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Random Attractor for Stochastic Partly Dissipative Systems on Unbounded Domains

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Abstract. In this paper, we consider the long time behaviors for the partly dissipative stochastic reaction diffusion equations. The existence of a bounded random absorbing set is firstly discussed for the systems and then an estimate on the solution is derived when the time is sufficiently large. Then, we establish the asymptotic compactness of the solution operator by giving uniform a priori estimates on the tails of solutions when time is large enough. In the last, we finish the proof of existence a pullback random attractor in $L^2(R^n)\times L^2(R^n)$. We also prove the upper semicontinuity of random attractors when the intensity of noise approaches zero. The long time behaviors are discussed to explain the corresponding physical phenomenon.

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Key Words: Reaction diffusion equation; random dynamical systems; random attractors; asymptotic compactness; Sobolev compact embedding.

1 Introduction

The aim of this work is to research the long time behavior of asymptotically compact random dynamical systems, which can be generated by solutions of the following stochastic partly dissipative reaction diffusion systems with additive white noise on unbounded domains [1],

$$\mathrm{d}u+(-\mu\Delta u+\lambda u+\alpha v)\mathrm{d}t=(h(u)+f(x))\mathrm{d}t+\varepsilon\sum_{j=1}^{m}h_j\mathrm{d}\omega_j, \tag{1.1}$$

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$$\mathrm{d}v+(\delta v-\beta u)\mathrm{d}t=g(x)\mathrm{d}t+\varepsilon\sum_{j=1}^{m}h_j^*\mathrm{d}\omega_j, \tag{1.2}$$

where μ, λ, α, δ, β are positive constants, f, g, h_j and h_j^* are given functions, $h(u)$ is a nonlinear function satisfying certain dissipative condition and $\{\omega_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be explain later. The without additive white noise equations [2] is often used to describe the signal transmission across axon and is a model of FitzHugh-Nagumo equation in neurobiology. The FitzHugh-Nagumo equation is obtained by simplifying the four variables HudgKin-Huxley equation. And the simplified H-H equation are very successful in many ways in describing the behavior of nerve fiber. The mathematical model has become an important branch of nonlinear science. These equations are known as an excitable system, we can refer to the literature [3–5]. However,deterministic models often ignore many small perturbation, the stochastic model can more accurately describe the physical phenomenon. In the past decades, consider the long time behavior of infinite dimensional dynamical system was one of the most important work in mathematical physics [6–8]. One of the important tasks of investigation of dissipative dynamical system is to find conditions for the existence of an random dynamical system [9]. A RDS on the phase space is said to be dissipative if and only if there exists a bounded random absorbing set [10–12]. The long time behavior of solutions from problem (1.1)-(1.2) in a bounded domain has been studied by several authors [1, 13], but little is known for unbounded domains. Existence of random attractor on unbounded domains for stochastic Benjamin-Bona-Mahony equation and Navier-Stokes equation have investigated distinguish in [14] and [15]. Here we prove the existence of such a random attractor [16] for the partly dissipative stochastic reaction diffusion systems (1.1)-(1.2). It is worth mentioning that many researcher are interested in this research area. As everyone knows that the sobolev embedding are no longer compact in the unboundedness of the domain, which form a major difficulty for proving the existence of an attractor [17]. So the asymptotic compactness of solutions cannot be acquired with the standard method. The energy equation approach is employed by some authors in the deterministic case on unboundedness domain [18]. In this paper, we provide uniform estimates on the far-field values of solutions, which can be used to circumvent the difficulty caused by the unboundedness of the domain. Some authors have used this method. The master devote in this essay is to develop the method of using tail estimates to the case of stochastic dissipative systems [9], and prove the existence of a random attractor for the stochastic partly dissipative reaction diffusion systems in $L^2(R^n)\times L^2(R^n)$.

The paper is made as follows. In the Section 2, we recall some main definitions and results concerning the existence of a random attractor for random dynamical systems. In Section 3, we transform (1.1)-(1.2) into a deterministic systems with random parameter and come into being a continuous random dynamical system. In Section 4, we devote to obtaining uniform estimates of solutions when $t\to\infty$. These estimates are necessary for proving the existence of bounded absorbing set and the asymptotic compactness of the equation. In the last section, we first establish the asymptotic compactness of the solution operator by giving uniform estimates on the tails of solutions, and then prove the existence of a pullback random attractor.

In the paper, we set $L^2(R^n)$, $H_0^1(R^n)$ and $E=L^2(R^n)\times L^2(R^n)$ with the following inner products and norms, respectively

$$(u,v)=\int_{R^n} uv\mathrm{d}x, \quad \|u\|=(u,u)^{\frac{1}{2}}, \qquad \forall u,v\in L^2(R^n).$$

$$((u,v))=\int_{R^n}\nabla u\nabla v\mathrm{d}x, \quad \|u\|_{H_1}=((u,u))^{\frac{1}{2}}=\left(\int_{R^n}(\nabla u)^2\mathrm{d}x\right)^{\frac{1}{2}}, \qquad \forall u,v\in H_0^1(R^n).$$

$$(y_1,y_2)_E=(u_1,u_2)+(v_1,v_2).$$

$$\|y_i\|_E=(y_i,y_i)_E^{\frac{1}{2}}=(\|u_i\|^2+\|v_i\|^2)^{\frac{1}{2}}, \qquad \forall y_i=(u_i,v_i)^\top\in E,\ i=1,2.$$

The letters c and $c_i(i=1,2,\cdots)$ are generic positive constants which may change their values from line to line or even in the same line.

2 Preliminaries on random dynamical systems

In this section, we introduce some basic concepts related to random attractors for stochastic dynamical systems [19]. To different concepts, We can consult these literature [20,21]. Let $(X,\ \|\cdot\|_X)$ be a separate Hilbert space with Borel σ-algebra $\mathcal{B}(X)$ and the three parts $(\Omega,\mathcal{F},P)$ is a probability space.

Definition 2.1. *A quadruple $(\Omega,\mathcal{F},P,(\theta_t)_{t\in R})$ is called a metric dynamical system if $\theta:R\times\Omega\to\Omega$ is $(\mathcal{B}(R)\times\mathcal{F},\mathcal{F})$ -measurable, θ_0 is the identity on Ω , $\theta_{s+t}=\theta_t\circ\theta_s$, for all $s,t\in R$ and $\theta_t P=P$ for all $t\in R$.*

Definition 2.2. *A continuous random dynamical system over $(\Omega,\mathcal{F},P,(\theta_t)_{t\in R})$ is a $(\mathcal{B}(R^+)\times\mathcal{F}\times\mathcal{B}(X),\mathcal{B}(X))$-measurable mapping. $S:R^+\times\Omega\times X\to X,(t,\omega,x)\to\varphi(t,\omega,x)$ and satisfies for P-a.e. $\omega\in\Omega$.*

(1)$S(0,\omega,\cdot)$ is the identify on X;

(2)$S(t+s,\omega,\cdot)=S(t,\theta_s\omega,\cdot)\circ S(s,\omega,\cdot)$ for all $s,t\geq 0$ and $\omega\in\Omega$.

(3)$S(t,\omega,\cdot):x\to x$ is continuous for all $t\in R^+$.

Definition 2.3. (1) *Let X is a Banach space, A set-valued mapping $\omega\to H(\omega):\Omega\to 2^X$ is said to be a random set if the mapping $\omega\to d(x,H(\omega))$ is measurable for any $x\in X$, If $H(\omega)$ is closed(compact) for each $\omega\in\Omega$, the mapping $\omega\to H(\omega)$ is called a random closed(compact) set. A random set $\omega\to H(\omega)$ is said to be bounded if there exist $x_0\in X$ and a random variable $R(\omega)>0$ such that $H(\omega)\subset\{x\in X:\|x-x_0\|\leq R(\omega)\}$ for all $\omega\in\Omega$.*

(2) *A random set* $\omega \to B(\omega)$ *is called tempered if for P-a.e.* $\omega \in \Omega$. $\lim_{t\to\infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0$ *for all* $\beta > 0$, *where* $d(B(\theta_{-t}\omega)) = \sup_{x\in B(\theta_{-t}\omega)} \|x\|_X$.

(3) *Let* $\mathcal{D}$ *be a collection of random subsets of X and* $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. *Then* $\{K(\omega)\}_{\omega\in\Omega}$ *is called a random absorbing set for S in* $\mathcal{D}$ *if for every* $B \in \mathcal{D}$ *and P-a.e.* $\omega \in \Omega$, *there exists* $t_B(\omega) > 0$ *such that* $S(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega)$ *for all* $t \geq t_B(\omega)$.

Definition 2.4. *Let* $\mathcal{D}$ *be a collection of random subsets of X. A random set* $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *of X is called a* $\mathcal{D}$*-pullback attractor(*$\mathcal{D}$*-random attractor) for S if the following conditions are satisfied for P-a.e.* $\omega \in \Omega$.
(i) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *is compact and* $\omega \to d(x,\mathcal{A}(\omega))$ *is measurable for* $x \in X$;
(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *is invariant, that is* $S(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega)$ *for all* $t \geq 0$;
(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *attracts every set in* $\mathcal{D}$, *that is, for every* $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} d_H(S(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d_H *is the Hausdorff semi-distance given by* $d(Y,Z) = \sup_{y\in Y} \inf_{z\in Z} \|y-z\|_X$ *for any* $Y \subseteq X$ *and* $Z \subseteq X$.

The following existence result for a random attractor for a continuous RDS can be found in [9, 14–16, 22]. First, recall that a collection $\mathcal{D}$ of random subsets is called inclusion closed if whenever $S(\omega)_{\omega\in\Omega}$ is an arbitrarily random set and $T(\omega)_{\omega\in\Omega}$ is in $\mathcal{D}$ with $S(\omega) \subset T(\omega)$ for all $\omega \in \Omega$, then $S(\omega)_{\omega\in\Omega}$ must belong to $\mathcal{D}$.

Definition 2.5. *A measurable RDS* (φ,θ) *on a metric space X over MDS* $(\theta_t)_{t\in R}$ *is said to be asymptotically compact, if and only if for any sequence* $\{t_n : n \in N\}, t_n \to \infty$ *and any bounded sequence* $\{x_n \in X : n \in N\}$, *the set* $\{\varphi(t_n, \theta_{-t_n}\omega)x_n : n \in N\}$ *is relatively compact in X for each* $\omega \in \Omega$.

