+2}}+\frac{4yy'}{A^{++2}}\right), \tag{3.18}$$

$$W_{-1,-1/2}(t,p,p')=c(xx')^{-1}(yy')^{-1/2}A^{++}A^{+-}\left(1+\frac{4xx'}{A^{+-2}}+\frac{4xx'}{A^{++2}}\right). \tag{3.19}$$

To compare the solutions given by the formulas (3.9)-(3.11) and (3.18)(3.19) we see that in the last two cases there is singularity in $(0,0)$.

The Maple plotting gives a much better idea of what happens for $p'=(1,1)$ (see figs. 1, 2, 3, 4 and 5 below).

4 Appendix

First of all we recall some facts about the Hankel transform (Burg et al [4]), (Planchon et al [7]).

When $\nu>-1$ we define the Hankel transform of order ν by.

$$(H_\nu f)(\omega)=\int_0^\infty (x\omega)^{1/2}J_\nu(x\omega)f(x)\mathrm{d}x, \tag{4.1}$$

where $f\in C_0^\infty(R_+^*)$ and J_ν is the Bessel function of the first kind and order ν.

Proposition 4.1. *(Burg et al* [4] *and Planchon et al* [7]*)*

$$\text{i)}\quad H_\nu^2=Id. \qquad \text{ii)}\quad H_\nu \text{ is self adjoint.} \qquad \text{iii)}\quad H_\nu A_\nu=-\omega^2 H_\nu,$$

where J_ν is the Bessel functions of the first kind.

The remaining of this section is devoted to the computation of the following double integral involving the product of Bessel functions, for $\nu>-1$, $\nu'>-1$, $\mu>-1/2$, $p=(x,y)$ and $p'=(x',y')$, $\omega=(\omega_1,\omega_2)$:

$$I^{\mu}_{(\nu,\nu')}(t,p,p')=\int_0^\infty\int_0^\infty|\omega|^{-\mu}J_\mu(t|\omega|)J_\nu(x\omega_1)\\ \times J_\nu(x'\omega_1)J_{\nu'}(y\omega_2)J_{\nu'}(y'\omega_2)\omega_1\omega_2\mathrm{d}\omega_1\mathrm{d}\omega_2. \tag{4.2}$$

Using respectively the formulas (Koornwinder [19]) and (Erdelyi et al [20], p.55):

$$J_\nu(x)J_\nu(y)=\frac{x^\nu y^\nu}{2^\nu\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^\pi\frac{J_\nu\left[\left(x^2+y^2+2xy\cos\theta\right)^{1/2}\right]}{\left(x^2+y^2+2xy\cos\theta\right)^{\nu/2}}(\sin\theta)^{2\nu}\mathrm{d}\theta, \tag{4.3}$$

$$\int_0^\infty(x^2+\beta^2)^{-\mu/2}J_\mu(a\sqrt{x^2+\beta^2})J_\nu(xy)x^{\nu+1}\mathrm{d}x\\ =\begin{cases}0, & \text{if } y>a,\\ a^{-\mu}y^\nu\beta^{-\mu+\nu+1}\left(a^2-y^2\right)^{\frac{\mu-\nu-1}{2}}J_{\mu-\nu-1}(\beta\sqrt{a^2-y^2}), & \text{if } 0<y<a.\end{cases} \tag{4.4}$$

we obtain

$$I^{\mu}_{(\nu,\nu')}=\frac{(xx')^\nu}{2^\nu\sqrt{\pi}\Gamma(\nu+1/2)}t^{-\mu}\int_0^\pi\sin^{2\nu}\theta_1\mathrm{d}\theta_1\left(t^2-A_1^2\right)^{(\mu-\nu-1)/2}\\ \times\int_0^\infty\omega_2^{-\mu+\nu+2}J_{\mu-\nu-1}\left(\omega_2\sqrt{t^2-A_1^2}\right)J_{\nu'}(y\omega_2)J_{\nu'}(y'\omega_2)\mathrm{d}\omega_2, \tag{4.5}$$

where $A_1=\left(x^2+x'^2+2xx'\cos\theta_1\right)^{1/2}$.

From the formulas (Koornwinder [19])

$$\int_0^\infty J_\alpha(\lambda x)J_\beta(\lambda y)J_\beta(\lambda z)\lambda^{1-\alpha}\mathrm{d}\lambda=\frac{x^\alpha y^\beta z^\beta}{2^\alpha\Gamma(\alpha+1)}K_{\alpha\beta}(x,y,z), \tag{4.6}$$

$$K_{\alpha\beta}(a,b,c)=\frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}a^{-2\alpha}(bc)^{\alpha-\beta-1}\int_{-1}^1(s-B)^{\alpha-\beta-1}(1-s^2)^{\beta-1/2}\mathrm{d}s, \tag{4.7}$$

$$\int_0^\infty J_\alpha(\lambda x)J_\beta(\lambda y)J_\beta(\lambda z)\lambda^{1-\alpha}\mathrm{d}\lambda=\frac{2^{-\beta}x^{-\alpha}(yz)^{\alpha-1}}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}\\ \times\int_{-1}^1(s-B)^{\alpha-\beta-1}(1-s^2)^{\beta-1/2}\mathrm{d}s, \tag{4.8}$$

with

$$B=\frac{y^2+z^2-x^2}{2yz}. \tag{4.9}$$

we can write

$$J:=\int_0^\infty \omega_2^{-\mu+\nu+2} J_{\mu-\nu-1}\left(\omega_2\sqrt{t^2-A_1^2}\right) J_{\nu'}(y\omega_2) J_\nu(y'\omega_2)\mathrm{d}\omega_2$$
$$=\frac{2^{-\nu'}(\sqrt{t^2-A_1^2})^{-\mu+\nu+1}(yy')^{\mu-\nu-2}}{\sqrt{\pi}\Gamma(\mu-\nu-\nu'-1)\Gamma(\nu'+1/2)}\int_{-1}^1 (s-B)^{\mu-\nu-\nu'-2}(1-s^2)^{\nu'-1/2}\mathrm{d}s. \quad (4.10)$$

Set $s=-\cos\theta_2$, we obtain

$$J=\frac{(yy')^{\nu'}(\sqrt{t^2-A_1^2})^{-\mu+\nu+1}}{2^{\nu'}\sqrt{\pi}\Gamma(\nu'+1/2)\Gamma(\mu-\nu-\nu'-1)}$$
$$\times\int_0^\pi \left(t^2-x^2-x'^2-2xx'\cos\theta_1-y^2-y'^2-2yy'\cos\theta_2\right)^{\mu-\nu-\nu'-2}\sin^{2\nu'}\theta_2\mathrm{d}\theta_2. \quad (4.11)$$

Inserting this in (4.5) we get

$$I^\mu_{(\nu,\nu')}=Ct^{-\mu}\left(xx'\right)^\nu\left(yy'\right)^{\nu'}\int_0^\pi\int_0^\pi \left(t^2-x^2-x'^2-2xx'\cos\theta_1-y^2-y'^2-2yy'\cos\theta_2\right)^{\mu-\nu-\nu'-2}$$
$$\times\sin^{2\nu}\theta_1\sin^{2\nu'}\theta_2\mathrm{d}\theta_1\mathrm{d}\theta_2, \quad (4.12)$$

where

$$C=\frac{2^{-\mu+2}}{\pi\Gamma(\nu+1/2)\Gamma(\nu'+1/2)\Gamma(\mu-\nu-\nu'-1)}.$$

Using the change of variables $\cos\theta_j=1-2\zeta_j$, $j=1,2$ and the integral representation of F_2 Appell hypergeometric function $F_2(\alpha,\beta,\beta',\gamma,\gamma',z;z')$ (Erdelyi et al [10], p.230) for $\Re e\beta>0$, $\Re e\beta'>0$, $\Re e(\gamma-\beta)>0$ and $\Re e(\gamma'-\beta')>0$.

$$F_2\left(\alpha,\beta,\beta',\gamma,\gamma',z,z'\right)=c\int_0^1\int_0^1(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}$$
$$\times u^{\beta-1}v^{\beta'-1}\left(1-uz-vz'\right)^{-\alpha}\mathrm{d}u\mathrm{d}v, \quad (4.13)$$

where

$$c=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}. \quad (4.14)$$

we get at last formally the formula (2.6). But in view of the asymtotiques formulas for the Bessel function of the first kind J_ν (Lebedev [14], *p*.134)

$$J_\nu(x)\approx\frac{x^\nu}{2^\nu\Gamma(1+\nu)},\quad (x\to 0), \quad (4.15)$$

$$J_\nu(x)\approx\sqrt{\frac{2}{\pi x}}\cos(x-(\nu\pi/2)-(\pi/4)),\quad (x\to\infty), \quad (4.16)$$

we can't use the Fubini Theorem.