Proposition 2.1. *Let* $\mathcal{D}$ *be an inclusion-closed collection of random subset of* $E = L^2(R^n) \times L^2(R^n)$ *and* ϕ *is a continuous RDS on E over* $(\Omega,\mathcal{F},P,(\theta_t)_{t\in R})$. *Suppose that* $\{S(\omega)\}_{\omega\in\Omega}$ *is a closed random absorbing set for* ϕ *in* $\mathcal{D}$ *and* ϕ *is* $\mathcal{D}$*-pullback asymptotically compact in E, then* ϕ *has a unique* $\mathcal{D}$*-random attractor* $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *which is definite by*

$$\{\mathcal{A}(\omega)\} = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t,\theta_{-t}\omega, S(\theta_{-t}\omega))}.$$

In this paper, we will take $\mathcal{D}$ as the collection of all tempered random subsets of E and prove the stochastic reaction-diffusion equation in E has a $\mathcal{D}$-random attractor.

3 The partly dissipative systems on $L^2(R^n)\times L^2(R^n)$ with additive noise

Here we show that there is a continuous random dynamical system generated by the stochastic partly dissipative systems defined on $L^2(R^n)\times L^2(R^n)$ with additive noise:

$$\mathrm{d}u+(-\mu\Delta u+\lambda u+\alpha v)\mathrm{d}t=(h(u)+f(x))\mathrm{d}t+\varepsilon\sum_{j=1}^{m}h_j\mathrm{d}\omega_j, \tag{3.1}$$

$$\mathrm{d}v+(\delta v-\beta u)\mathrm{d}t=g(x)\mathrm{d}t+\varepsilon\sum_{j=1}^{m}h_j^*\mathrm{d}\omega_j, \tag{3.2}$$

with the initial conditions

$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x). \tag{3.3}$$

Here $\mu,\lambda,\alpha,\delta,\beta$ are positive constants, f,g,h_j and h_j^* are given functions, $h(u)$ is a nonlinear function satisfying the following condition:

$$h(t)t\le-\alpha_1|t|^q,\quad |h(t)|\le\alpha_2|t|^{q-1},\quad \frac{\partial h(s)}{\partial s}\le\alpha_3. \tag{3.4}$$

where α_1,α_2 and α_3 are positive constants.

In the sequel, we consider the probability space $(\Omega,\mathcal{F},P)$ where

$$\Omega=\{\omega=(\omega_1,\omega_2,\cdots),\ \omega_m\in C(R,R^m):\ \omega(0)=0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega,\mathcal{F})$, then we will identify ω with

$$W(t)\equiv(\omega_1(t),\omega_2(t),\cdots,\omega_m(t))=\omega(t)\qquad \text{for } t\in R.$$

We definite the time shift by $\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)$, $\omega\in\Omega,t\in R$. Then $(\Omega,\mathcal{F},P,(\theta_t)_{t\in R})$ is a metric dynamical system. To this end, we need to convert the stochastic systems with a random additive term into a deterministic systems with random parameter.

Given $j=1,2,\cdots,m$, we consider the one-dimensional Ornstein-Uhlenbeck equation [23],

$$\mathrm{d}z_j+\lambda z_j\mathrm{d}t=\mathrm{d}\omega_j(t). \tag{3.5}$$

The equation have a solution $z_j(t)=z_j(\theta_t\omega_j)\equiv-\lambda\int_{-\infty}^0 e^{\lambda\tau}(\theta_t\omega_j)(\tau)\mathrm{d}\tau,t\in R$. Note that the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t\omega_j)$ is P-a.e. continuous. So there exists a tempered function $r(\omega)>0$ such that

$$\sum_{j=1}^{m}(|z_j(\omega_j)|^2+|z_j(\omega_j)|^p)\le r(\omega), \tag{3.6}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega\in\Omega$,

$$r(\theta_t\omega)\le e^{\epsilon|t|}r(\omega),\quad \text{for } t\in R,\ \omega>0. \tag{3.7}$$

Then it get from(3.6)-(3.7) that, for P-a.e. $\omega\in\Omega$,

$$\sum_{j=1}^{m}(|z_j(\theta_t\omega_j)|^2+|z_j(\theta_t\omega_j)|^p)\le e^{\epsilon|t|}r(\omega),\quad t\in R. \tag{3.8}$$

We can refer to literature [1] to acquaintance more.

Let $z(\theta_t\omega)=\varepsilon\sum_{j=1}^m h_j z_j(\theta_t\omega_j)$, $z^*(\theta_t\omega)=\varepsilon\sum_{j=1}^m h_j^* z_j(\theta_t\omega_j)$, by(3.5) we have

$$\mathrm{d}z+\lambda z\mathrm{d}t=\varepsilon\sum_{j=1}^{m}h_j\mathrm{d}\omega_j,\quad \mathrm{d}z^*+\lambda z^*\mathrm{d}t=\varepsilon\sum_{j=1}^{m}h_j^*\mathrm{d}\omega_j. \tag{3.9}$$

Lemma 3.1. *Suppose $h_j,h_j^*\in H^2\bigcap W^{2,q}(D),j=1,2,\cdots,m$. For $\epsilon>0$, there is a constant $c>0$ such that*

$$\|z(\theta_t\omega)\|_q^q+\|z(\theta_t\omega)\|^2+\|z^*(\theta_t\omega)\|^2+\|\nabla z(\theta_t\omega)\|^2\le l_1 e^{\epsilon|t|}r(\omega), \tag{3.10}$$

$$\|\Delta z(\theta_t\omega)\|_q^q+\|\Delta z(\theta_t\omega)\|^2+\|z^*(\theta_t\omega)\|^2\le l_2 e^{\epsilon|t|}r(\omega), \tag{3.11}$$

$$\|\nabla z(\theta_t\omega)\|^2+\|\nabla z^*(\theta_t\omega)\|^2\le l_3 e^{\epsilon|t|}r(\omega), \tag{3.12}$$

for all $t\in R,\omega\in\Omega$, where

$$l_1=(\sum_{j=1}^{m}\|h_j\|_q^{\frac{q}{q-1}})^{q-1}+\sum_{j=1}^{m}(\|h_j\|^2+\|\nabla h_j^*\|^2\|h_j\|^2),$$

$$l_2=(\sum_{j=1}^{m}\|\Delta h_j\|_q^{\frac{q}{q-1}})^{q-1}+\sum_{j=1}^{m}(\|h_j\|^2+\|h_j^*\|^2),\qquad \text{and}\ \ l_3=\sum_{j=1}^{m}(\|\nabla h_j\|^2+\|\nabla h_j^*\|^2).$$

Proof. Since $z(\theta_t\omega)=\sum_{j=1}^{m}h_j z_j(\theta_t\omega_j)$, we get

$$\|z(\theta_t\omega)\|_q\le\sum_{j=1}^{m}\|h_j\|_q|z_j(\theta_t\omega_j)|\le(\sum_{j=1}^{m}\|h_j\|_q^{\frac{q}{q-1}})^{\frac{q-1}{q}}(\sum_{j=1}^{m}|z_j(\theta_t\omega_j)|^q)^{\frac{1}{q}}.$$

We have known that

$$\|z(\theta_t\omega)\|_q^q\le((\sum_{j=1}^{m}\|h_j\|_q^{\frac{q}{q-1}})^{q-1})e^{\epsilon|t|}r(\omega),$$

similarly,

$$\|z(\theta_t\omega)\|^2\le(\sum_{j=1}^{m}\|h_j\|^2)e^{\epsilon|t|}r(\omega),\quad \|z^*(\theta_t\omega)\|^2\le(\sum_{j=1}^{m}\|h_j^*\|^2)e^{\epsilon|t|}r(\omega),$$

and

$$\|\nabla z(\theta_t\omega)\|^2 \le (\sum_{j=1}^{m} \|\nabla h_j\|^2) e^{\epsilon|t|} r(\omega).$$

Adding the above three inequalities implies (3.10) holds. Similar to the prove of (3.10), we can prove the other two inequalities. □

The existence of a solution to the stochastic partly dissipative systems (3.1)-(3.2), with initial condition (3.3) have proved. To show (3.1)-(3.2) generates a random dynamical system, we let $n(t) = u(t) - z(\theta_t\omega)$, $m(t) = v(t) - z^*(\theta_t\omega)$, where (u,v) is a solution of problem (3.1)-(3.2), then $n(t), m(t)$ satisfies

$$\frac{\partial n}{\partial t} - \mu\Delta n + \lambda n + \alpha m = \mu\Delta z(\theta_t\omega) - \alpha z^*(\theta_t\omega) + h(u) + f(x), \tag{3.13}$$

$$\frac{\partial m}{\partial t} + \delta m - \beta n = \beta z(\theta_t\omega) + (\lambda - \delta) z^*(\theta_t\omega) + g(x), \tag{3.14}$$

with the initial date $(n_0, m_0) = (u_0 - z(\omega), v_0 - z^*(\omega))$ and homogeneous boundary conditions. For each invariability $\omega \in \Omega$, (3.13)-(3.14) is a deterministic differential equations. By a Galerkin method, one can show that if h satisfies (3.4), then (3.13)-(3.14) have a unique solution $(n,m) \in C([0,\infty); L^2 \times L^2) \bigcap L^2((0,T); H^1 \times L^2)$ with (n_0, m_0) for every $T \ge 0$, Let $\varphi_0 = (n_0, m_0) = (u_0 - z(\omega), v_0 - z^*(\omega))$ and $\varphi(t,\omega,\varphi_0) = (n(t,\omega,n_0), m(t,\omega,m_0))$, then the process $\phi = \varphi + (z(\theta_t\omega), z^*(\theta_t\omega))$ is the solution of problem (3.1)-(3.3). Therefore, ϕ is a continuous random dynamical system associated with the stochastic partly dissipative reaction-diffusion equations. In the next section, we establish uniform estimates for the solutions of problem (3.1)-(3.3), and prove the existence of a random attractor for ϕ.

4 Uniform estimates of solutions

Let $\varphi = (n,m)$ be the solution of (3.13)-(3.14). For $\omega \in \Omega$, we need the priori estimates of the solution $\varphi = (n,m)$ in $E = L^2(R^n) \times L^2(R^n)$. From now on, we always assume that $\mathcal{D}$ is the collection of all tempered subset of E with respect to $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in R})$, the next lemma will show that ϕ has a random absorbing set in $\mathcal{D}$.