References

[1] Taylor M. E., Partial Differential Equations I, Springer, New York, Berlin, Heidelberg, 1996.

[2] Reed M. and Simon B., Methods of Moderne Mathematical Physics vol. II, Academic press, New-York, 1979.

[3] Case J. K. M., Singular potential. *Phys. Rev.*, **80** (1950), 797-806.

[4] Burg N. Planchon F., Stalker J. and Shadi Tahvildar-Zadeh A., Strichartz estimate for the wave equation with the inverse square potential. arXiv:math.AP/0207152v327 Aug., 2002.

[5] Cheeger J., Taylor M., On the diffraction of waves by canonical singularites I. *Comm. Pure Appl. Math.*, **35** (3) (1982), 275-331.

[6] Lamb H., On the theory of waves propogated vertically in the atmosphere. *Proc. Lonndon Math. Soc.*, **2** (7) (1909), 122-141.

[7] Planchon F., Stalker J. and Shadi Tahvildar-Zadeh A., Dispersive estimate for the wave equation with the inverse square potential, *Discr. Contin. Dyna. Syst.*, **9** (6) (2003), 1337-1400.

[8] Boyer C. P., Lie theory and separation of variables for the equation $iU_t+(\frac{\alpha}{x_1^2}+\frac{\beta}{x_2^2})U=0$. *SIAM J. Math. Anal.*, **7** (1976), 230-263.

[9] Folland G. B., Introduction to Partial Differential Equations, Princeton university press, Princeton N. J., 1976.

[10] Erdélyi A., Magnus F., Oberhettinger F. and Tricomi F. G., Higher Transcendental Functions, Tome *I* New York, Toronto, London, INC 1954.

[11] Anvar H., Fundamental solutions of generalized bi-axially symmetric Helmoltz equation. *Complex Variable and Elliptic Equations*, **52** (2007), 673-683.

[12] Magnus F., Oberhettinger and Soni R. P., Formulas and Theorems for Special Functions of Mathematical Physics, Third enlarged edition, Springer-Verlag Berlin Heidelberg New York, 1966.

[13] Srivastava H. M., Morgautown W. Va., Infinite series of certains products involving Appells double hypergeometric functions. *Giasnik Matematicki*, **24** (1) 1969, 67-72.

[14] Lebedev N. N., Special Functions and Their Applications, Dover Publications INC New York, 1972.

[15] Bell W. W., Special Functions for Scientists and Engineers, D. Van Nostrand Company LTD, London 1967.

[16] Murley J., Saad N., Tables of the Appell Hypergeometric functions F_2. arXiv: 0809.5203v3 [math-ph] 28 oct. 2008.

[17] Saad N., Richard L. Hall, Integrals containing hypergeometric functions with applications to perturbed singular potentials. arXiv:math: ph/0306043 v1 16 Jun, 2003.

[18] Mourad E. H. Ismail and Pitman J., Algebric evaluations of some Euler integrals, duplication formulae for Appells hypergeometric function F1, and Brownian variations. *Canad. J. Math.*, **52** (5) 2000, 961-981. Report No. 554 University of California (1999).

[19] Koornwinder T., Jacobi polynomials, II An analytic proof of the product formula. *SIAM J. Math. Anal.*, **5** (1) (1974), 125-137.

[20] Erdélyi A., Magnus F., Oberhettinger F. and Tricomi F. G., Higher Transcendental Functions, Table of integral Transforms Tome *II* New York, Toronto, London, INC 1954.

[21] Appell A. et Kampé de Fériet J., Fonctions Hypergéométriques et Hypersphériques, Polynomes d'Hermite, Gauthier-Villars Paris, 1926.

[22] Exton H., Multiples Hypergeometric Functions and Applications, Ellis Horwood Chichester, 1976.