Lemma 4.1. *Assume that $f, g \in L^2(R^n)$ and* (3.4) *hold, Let$B = B(\omega)_{\omega\in\Omega} \in \mathcal{D}$ which is the collection of all tempered subsets of E, and $\phi_0(\omega) = (u_0(\omega), v_0(\omega)) \in B(\omega)$, Then for P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$, such that for all $t \ge T_B(\omega)$.*

$$\phi(t, \theta_{-t}(\omega), \phi_0(\theta_{-t}\omega)) \subseteq K(\omega),$$

where c is a positive deterministic constant independent of $T_B(\omega)$ and $r(\omega)$ is tempered function.

Proof. Taking the inner product of both sides of (3.13) with βn, similarity, taking the inner product of both sides of (3.14) with αm, then we add the two equation together, we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\beta\|n\|^2+\alpha\|m\|^2)+\mu\beta\|\nabla n\|^2+\lambda\beta\|n\|^2+\delta\alpha\|m\|^2\\ =&\beta(h(u),n)+\beta\mu(\Delta z(\theta_t\omega),n)-\alpha\beta(z^*(\theta_t\omega),n)+\alpha\beta(z(\theta_t\omega),m)\\ &+(\lambda-\delta)(z^*(\theta_t\omega),m)+\beta(f,n)+\alpha(g,m).\end{aligned} \tag{4.1}$$

Now, we start to estimate the above equation, firstly we magnify the equation's terms as follows

$$\begin{aligned}&\beta(h(u),n)=\beta\int_{R^n}h(u)n\mathrm{d}x=\beta\int_{R^n}h(u)u\mathrm{d}x-\beta\int_{R^n}h(u)z(\theta_t\omega)\mathrm{d}x\\ \le&-\beta\alpha_1\int_{R^n}|u|^q\mathrm{d}x+\beta\alpha_2\int_{R^n}|u|^{q-1}|z(\theta_t\omega)|\mathrm{d}x\le-\frac{1}{2}\beta\alpha_1\|u\|_q^q+c\|z(\theta_t\omega)\|_q^q,\end{aligned} \tag{4.2}$$

$$|\beta\mu(\nabla z(\theta_t\omega),n)|=|\beta\mu\int_{R^n}\Delta z(\theta_t\omega)n\mathrm{d}x|\le\frac{1}{2}\beta\mu\|\nabla n\|^2+\frac{1}{2}\beta\mu\|\nabla z(\theta_t\omega)\|^2, \tag{4.3}$$

$$|-\alpha\beta(z^*(\theta_t\omega),n)|=|\alpha\beta\int_{R^n}z^*(\theta_t\omega)n\mathrm{d}x|\le\frac{\lambda\beta}{4}\|n\|^2+\frac{1}{\lambda}\beta\alpha^2\|z^*(\theta_t\omega)\|^2, \tag{4.4}$$

$$|\alpha\beta(z(\theta_t\omega),m)|\le\frac{1}{8}\alpha\delta\|m\|^2+\frac{2}{\delta}\alpha\beta^2\|z(\theta_t\omega)\|^2, \tag{4.5}$$

$$|\alpha(\lambda-\delta)(z^*(\theta_t\omega),m)|\le\frac{1}{8}\alpha\delta\|m\|^2+\frac{2}{\delta}\alpha(\lambda-\delta)^2\|z^*(\theta_t\omega)\|^2, \tag{4.6}$$

$$|\beta(f,n)|=|\beta\int_{R^n}fn\mathrm{d}x|\le\frac{1}{4}\beta\lambda\|n\|^2+\frac{1}{\lambda}\beta\|f\|^2, \tag{4.7}$$

$$|\alpha(g,m)|=|\alpha\int_{R^n}gm\mathrm{d}x|\le\frac{1}{8}\alpha\delta\|m\|^2+\frac{2}{\delta}\alpha\|g\|^2. \tag{4.8}$$

By(4.1)-(4.8), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\beta\|n\|^2+\alpha\|m\|^2)+\mu\beta\|\nabla n\|^2+\lambda\beta\|n\|^2+\delta\alpha\|m\|^2+\beta\alpha_1\|u\|_q^q\\ \le&c(\|z(\theta_t\omega)\|_q^q+\|z(\theta_t\omega)\|^2+\|z^*(\theta_t\omega)\|^2+\|\nabla z(\theta_t\omega)\|^2)+c\le p_0(\theta_t\omega)+c,\end{aligned} \tag{4.9}$$

where $p_0(\theta_t\omega)=c(\|z(\theta_t\omega)\|_q^q+\|z(\theta_t\omega)\|^2+\|z^*(\theta_t\omega)\|^2+\|\nabla z(\theta_t\omega)\|^2)$, we let $\nu=\min\{\delta,\lambda\}$, $\sigma=\min\{\alpha,\beta\}$ and $\gamma=\max\{\alpha,\beta\}$, and $\|\varphi\|_E^2=\|m\|^2+\|n\|^2$ then we find,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma\|\varphi\|_E^2)+\nu\sigma(\|\varphi\|_E^2)\le p_0(\theta_t\omega)+c. \tag{4.10}$$

Applying Gronwall's lemma, we find that, for all $t\ge 0$

$$\|\varphi(t,\omega,\varphi_0(\omega))\|_E^2\le\frac{1}{\sigma}\left(\gamma e^{-\nu t}\|\varphi_0(\omega)\|_E^2+\int_0^t e^{\nu(\tau-t)}p_0(\theta_\tau\omega)\mathrm{d}\tau+\frac{c}{\nu}\right). \tag{4.11}$$

By replacing ω by $\theta_{-t}(\omega)$ in (4.11), and by Lemma 3.1 with $\epsilon=\nu/2$, we obtain, for all $t\geq 0$,

$$\begin{aligned}
&\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2\\
\leq&\frac{1}{\sigma}\left(\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2+\int_0^t e^{\nu(\tau-t)}p_0(\theta_{\tau-t}\omega)\mathrm{d}\tau+\frac{c}{\nu}\right)\\
\leq&\frac{1}{\sigma}\left(\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2+\int_0^{-t} e^{\nu\tau}p_0(\theta_{\tau}\omega)\mathrm{d}\tau+\frac{c}{\nu}\right)\\
\leq&\frac{1}{\sigma}\left(\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2+cl_1\int_0^{-t} e^{\frac{1}{2}\nu\tau}r(\omega)\mathrm{d}\tau+\frac{c}{\nu}\right)\\
\leq&\frac{1}{\sigma}\left(\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2+\frac{2cl_1}{\nu}r(\omega)+\frac{c}{\nu}\right),
\end{aligned}\tag{4.12}$$

Because $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ is tempered and $\|z(\omega)\|^2,\|z^*(\omega)\|^2$ is also tempered, therefore, then there exist $T_B(\omega)>0$, such that for all $t\geq T_B(\omega)$,

$$\begin{aligned}
\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2\leq&\gamma e^{-\nu t}(\|u_0(\theta_{-t}\omega)\|^2+\|z(\theta_{-t}\omega)\|^2+\|v_0(\theta_{-t}\omega)\|^2+\|z^*(\theta_{-t}\omega)\|^2)\\
\leq&\frac{2cl_1}{\nu}r(\omega)+\frac{c}{\nu}.
\end{aligned}\tag{4.13}$$

It follows from (4.12) and (4.13) that, we have

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2\leq\frac{1}{\nu\sigma}(2cl_1+c)(1+r(\omega)).\tag{4.14}$$

Attention that $\phi=\varphi+(z(\theta_t\omega),z^*(\theta_t\omega))$, and in Lemma 3.1 $\|z(\omega)\|^2+\|z^*(\omega)\|^2\leq l_1r(\omega)$, we have

$$\|u\|^2+\|v\|^2\leq\frac{2}{\nu\sigma}(2cl_1+l_1+c)(1+r(\omega)).$$

Denote by $K(\omega)=\{(u,v)\in L^2(R^n)\times L^2(R^n):\|u\|^2+\|v\|^2\leq\frac{2}{\nu\sigma}(2cl_1+l_1+c)(1+r(\omega))\}$, then $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ is a random bounded absorbing set. □

Lemma 4.2. *Assume that $f,g\in L^2(R^n)$ and* (3.4) *hold, Let $B=B(\omega)_{\omega\in\Omega}\in\mathcal{D}$ which is the collection of all tempered subsets of E, and $\phi_0(\omega)=(u_0(\omega),v_0(\omega))\in B(\omega)$. Then for P-a.e. $\omega\in\Omega$, there exists $T_B(\omega)>0$, such that the solution (u,v) of problem* (3.1)-(3.3) *and (n,m) of* (3.13)-(3.14) *satisfy, for $t\geq T$,*

$$\int_T^t e^{\nu(s-t)}\|\varphi(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2\mathrm{d}s\leq c(1+r(\omega)+\|\varphi_0(\theta_{-t}\omega)\|_E^2e^{-\nu t})(t-T),\tag{4.15}$$

$$\begin{aligned}
&\int_T^t e^{\nu(s-t)}(\|\nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2+\|u(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_q^q)\mathrm{d}s\\
\leq&c(1+r(\omega)+\|\varphi_0(\theta_{-t}\omega)\|_E^2e^{-\nu t}),
\end{aligned}\tag{4.16}$$

$$\int_T^t e^{\nu(s-t)}(\|\nabla u(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2\mathrm{d}s\leq c(1+r(\omega)+\|\varphi_0(\theta_{-t}\omega)\|_E^2e^{-\nu t}),\tag{4.17}$$

where $\varphi_0(\omega)=\phi_0(\omega)-(z(\omega),z^*(\omega))$, c is a positive constant and $r(\omega)$ is a tempered function.

Proof. First, replacing t by T and ω by $\theta_{-t}\omega$ in (4.11), we obtain

$$\|\varphi(T,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 \leq \frac{1}{\sigma}\left(\gamma e^{-\nu T}\|\varphi_0(\theta_{-t}\omega)\|_E^2 + \int_0^T e^{\nu(\tau-T)} p_0(\theta_{\tau-t}\omega)\mathrm{d}\tau + \frac{c}{\nu}\right). \tag{4.18}$$

By $e^{\nu(T-t)}$ to multiply the above inequality and by lemma 3.1 with $\epsilon=\nu/2$, we can get

$$\begin{aligned}
&e^{\nu(T-t)}\|\varphi(T,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 \\
\leq& \frac{1}{\sigma}\left(\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2 + \int_{-t}^{T-t} e^{\nu\tau} p_0(\theta_\tau\omega)\mathrm{d}\tau + \frac{c}{\nu}e^{\nu(T-t)}\right) \\
\leq& \frac{1}{\sigma}\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2 + \frac{1}{\sigma}\left(l_1\int_{-t}^{T-t} e^{\frac{1}{2}\nu\tau} r(\omega)\mathrm{d}\tau + \frac{c}{\nu}\right) \\
\leq& \frac{1}{\sigma}\gamma e^{-\nu t}\|\varphi_0(\theta_{-t}\omega)\|_E^2 + \frac{1}{\sigma}\left(\frac{2}{\nu}l_1 r(\omega) + \frac{c}{\nu}\right) \\
\leq& c(1+r(\omega)+\|\varphi_0(\theta_{-t}\omega)\|_E^2 e^{-\nu t}).
\end{aligned} \tag{4.19}$$

Integral above inequality from T to t, we can get the (4.15). By (4.9) and (4.10), we can obtain that, for all $t\geq T$,

$$\begin{aligned}
&\mu\beta\int_T^t e^{\nu(s-t)}\|\nabla n(s,\omega,\varphi_0(\omega))\|^2\mathrm{d}s + \beta\alpha_1\int_T^t e^{\nu(s-t)}\|u(s,\omega,\varphi_0(\omega))\|_q^q\mathrm{d}s \\
\leq& \sigma e^{\nu(T-t)}\|\varphi(T,\omega,\varphi_0(\omega))\|_E^2 + \int_T^t e^{\nu(s-t)} p_0(\theta_s\omega)\mathrm{d}s + c\int_T^t e^{\nu(s-t)}\mathrm{d}s.
\end{aligned} \tag{4.20}$$

Replacing ω by $\theta_{-t}\omega$ in above inequality, we get that, for all $t\geq T$,

$$\begin{aligned}
&\mu\beta\int_T^t e^{\nu(s-t)}\|\nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2\mathrm{d}s + \beta\alpha_1\int_T^t e^{\nu(s-t)}\|u(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_q^q\mathrm{d}s \\
\leq& \sigma e^{\nu(T-t)}\|\varphi(T,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 + \int_T^t e^{\nu(s-t)} p_0(\theta_{s-t}\omega)\mathrm{d}s + c\int_T^t e^{\nu(s-t)}\mathrm{d}s \\
\leq& \sigma e^{\nu(T-t)}\|\varphi(T,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 + \int_{T-t}^0 e^{\nu s} p_0(\theta_s\omega)\mathrm{d}s + \frac{c}{\nu} \\
\leq& \sigma e^{\nu(T-t)}\|\varphi(T,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 + \frac{2}{\nu}l_1 r(\omega) + \frac{c}{\nu}.
\end{aligned} \tag{4.21}$$

It follows from (4.19) and (4.21), we can obtain the (4.16). By the inequality (4.16), we can obtain

$$\int_T^t e^{\nu(s-t)}\|\nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2\mathrm{d}s \leq c(1+r(\omega)+\|\varphi_0(\theta_{-t}\omega)\|_E^2 e^{-\nu t}).$$

We know that $\nabla u(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)) = \nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)) + \nabla z(\theta_{s-t}\omega)$.

So $\|\nabla u(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2 \leq 2\|\nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2 + 2\|\nabla z(\theta_{s-t}\omega)\|^2$. By the inequality (3.6) we know $\|\nabla z(\theta_t\omega)\|^2 \leq l_3 e^{\epsilon|t|} r(\omega)$. From above, we can get (4.17). □

Lemma 4.3. *Assume that* $f,g\in L^2(R^n)$ *and* (3.4) *hold, Let* $B=B(\omega)_{\omega\in\Omega}\in\mathcal{D}$ *which is the collection of all tempered subsets of* E*, and* $\phi_0(\omega)=(u_0(\omega),v_0(\omega))\in B(\omega)$*. Then for P-a.e.* $\omega\in\Omega$*, there exists* $T_B(\omega)>0$*, such that the solution* (u,v) *of problem* (3.1)-(3.3) *and* (n,m) *of* (3.13)-(3.14) *satisfy, for* $t\geq T_B(\omega)$*,*

$$\int_t^{t+1}\|\varphi(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\mathrm{d}s\leq c(1+r(\omega)), \tag{4.22}$$

$$\begin{aligned}&\int_t^{t+1}(\|\nabla n(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2+\|u(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|_q^q)\mathrm{d}s\\ \leq& c(1+r(\omega)),\end{aligned} \tag{4.23}$$

$$\int_t^{t+1}\|\nabla u(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\mathrm{d}s\leq c(1+r(\omega)), \tag{4.24}$$

where c is a positive deterministic constant independent of $T_B(\omega)$ *and* $r(\omega)$ *is tempered function.*

Proof. First replacing t by $t+1$ and then replacing T by t in inequality (4.15), we obtain

$$\begin{aligned}&\int_t^{t+1}e^{\nu(s-t-1)}\|\varphi(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|_E^2\mathrm{d}s\\ \leq& c(1+r(\omega)+\|\varphi_0(\theta_{-t-1}\omega)\|_E^2e^{-\nu(t+1)}).\end{aligned} \tag{4.25}$$

We know that

$$\begin{aligned}\|\varphi_0(\theta_{-t-1}\omega)\|_E^2=&\|n_0(\theta_{-t-1}\omega)\|^2+\|m_0(\theta_{-t-1}\omega)\|^2\\ \leq& 2\|u_0(\theta_{-t-1}\omega)\|^2+2\|z(\theta_{-t-1}\omega)\|^2+2\|v_0(\theta_{-t-1}\omega)\|^2+2\|z^*(\theta_{-t-1}\omega)\|^2.\end{aligned}$$

And $\|u_0(\theta_{-t}\omega)\|^2,\|z(\theta_{-t}\omega)\|^2,\|v_0(\theta_{-t}\omega)\|^2$ and $\|z^*(\theta_{-t}\omega)\|^2$ are tempered, there is $T_B(\omega)>0$ such that for $t\geq T_B(\omega)$

$$\|\varphi_0(\theta_{-t-1}\omega)\|_E^2e^{-\nu(t+1)}\leq c(1+r(\omega)).$$

Hence, from (4.25), we have, for all $t\geq T_B(\omega)$

$$\int_t^{t+1}\|\varphi(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\mathrm{d}s\leq c(1+r(\omega)).$$

By (4.16), we can also find that for all $t\geq T_B(\omega)$, so the (4.23) is established.

By means of above, we know that

$$\begin{aligned}&\|\nabla u(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\\ =&\|\nabla n(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))+\nabla z(\theta_{s-t-1}\omega)\|^2\\ \leq&2(\|\nabla n(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2+\|\nabla z(\theta_{s-t-1}\omega)\|^2).\end{aligned} \tag{4.26}$$

By Lemma 3.1, when $\epsilon=\nu/2$, we obtain

$$\|\nabla z(\theta_{s-t-1}\omega)\|^2\le l_3 e^{\frac{\nu}{2}(t+1-s)}r(\omega)\le l_3 e^{\frac{\nu}{2}}r(\omega). \tag{4.27}$$

From inequality (4.27) and integrate (4.26), we can have

$$\begin{aligned}&\int_t^{t+1}\|\nabla u(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\mathrm{d}s\\ \le&2\left(\int_t^{t+1}\|\nabla n(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\mathrm{d}s+\int_t^{t+1}\|\nabla z(\theta_{s-t-1}\omega)\|^2\mathrm{d}s\right)\\ \le&2c(1+r(\omega))+2l_3e^{\frac{\nu}{2}}r(\omega)\le c(1+r(\omega)).\end{aligned} \tag{4.28}$$

The result (4.24) hold from (4.28). □

Lemma 4.4. *Assume that* $f,g\in L^2(R^n)$ *and* (3.4) *hold, let* $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ *which is the collection of all tempered subsets of* E *and* $\phi_0(\omega)=(u_0(\omega),v_0(\omega))\in B(\omega)$, *then for* P*-a.e.* $\omega\in\Omega$, *there exists* $T_B(\omega)>0$ *such that for* $t\ge T_B(\omega)$

$$\|\nabla u(t,\theta_{-t}(\omega),\phi_0(\theta_{-t}\omega))\|^2\le c(1+r(\omega)).$$

Proof. Taking the inner product of (3.13) with $-\Delta n$ in L^2, we get that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|^2+\mu\|\Delta n\|^2+\lambda\|\nabla n\|^2+\alpha(m,-\Delta n)\\ =&(h(u),-\Delta n)+\mu(\Delta z,-\Delta n)+(f,-\Delta n)-\alpha(z^*,-\Delta n).\end{aligned} \tag{4.29}$$

Note that

$$\begin{aligned}&-\int_{R^n}h(u)\Delta n\mathrm{d}x=-\int_{R^n}h(u)\Delta u\mathrm{d}x+\int_{R^n}h(u)\Delta z(\theta_t\omega)\mathrm{d}x\\ \le&\int_{R^n}h'(u)|\nabla u|^2\mathrm{d}x+\int_{R^n}|h(u)\Delta z(\theta_t\omega)|\mathrm{d}x\le\alpha_3\|\nabla u\|^2+\alpha_2\int_{R^n}|u|^{q-1}|\Delta z(\theta_t\omega)|\mathrm{d}x\\ \le&c(\|\nabla u\|^2+\|u\|_q^q)+c\|\Delta z(\theta_t\omega)\|_q^q,\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}&|-\alpha(m,-\Delta n)+\mu(\Delta z,-\Delta n)+(f,-\Delta n)-\alpha(z^*,-\Delta n)|\\ \le&\frac{\mu}{2}\|\Delta n\|^2+2\frac{\alpha^2}{\mu}\|m\|^2+2\mu\|\Delta z(\theta_t\omega)\|^2+2\frac{1}{\mu}\|f\|^2+2\frac{\alpha^2}{\mu}\|z^*(\theta_t\omega)\|^2.\end{aligned} \tag{4.31}$$

It follows from(4.29)-(4.31), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla n\|^2\le c(\|m\|^2+\|\nabla u\|^2+\|u\|_q^q)+p_1(\theta_t\omega), \tag{4.32}$$

where $p_1(\theta_t\omega)$ is a tempered function, and $p_1(\theta_t\omega)\le cr(\omega)e^{-\frac{1}{2}\nu t}+c$, let $T_B(\omega)$ is the positive constant in Lemma 4.1, take $t\ge T_B(\omega)$ and $s\in(t,t+1)$, then integrate (4.32) over $(s,t+1)$ to get

$$\begin{aligned}&\|\nabla n(t+1),\omega,\varphi_0(\omega)\|^2-\|\nabla n(s,\omega,\varphi_0(\omega))\|^2\\ \le&\int_s^{t+1}p_1(\theta_\tau\omega)\mathrm{d}\tau+c\int_s^{t+1}\|m(\tau,\omega,\varphi_0(\omega))\|^2\mathrm{d}\tau\\ &+c\int_s^{t+1}(\|\nabla u(\tau,\omega,\varphi_0(\omega))\|^2+\|u(\tau,\omega,\varphi_0(\omega))\|_q^q)\mathrm{d}\tau\\ \le&\int_t^{t+1}p_1(\theta_\tau\omega)\mathrm{d}\tau+c\int_t^{t+1}\|m(\tau,\omega,\varphi_0(\omega))\|^2\mathrm{d}\tau\\ &+c\int_t^{t+1}(\|\nabla u(\tau,\omega,\varphi_0(\omega))\|^2+\|u(\tau,\omega,\varphi_0(\omega))\|_q^q)\mathrm{d}\tau. \qquad (4.33)\end{aligned}$$

Now replacing ω by $\theta_{-t-1}\omega$, and integrating the inequality with respect to s over $(t,t+1)$, we obtain that

$$\begin{aligned}&\|\nabla n(t+1),\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega)\|^2-\int_t^{t+1}\|\nabla n(s,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\mathrm{d}s\\ \le&\int_t^{t+1}p_1(\theta_{\tau-t-1}\omega)\mathrm{d}\tau+c\int_t^{t+1}\|m(\tau,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\mathrm{d}\tau\\ &+c\int_t^{t+1}(\|\nabla u(\tau,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2+\|u(\tau,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|_q^q)\mathrm{d}\tau. \quad (4.34)\end{aligned}$$

Since $\|m(\tau,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\le\|\varphi(\tau,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2$, by Lemma 3.1, Lemma 4.2, and Lemma 4.3, and it follows from (4.34) that for all $t\ge T_B(\omega)$,

$$\|\nabla n(t+1,\theta_{-t-1}\omega,\varphi_0(\theta_{-t-1}\omega))\|^2\le c(1+r(\omega)).$$

So the result is accomplished. □

In the following, we prove v is precompact in $L^2(R^n)$, we decompose $v=m_1+m_2+z^*(\theta_t\omega),m_i(i=1,2)$ solves respectively,

$$\frac{\partial}{\partial t}m_1+\delta m_1=0, \qquad (4.35)$$

$$m_{1,0}(0)=m_0=v_0-z^*(\omega), \qquad (4.36)$$

and

$$\frac{\partial}{\partial t}m_2+\delta m_2=\beta u+g(x)+(\lambda-\delta)z^*(\theta_t\omega), \qquad (4.37)$$

$$m_{2,0}(0)=0. \qquad (4.38)$$

For m_1,m_2, we have the following lemma.

Lemma 4.5. *Assume that* $f,g\in L^2(R^n)$ *and* (3.4) *hold, let* $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ *which is the collection of all tempered subsets of E and* $\phi_0(\omega)=(u_0(\omega),v_0(\omega))\in B(\omega)$*, then for P-a.e.* $\omega\in\Omega$*, there exists* $T_B(\omega)>0$ *such that for* $t\geq T_B(\omega)$

$$\|\nabla v(t,\theta_{-t}\omega,z^*(\theta_{-t}\omega))\|^2\leq c(1+r(\omega)), \tag{4.39}$$

where c is a positive deterministic constant independent of $T_B(\omega)$ *and* $r(\omega)$ *is tempered function.*

Proof. Taking the inner product of (4.35) with m_1 in L^2, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m_1\|+2\delta\|m_1\|=0. \tag{4.40}$$

Applying Gronwall's Lemma, we find that, for all $t\geq 0$

$$\|m_1(t,\omega,m_0(\omega))\|^2=e^{-2\delta t}\|m_0(\omega)\|^2. \tag{4.41}$$

Replacing ω by $\theta_{-t}\omega$, we have

$$\|m_1(t,\theta_{-t}\omega,m_0(\theta_{-t}\omega))\|^2=e^{-2\delta t}\|m_0(\theta_{-t}\omega)\|^2. \tag{4.42}$$

Multiplying (4.37) by $-\Delta m_2$ and integrating over $(0,t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla m_2\|^2+\delta\|\nabla m_2\|^2\\=&\beta(u,-\Delta m_2)+(g,-\Delta m_2)+(\lambda-\delta)(z^*(\theta_t\omega),-\Delta m_2)\\\leq&\frac{\delta}{2}\|\nabla m_2\|^2+\frac{2\beta^2}{\delta}\|\nabla u\|^2+\frac{2}{\delta}\|\nabla g\|^2+\frac{2}{\delta}(\lambda-\delta)^2\|z^*(\theta_t\omega)\|^2.\end{aligned} \tag{4.43}$$

Since $\nu=\min\{\delta,\mu\}$, afterwards, by Gronwall's inequality and replacing ω by $\theta_{-t}\omega$, we can obtain(refer to literature [1] to acquaintance more)

$$\begin{aligned}\|\nabla m_2(t,\theta_{-t}\omega)\|^2\leq&\frac{4}{\nu}\beta^2c(1+r(\omega)+\|\varphi_0(\theta_{-T}\omega)\|_E^2e^{-\nu t})\\&+\frac{4}{\nu^2}\|\nabla g\|^2+\frac{8}{\nu^2}(\lambda-\delta)^2l_1r(\omega),\end{aligned} \tag{4.44}$$

where we have used Lemma 3.1 and (4.17)with $T=0$, then there exists $T_B(\omega)>0$, such that for $t\geq T_B(\omega)$, we obtain

$$\|\nabla m_2(t,\theta_{-t}\omega)\|^2\leq c(1+r(\omega)). \tag{4.45}$$

$$\|\nabla v\|^2=\|\nabla(m_1+m_2+z^*(\theta_t\omega))\|^2\leq 2\|\nabla m_1\|^2+2\|\nabla(m_2+z^*(\theta_t\omega))\|^2. \tag{4.46}$$

The result hold from (4.42), (4.45) and (4.46), we completed the proof. $\square$

Lemma 4.6. *Assume that* $f,g\in L^2(R^n)$ *and* (3.4) *hold, let* $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ *which is the collection of all tempered subsets of E and* $u_0(\omega)\in B(\omega)$. *Then for every* $\epsilon>0$ *and P-a.e.* $\omega\in\Omega$, *there exist* $T^*=T_{B^*}(\omega,\epsilon)>0$ *and* $R^*=R^*(\omega,\epsilon)>0$ *such that the solution* $\varphi(t,\omega,v_0(\omega))$ *of* (3.13)-(3.14) *with* $\varphi_0(\omega)=\phi_0(\omega)-(z(\omega),z^*(\omega))$ *satisfies, for all* $t\geq T^*=T^*_B(\omega,\epsilon)$.

$$\int_{|x|\geq R^*}|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))(x)|^2\mathrm{d}x\leq\epsilon.$$

Proof. Let ρ be a smooth function defined on R^+, such that $0\leq\rho(s)\leq 1$ for all $s\in R^+$, and

$$\rho(s)=\begin{cases}0, & \text{for } 0\leq s\leq 1,\\ 1, & \text{for } s\geq 2.\end{cases}\tag{4.47}$$

Then there exist a positive constant c such that $|\rho'(s)|\leq c$ for all $s\in R^+$. Taking the inner product of (3.13) with $\beta\rho(|x|^2/k^2)n$ in $L^2(R^n)$, we get that

$$L_1(t,\omega)=R_1(t,\omega),\tag{4.48}$$

where

$$\begin{aligned}L_1(t,\omega)=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n}\beta\rho\left(\frac{|x|^2}{k^2}\right)|n|^2\mathrm{d}x-\int_{R^n}\mu\beta\rho\left(\frac{|x|^2}{k^2}\right)\Delta n n\mathrm{d}x+\int_{R^n}\lambda\beta\rho\left(\frac{|x|^2}{k^2}\right)|n|^2\mathrm{d}x\\&+\int_{R^n}\alpha\beta\rho\left(\frac{|x|^2}{k^2}\right)mn\mathrm{d}x.\\R_1(t,\omega)=&\int_{R^n}\mu\beta\rho\left(\frac{|x|^2}{k^2}\right)n\Delta z(\theta_t\omega)\mathrm{d}x-\int_{R^n}\alpha\beta\rho\left(\frac{|x|^2}{k^2}\right)nz^*(\theta_t\omega)\mathrm{d}x\\&+\int_{R^n}\beta\rho\left(\frac{|x|^2}{k^2}\right)n(h(u)+f(x))\mathrm{d}x.\end{aligned}$$

Equally important, we taking the inner product of (3.14) with $\alpha\rho(|x|^2/k^2)m$ in $L^2(R^n)$, we get that

$$L_2(t,\omega)=R_2(t,\omega),\tag{4.49}$$

where

$$\begin{aligned}L_2(t,\omega)=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n}\alpha\rho\left(\frac{|x|^2}{k^2}\right)|m|^2\mathrm{d}x+\int_{R^n}\delta\alpha\rho\left(\frac{|x|^2}{k^2}\right)|m|^2\mathrm{d}x-\int_{R^n}\alpha\beta\rho\left(\frac{|x|^2}{k^2}\right)mn\mathrm{d}x.\\R_2(t,\omega)=&\int_{R^n}\alpha\beta\rho\left(\frac{|x|^2}{k^2}\right)mz(\theta_t\omega)\mathrm{d}x+\int_{R^n}\alpha(\lambda-\delta)\rho\left(\frac{|x|^2}{k^2}\right)mz^*(\theta_t\omega)\mathrm{d}x\\&+\int_{R^n}\alpha\rho\left(\frac{|x|^2}{k^2}\right)mg(x)\mathrm{d}x.\end{aligned}$$

Summing up (4.48) and (4.49) two equation, we have $U_1(t,\omega)=U_2(t,\omega)$.

$$U_1(t,\omega)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n}(\beta n^2+\alpha m^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x-\int_{R^n}\mu\beta\Delta n\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x$$

$$+\int_{R^n}\lambda\beta\rho\left(\frac{|x|^2}{k^2}\right)|n|^2\mathrm{d}x+\int_{R^n}\delta\alpha\rho\left(\frac{|x|^2}{k^2}\right)|m|^2\mathrm{d}x.$$

$$\begin{aligned}U_2(t,\omega)=&\int_{R^n}\mu\beta\Delta z(\theta_t\omega)\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x-\int_{R^n}\alpha\beta z^*(\theta_t)\omega)\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x\\&+\int_{R^n}(h(u)+f(x))\beta\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x+\int_{R^n}\alpha\beta z(\theta_t\omega)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x\\&+\int_{R^n}\alpha(\lambda-\delta)z^*(\theta_t\omega)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x+\int_{R^n}\alpha g(x)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x.\end{aligned}$$

Let $\sigma=\min\{\alpha,\beta\},\gamma=\max\{\alpha,\beta\},\nu=\min\{\delta,\lambda\}$, we can get that $V_1(t,\omega)\le V_2(t,\omega)$.

$$\begin{aligned}V_1(t,\omega)=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n}\sigma(m^2+n^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\int_{R^n}(n^2+m^2)\sigma\nu\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x.\\V_2(t,\omega)=&\int_{R^n}\mu\beta\Delta n\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x+\int_{R^n}(\mu\beta\Delta z(\theta_t\omega)-\alpha\beta)z^*(\theta_t\omega)\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x\\&+\int_{R^n}(h(u)+f(x))\beta\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x+\int_{R^n}\alpha g(x)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x\\&+\int_{R^n}(\alpha\beta z(\theta_t\omega)+\alpha(\lambda-\delta)z^*(\theta_t\omega))\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x.\end{aligned}$$

We now estimate the terms in $V_2(t,\omega)$ as follows. First we have

$$\begin{aligned}&-\int_{R^n}\mu\beta\rho\left(\frac{|x|^2}{k^2}\right)\Delta n\cdot n\mathrm{d}x=-\int_{R^n}\mu\beta\Delta n\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x=\int_{R^n}\mu\beta\nabla n\left(\rho\left(\frac{|x|^2}{k^2}\right)n\right)'\mathrm{d}x\\&=\int_{R^n}\mu\beta|\nabla n|^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\int_{R^n}\mu\beta n\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\nabla n\mathrm{d}x\\&=\int_{R^n}\mu\beta|\nabla n|^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\int_{k\le|x|\le\sqrt{2}k}\mu\beta n\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\nabla n\mathrm{d}x.\end{aligned}\tag{4.50}$$

Attention that the second term on the right-hand side of (4.50) is bounded by

$$\begin{aligned}&\left|\int_{k\le|x|\le\sqrt{2}k}\mu\beta n\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\nabla n\mathrm{d}x\right|\le\frac{2\sqrt{2}}{k}\int_{k\le|x|\le\sqrt{2}k}\mu\beta|n||\rho'\left(\frac{|x|^2}{k^2}\right)||\nabla n|\mathrm{d}x\\&\le\frac{c2\sqrt{2}\mu\beta}{k}\int_{R^n}|n||\nabla n|\mathrm{d}x\le\frac{c}{k}(\|n\|^2+\|\nabla n\|^2).\end{aligned}\tag{4.51}$$

By (4.50) and (4.51), we find that

$$-\int_{R^n}\mu\beta\rho\left(\frac{|x|^2}{k^2}\right)\Delta n\cdot n\mathrm{d}x\ge\int_{R^n}\mu\beta|\nabla n|^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x-\frac{c}{k}(\|n\|^2+\|\nabla n\|^2).\tag{4.52}$$

Now we estimate the second term in $V_2(t,\omega)$, by Cauchy-Schwarz inequality we can get that

$$\begin{aligned}&\int_{R^n}(\mu\Delta z(\theta_t\omega)-\alpha z^*(\theta_t\omega))\beta\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x\\ \le&\frac{1}{8}\lambda\beta\int_{R^n}\rho\left(\frac{|x|^2}{k^2}\right)|n|^2\mathrm{d}x+\frac{2\beta}{\lambda}\int_{R^n}(\mu\Delta z(\theta_t\omega)-\alpha z^*(\theta_t\omega))^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\ \le&\frac{1}{8}\lambda\beta\int_{R^n}\rho\left(\frac{|x|^2}{k^2}\right)|n|^2\mathrm{d}x+\frac{2\beta}{\lambda}\int_{R^n}[\mu^2(\Delta z(\theta_t\omega))^2+\alpha^2(z^*(\theta_t\omega))^2]\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x. \qquad (4.53)\end{aligned}$$

By(4.2) and (4.7) we can estimate the third terms in $V_2(t,\omega)$, and we can get that

$$\begin{aligned}&\int_{R^n}(h(u)+f(x))\beta\rho\left(\frac{|x|^2}{k^2}\right)n\mathrm{d}x\\ =&\int_{R^n}(h(u)+f(x))\beta\rho\left(\frac{|x|^2}{k^2}\right)u\mathrm{d}x-\int_{R^n}(h(u)+f(x))\beta\rho\left(\frac{|x|^2}{k^2}\right)z(\theta_t)\omega\mathrm{d}x\\ =&\int_{R^n}(h(u)u-h(u)z(\theta_t\omega))\beta\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\int_{R^n}[f(x)u-f(x)z(\theta_t\omega)]\beta\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\ \le&\int_{R^n}(-\frac{1}{2}\beta\alpha_1|u|^q+c|z(\theta_t\omega)|^q)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\int_{R^n}(\frac{1}{4}\beta\lambda|n|^2+\frac{1}{\lambda}\beta|f|^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x. \qquad (4.54)\end{aligned}$$

By (4.5) and (4.6), (4.8) we can estimate the last terms in $V_2(t,\omega)$, and we can get that

$$\begin{aligned}&\int_{R^n}(\alpha\beta z(\theta_t\omega)+\alpha(\lambda-\delta)z^*(\theta_t\omega))\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x+\int_{R^n}\alpha g(x)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x\\ =&\int_{R^n}(\alpha\beta z(\theta_t\omega)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x+\int_{R^n}\alpha(\lambda-\delta)z^*(\theta_t\omega)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x\\ &\quad+\int_{R^n}\alpha g(x)\rho\left(\frac{|x|^2}{k^2}\right)m\mathrm{d}x\\ \le&\int_{R^n}(\frac{1}{8}\alpha\delta|m|^2+\frac{2}{\delta}\alpha\beta^2z^2(\theta_t\omega))\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\ &\quad+\int_{R^n}(\frac{\alpha}{8}\delta|m|^2+\frac{2\alpha}{\delta}(\lambda-\delta)^2(z^*(\theta_t\omega))^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\ &\quad+\int_{R^n}(\frac{1}{8}\alpha\delta|m|^2+\frac{2}{\delta}\alpha(g(x))^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\ \le&\int_{R^n}(\frac{3}{8}\alpha\delta|m|^2+c(z(\theta_t\omega))^2+c(z^*(\theta_t\omega))^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\int_{R^n}\frac{2}{\delta}\alpha g^2(x)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x. \qquad (4.55)\end{aligned}$$

Finally, from (4.52)-(4.55) we can get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n}\sigma\varphi^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\frac{5}{8}\sigma\nu\int_{R^n}\varphi^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x$$

$$\begin{aligned}&\leq\frac{c}{k}(\|n\|^2+\|\nabla n\|^2)+c\int_{R^n}(2f^2(x)+g^2(x))\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\&\quad+c\int_{R^n}(|\Delta z(\theta_t\omega)|^2+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^q+|z^*(\theta_t\omega)|^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x.\end{aligned}\tag{4.56}$$

Attention that (4.56) implies that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n}\varphi^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x+\frac{5}{4}\nu\int_{R^n}\varphi^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\&\leq\frac{c}{k}(\|n\|^2+\|\nabla n\|^2)+c\int_{R^n}(2f^2(x)+g^2(x))\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\&\quad+c\int_{R^n}(|\Delta z(\theta_t\omega)|^2+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^q+|z^*(\theta_t\omega)|^2)\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x.\end{aligned}\tag{4.57}$$

By Lemmas 4.1 and 4.5, there is $T_1=T_1(B,\omega)>0$ such that for all $t\geq T_1$

$$\|n(t,\omega,\varphi_0(\omega))\|^2_{H^1(R^n)}\leq c(1+r(\omega)).$$

Now we integrate the (4.57) over (T_1,t), and we can get that for all $t\geq T_1$

$$\begin{aligned}&\int_{R^n}|\varphi(t,\omega,\varphi_0(\omega))|^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\&\leq e^{\lambda(T_1-t)}\int_{R^n}\rho\left(\frac{|x|^2}{k^2}\right)|\varphi(T_1,\omega,\varphi_0(\omega))|^2\mathrm{d}x\\&\quad+\frac{c}{k}\int_{T_1}^t e^{\lambda(s-t)}(\|\nabla n(s,\omega,\varphi_0(\omega))\|^2+\|n(s,\omega,\varphi_0(\omega))\|^2)\mathrm{d}s\\&\quad+c\int_{T_1}^t e^{\lambda(s-t)}\int_{R^n}(2f^2(x)+g^2(x))\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\mathrm{d}s\\&\quad+c\int_{T_1}^t e^{\lambda(s-t)}\int_{R^n}V_3(t,\omega)\mathrm{d}x\mathrm{d}s,\end{aligned}\tag{4.58}$$

where $V_3(t,\omega)=(|\Delta z(\theta_s\omega)|^2+|z(\theta_s\omega)|^2+|z^*(\theta_s\omega)|^2+|z(\theta_s\omega)|^q)\rho\left(|x|^2/k^2\right)$. Replacing ω by $\theta_{-t}\omega$, we obtain from (4.58) that for all $t\geq T_1$,

$$\begin{aligned}&\int_{R^n}|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))|^2\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\\&\leq e^{\lambda(T_1-t)}\int_{R^n}\rho\left(\frac{|x|^2}{k^2}\right)|\varphi(T_1,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))|^2\mathrm{d}x\\&\quad+\frac{c}{k}\int_{T_1}^t e^{\lambda(s-t)}(\|\nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2+\|n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2)\mathrm{d}s\\&\quad+c\int_{T_1}^t e^{\lambda(s-t)}\int_{R^n}(2f^2(x)+g^2(x))\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\mathrm{d}s\end{aligned}$$

$$+c\int_{T_1}^{t} e^{\lambda(s-t)}\int_{R^n} V_4(t,\omega)\mathrm{d}x\mathrm{d}s, \tag{4.59}$$

where $V_4(t,\omega)=(|\Delta z(\theta_{s-t}\omega)|^2+|z(\theta_{s-t}\omega)|^2+|z^*(\theta_{s-t}\omega)|^2+|z(\theta_{s-t}\omega)|^q)\rho(|x|^2/k^2)$. Now we estimate the terms in (4.59). First replacing t by T_1, and then replacing ω by $\theta_{-t}\omega$ in (4.13), we have the following bounds for the first term

$$\begin{aligned}
&e^{\lambda(T_1-t)}\int_{R^n}\rho\left(\frac{|x|^2}{k^2}\right)|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))|^2\mathrm{d}x\\
\leq\ & e^{\lambda(T_1-t)}(e^{-\lambda T_1}\|\varphi_0(\theta_{-t}\omega)\|^2+\int_0^{T_1} e^{\lambda(s-T_1)}p_0(\theta_{s-t}\omega)\mathrm{d}s+c)\\
\leq\ & e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|^2+ce^{\lambda(T_1-t)}+\int_{-t}^{T_1-t} e^{\lambda\tau}p_0(\theta_\tau\omega)\mathrm{d}\tau\\
\leq\ & e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|^2+ce^{\lambda(T_1-t)}+\int_{-t}^{T_1-t} ce^{\frac{1}{2}\lambda\tau}r(\omega)\mathrm{d}\tau\\
\leq\ & e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|^2+ce^{\lambda(T_1-t)}+\frac{2}{\lambda}cr(\omega)e^{\frac{1}{2}\lambda(T_1-t)},
\end{aligned}\tag{4.60}$$

then we have found that, given $\epsilon>0$, there is $T_2(B,\omega,\epsilon)>T_1$ such that for all $t\geq T_2$

$$e^{\lambda(T_1-t)}\int_{R^n}\rho\left(\frac{|x|^2}{k^2}\right)|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))|^2\mathrm{d}x\leq\epsilon. \tag{4.61}$$

By Lemma 4.2, there is $T_3=T_3(B,\omega)>T_1$ such that the term satisfies

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\|\nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2\mathrm{d}s\leq\frac{c}{k}(1+r(\omega)).$$

Hence, there is $R_1=R_1(\omega,\epsilon)>0$, such that for all $t\geq T_3$ and $k\geq R_1$

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\|\nabla n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2\mathrm{d}s\leq\epsilon. \tag{4.62}$$

Estimate the next term

$$\begin{aligned}
&\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\|n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2\mathrm{d}s\\
\leq&\frac{c}{k}\int_{T_1}^{t} e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|^2\mathrm{d}s+\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\int_0^s e\lambda(\tau-s)p_0(\theta_{\tau-t}\omega)\mathrm{d}\tau\mathrm{d}s+\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\mathrm{d}s\\
\leq&\frac{c}{k}e^{-\lambda t}(t-T_1)\|\varphi_0(\theta_{-t}\omega)\|^2+\frac{c}{k}+\frac{c}{k}\int_{T_1}^{t}\int_0^s e^{\lambda(\tau-t)}p_0(\theta_{\tau-t}\omega)\mathrm{d}\tau\mathrm{d}s\\
\leq&\frac{c}{k}e^{-\lambda t}(t-T_1)\|\varphi_0(\theta_{-t}\omega)\|^2+\frac{c}{k}+\frac{c}{k}\int_{T_1}^{t}\int_{-t}^{s-t} e^{\lambda\tau}p_0(\theta_\tau\omega)\mathrm{d}\tau\mathrm{d}s
\end{aligned}$$

$$\leq \frac{c}{k}e^{-\lambda t}(t-T_1)\|\varphi_0(\theta_{-t}\omega)\|^2+\frac{c}{k}+\frac{c}{k}lr(\omega)\int_{T_1}^{t}\int_{-t}^{s-t}e^{\frac{1}{2}\lambda\tau}\mathrm{d}\tau\mathrm{d}s$$
$$\leq \frac{c}{k}e^{-\lambda t}(t-T_1)\|\varphi_0(\theta_{-t}\omega)\|^2+\frac{c}{k}+\frac{4c}{\lambda^2 k}lr(\omega), \tag{4.63}$$

where $l>0$, this implies that there exist $T_4=T_4(B,\omega,\epsilon)>T_1$ and $R_2=R_2(\omega,\epsilon)$ such that for all $t\geq T_4$ and $k\geq R_2$

$$\frac{c}{k}\int_{T_1}^{t}e^{\lambda(s-t)}\|n(s,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|^2\mathrm{d}s\leq\epsilon. \tag{4.64}$$

Attention that $f(x),g(x)\in L^2(R^n)$, therefore, there is $R_3=R_3(\epsilon)$, such that for all $k\geq R_3$,

$$\int_{|x|\geq k}(2f^2(x)+g^2(x))\mathrm{d}x\leq\lambda\epsilon.$$

Then for the fourth term on the equation's the right-hand side of (4.59), we have found

$$c\int_{T_1}^{t}e^{\lambda(s-t)}\int_{R^n}(2f^2(x)+g^2(x))\rho\left(\frac{|x|^2}{k^2}\right)\mathrm{d}x\mathrm{d}s\leq c\int_{T_1-t}^{0}e^{\lambda\zeta}\int_{|x|\geq k}(2f^2(x)+g^2(x))\mathrm{d}x\mathrm{d}\zeta$$
$$\leq c\lambda\epsilon\int_{T_1-t}^{0}e^{\lambda\zeta}\mathrm{d}\zeta\leq\epsilon. \tag{4.65}$$

Note that $z(\theta_t\omega)=\sum_{j=1}^m h_j z_j(\theta_t\omega_j), z^*(\theta_t\omega)=\sum_{j=1}^m h_j^* z_j^*(\theta_t\omega_j)$ and $h_j,h_j^*\in H^2(R^n)\bigcap W^{2,q}(R^n)$, hence there is $R_4=R_4(\omega,\epsilon)$, such that for all $k\geq R_4$ and $j=1,2,\cdots,m$,

$$\int_{|x|\geq k}(|h_j(x)|^2+|h_j(x)|^q+|\Delta h_j(x)|^2+|h_j^*(x)|^2)\mathrm{d}x\leq\min\left\{\frac{\lambda\epsilon}{4cm^q r(\omega)},\quad\frac{\epsilon}{2m^2 r(\omega)}\right\}, \tag{4.66}$$

where $r(\omega)$ is the tempered function and c is the positive constant. By above and (3.6), (3.7), we have the following bounds for the last term on the right hand side of (4.59),

$$\begin{aligned}
&c\int_{T_1}^{t}e^{\lambda(s-t)}\int_{R^n}V_4\mathrm{d}x\mathrm{d}s\\
\leq& c\int_{T_1-t}^{0}e^{\lambda\zeta}\int_{|x|\geq k}(|\Delta z(\theta_\zeta\omega)|^2+|z(\theta_\zeta\omega)|^2+|z^*(\theta_\zeta\omega)|^2+|z(\theta_\zeta\omega)|^q))\mathrm{d}x\mathrm{d}\zeta\\
\leq& cm^q\int_{T_1-t}^{0}e^{\lambda\zeta}\sum_{j=1}^{m}\int_{|x|\geq k}(|\Delta h_j|^2|\Delta z_j(\theta_\zeta\omega_j)|^2+|h_j|^2|z_j(\theta_\zeta\omega_j)|^2+|h_j^*|^2|z_j^*(\theta_\zeta\omega_j)|^2\\
&\quad+|h_j|^q|z_j(\theta_\zeta\omega_j)|^q)\mathrm{d}x\mathrm{d}\zeta\\
\leq&\frac{\lambda\epsilon}{2r(\omega)}\int_{T_1-t}^{0}e^{\lambda\zeta}\sum_{j=1}^{m}(|\Delta z_j(\theta_\zeta\omega_j)|^2+|z_j(\theta_\zeta\omega_j)|^2+|z_j^*(\theta_\zeta\omega_j)|^2+|z_j(\theta_\zeta\omega_j)|^q)\mathrm{d}\zeta\\
\leq&\frac{\lambda\epsilon}{2r(\omega)}\int_{T_1-t}^{0}e^{\lambda\zeta}r(\theta_\zeta\omega)\mathrm{d}\zeta\leq\frac{\lambda\epsilon}{2r(\omega)}\int_{T_1-t}^{0}e^{\lambda\zeta}r(\theta_\zeta\omega)\mathrm{d}\zeta
\end{aligned}$$

$$\leq \frac{\lambda\epsilon}{2r(\omega)}\int_{T_1-t}^{0} e^{\frac{1}{2}\lambda\zeta} r(\omega)\mathrm{d}\zeta \leq \epsilon. \tag{4.67}$$

Let $T_5 = T_5(B,\omega,\epsilon) = \max\{T_1,T_2,T_3,T_4\}$ and $R_5 = R_5(\omega,\epsilon) = \max\{R_1,R_2,R_3,R_4\}$, then it follows from above all, we have

$$\int_{R^n} \rho\left(\frac{|x|^2}{k^2}\right) |\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))|^2 \mathrm{d}x \leq 5\epsilon,$$

which shows that for all $t \geq T_5$, and $k \geq R_5$.

$$\int_{|x|\geq\sqrt{2}k} |\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))|^2 \mathrm{d}x \leq \int_{R^n} \rho\left(\frac{|x|^2}{k^2}\right) |\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))|^2 \mathrm{d}x \leq 5\epsilon.$$

We have completed the proof. □

5 The asymptotic compactness and existence of random attractor

In this section, we will prove the existence of a random attractor for the random dynamical system ϕ which is associated with stochastic partly dissipative reaction diffusion systems (3.1)-(3.3) in $L^2(R^n)\times L^2(R^n)$, It follows from Lemma 4.1 that ϕ has a closed random absorbing set. The follow will be using the uniform estimate on the tails of solution to get the D-pullback asymptotic compactness of ϕ. Finally, we will get the existence of the random attractor.

Lemma 5.1. *Assume that $f,g \in L^2(R^n)$ and* (3.4) *hold, let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ which is the collection of all tempered subset of E, and $\phi_0(\omega) = (u_0(\omega),v_0(\omega)) \in B(\omega)$, then for every $\epsilon > 0$, and P-a.e. $\omega \in \Omega$, there exist $T^* = T_B^*(\omega,\epsilon) > 0$ and $R^* = R^*(\omega,\epsilon) > 0$ such that for all $t \geq t^*$*

$$\int_{|x|\geq R^*} |\phi(t,\theta_{-t}\omega,\phi_0(\theta_{-t}\omega))(x)|^2 \mathrm{d}x \leq \epsilon.$$

Proof. Let T^* and R^* be the constants in Lemma 4.6, By (3.6) and (4.66) we have for all $t \geq T^*$ and $|x| \geq R^*$,

$$\begin{aligned}\int_{|x|\geq R^*} |z(\omega)|^2 \mathrm{d}x &= \int_{|x|\geq R^*} |\sum_{j=1}^{m} h_j z_j(\omega_j)|^2 \mathrm{d}x \leq m^2 \int_{|x|\geq R^*} \sum_{j=1}^{m} |h_j|^2 |z_j(\omega_j)|^2 \mathrm{d}x \\ &\leq \frac{\epsilon}{2r(\omega)} \sum_{j=1}^{m} |z_j(\omega_j)|^2 \leq \frac{\epsilon}{2r(\omega)} r(\omega) \leq \frac{1}{2}\epsilon. \end{aligned}\tag{5.1}$$

Then we can also obtain that,

$$\int_{|x|\geq R^*} |z^*(\omega)|^2 \mathrm{d}x \leq \frac{\epsilon}{2}. \tag{5.2}$$

By (5.1)-(5.2) and Lemma 4.6, we get that, for all $t\geq t^*$, and $|x|\geq R^*$,

$$\begin{aligned}
&\int_{|x|\geq R^*}|\phi(t,\theta_{-t}\omega,\phi_0(\theta_{-t}\omega))(x)|^2\mathrm{d}x\\
=&\int_{|x|\geq R^*}|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))+(z(\omega),z^*(\omega))|^2\mathrm{d}x\\
\leq&2\int_{|x|\geq R^*}\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))^2\mathrm{d}x+2\int_{|x|\geq R^*}|(z(\omega),z^*(\omega))|^2\mathrm{d}x\\
=&2\int_{|x|\geq R^*}\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))^2\mathrm{d}x+2\int_{|x|\geq R^*}z(\omega)^2\mathrm{d}x+2\int_{|x|\geq R^*}z^*(\omega)^2\mathrm{d}x\\
\leq&2\epsilon+2\frac{\epsilon}{2}+2\frac{\epsilon}{2}=4\epsilon.
\end{aligned}\tag{5.3}$$

The proof is completed. □

Lemma 5.2. *Assume that* $f,g\in L^2(R^n)$ *and* (3.4) *hold, then the random dynamical system* ϕ *is* $\mathcal{D}$*-pullback asymptotically compact in* $E=L^2(R^n)\times L^2(R^n)$*; that is for P-a.e.* $\omega\in\Omega$*, the sequence* $\{\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^{\infty}$ *has a convergent subsequence in E, provided* $t_n\to\infty, B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ *and* $\phi_{0,n}(\theta_{-t_n}\omega)=(u_{0,n}(\theta_{-t_n}\omega),v_{0,n}(\theta_{-t_n}\omega))\in B(\theta_{-t}\omega)$.

Proof. Let $t_n\to\infty, B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and $\phi_{0,n}(\theta_{-t_n}\omega)\in B(\theta_{-t}\omega)$. Then by Lemma 4.1 for P-a.e. $\omega\in\Omega$, we have known that the sequence $\{\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^{\infty}$ is bounded in $E=L^2(R^n)\times L^2(R^n)$, hence, there is $\xi\in L^2(R^n)\times L^2(R^n)$, such that, up to a subsequence

$$\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))\to\xi \text{ weakly in } L^2(R^n)\times L^2(R^n).\tag{5.4}$$

Next, we prove the weak convergence of (5.4) is actually strong convergence, given $\epsilon>0$, by Lemma(5.1), there is $T_1=T_1(B,\omega,\epsilon)$ and $R_1=R_1(\omega,\epsilon)$ such that for all $t\geq T_1$,

$$\int_{|x|\geq R_1}|\phi(t,\theta_{-t}\omega,\phi_0(\theta_{-t}\omega))|^2\mathrm{d}x\leq\epsilon.\tag{5.5}$$

Since $t_n\to\infty$, there is $N_1=N_1(B,\omega,\epsilon)$ such that $t_n\geq T_1$ for every $n\geq N_1$, hence, it follows from (5.5) that for all $n\geq N_1$,

$$\int_{|x|\geq R_1}|\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))|^2\mathrm{d}x\leq\epsilon.\tag{5.6}$$

On the other hand, by Lemmas 4.1 and 4.5, there is $T_2=T_2(B,\omega)$, such that for all $t\geq T_2$,

$$\|\phi(t,\theta_{-t}\omega,\phi_0(\theta_{-t}\omega))\|^2_{H^1(R^n)\times H^1(R^n)}\leq c(1+r(\omega)).\tag{5.7}$$

Let $N_2=N_2(B,\omega)$ be large enough such that $t_n\geq T_2$ for $n\geq N_2$, then by (5.7) we find that, for all $n\geq N_2$,

$$\|\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))\|^2_{H^1(R^n)\times H^1(R^n)}\leq c(1+r(\omega)).\tag{5.8}$$

Denote by Q_{R_1} the set $\{x\in R^n: |x|\le R_1\}$. By the compactness of embedding $H^1(Q_{R_1})\times H^1(Q_{R_1})\hookrightarrow L^2(Q_{R_1})\times L^2(Q_{R_1})$, it follows from (5.8) that, up to a subsequence,

$$\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))\to\xi \text{ strongly in } L^2(Q_{R_1})\times L^2(Q_{R_1}),$$

which shows that for the given $\epsilon>0$, there exists $N_3=N_3(B,\omega,\epsilon)$ such that for all $n\ge N_3$,

$$\|\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))-\xi\|^2_{L^2(Q_{R_1})\times L^2(Q_{R_1})}\le\epsilon. \tag{5.9}$$

Attention that $\xi\in L^2(R^n)\times L^2(R^n)$, there for there exists $R_2=R_2(\epsilon)$ such that,

$$\int_{|x|\ge R_2}|\xi(x)|^2\mathrm{d}x\le\epsilon. \tag{5.10}$$

Let $R_3=\max\{R_1,R_2\}$ and $N_4=\max\{N_1,N_3\}$, by (5.6), (5.9) and (5.10), we find that for all $n\ge N_4$,

$$\begin{aligned}&\|\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))-\xi\|^2_{L^2(R^n)\times L^2(R^n)}\\ \le&\int_{|x|\le R_3}|\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))-\xi|^2\mathrm{d}x+\int_{|x|\ge R_3}|\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))-\xi|^2\mathrm{d}x\le2\epsilon,\end{aligned}$$

which show that

$$\phi(t_n,\theta_{-t_n}\omega,\phi_{0,n}(\theta_{-t_n}\omega))\to\xi \text{ strongly in } L^2(R^n)\times L^2(R^n),$$

which is desiring. □

We are now in a position to present our main result: the existence of a $\mathcal{D}$-random attractor for ϕ in $L^2(R^n)\times L^2(R^n)$.

Theorem 5.1. *Assume that $f,g\in L^2(R^n)$ and* (3.4) *hold, then the random dynamical system ϕ has a unique $\mathcal{D}$-random attractor in $L^2(R^n)\times L^2(R^n)$.*

Proof. Attention that ϕ has a bounded random absorbing set $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ by Lemma 4.1, and ϕ is $\mathcal{D}$-pullback asymptotically compact in $L^2(R^n)\times L^2(R^n)$ by Lemmas 5.1 and 5.2, hence the existence of a unique $\mathcal{D}$-random attractor for ϕ allows from Proposition 2.1 immediately. □

6 Upper semicontinuity of attractors

In this section, we consider the upper semicontinuity of attractors of system (3.1)-(3.3) when $\varepsilon\to0$. To show the dependence of solution on ε, we will write the solution of system (3.1)-(3.2) as $(u_\varepsilon,v_\varepsilon)$, and we define the corresponding random dynamical system as ϕ_ε. When $\varepsilon=0$, the stochastic system (3.1)-(3.2) transform the following deterministic autonomous one:

$$\frac{\mathrm{d}u}{\mathrm{d}t}-\mu\Delta u+\lambda u+\alpha v=h(u)+f(x), \tag{6.1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\delta v-\beta u=g(x). \tag{6.2}$$

By ahead Theorem we know that given $\varepsilon\in[0,1]$, ϕ_ε has a $\mathcal{D}$-pullback attractor $\mathcal{A}_\varepsilon\in\mathcal{D}$. Let ϕ_0 be the dynamical system associated with system (6.1)-(6.2) in $L^2(R^n)\times L^2(R^n)$ over R. Denote by $\mathcal{D}_0$ the collection of families $D=\{D(\tau)\subseteq L^2(R^n)\times L^2(R^n):\tau\in R\}$ satisfying the condition: there exists $\delta_0=\delta_0(D)\in(0,\delta)$ such that,

$$\lim_{t\to\infty}e^{\delta_0 t}\|D(\tau+t)\|^2=0.$$

Let $\mathcal{A}_0$ be the $\mathcal{D}_0$-global attractor of ϕ_0. Given $0<\varepsilon\le 1$, for $\omega\in\Omega$, let K_ε be the $\mathcal{D}$-pullback absorbing set of ϕ_ε, and we let K_0 is a $\mathcal{D}_0$-bounded absorbing set of ϕ_0 when $\varepsilon=0$.

Given $\omega\in\Omega$, denote by

$$B(\omega)=\{(u_\varepsilon,v_\varepsilon)\in L^2(R^n)\times L^2(R^n):\ \|u_\varepsilon\|^2+\|v_\varepsilon\|^2\le R(\omega)\}.$$

So we know that $K_\varepsilon(\omega)\subseteq B(\omega)$ for all $\varepsilon\in(0,1]$, and $\omega\in\Omega$, this implies that for every $\omega\in\Omega$,

$$\bigcup_{0<\varepsilon\le 1}\mathcal{A}_\varepsilon(\omega)\subseteq\bigcup_{0<\varepsilon\le 1}K_\varepsilon(\omega)\subseteq R(\omega). \tag{6.3}$$

From ahead lemma we can find for all $\varepsilon\in(0,1]$ and $\omega\in\Omega$, $\|\widehat{u}\|^2_{H^1(R^n)}+\|\widetilde{v}\|^2_{H^1(R^n)}\le R(\omega)$, for all $(\widetilde{u},\widetilde{v})\in\mathcal{A}_\varepsilon(\omega)$, and we can also know that the set $\bigcup_{0<\varepsilon\le 1}\mathcal{A}_\varepsilon(\omega)$ of pullback attractors is precompact in $L^2(R^n)\times L^2(R^n)$, we can refer literature [24] to know more.

Lemma 6.1. *Suppose ahead information is hold, let* $(u_\varepsilon,v_\varepsilon)$ *and* $(\widetilde{u},\widetilde{v})$ *be the solutions of* (1.1)-(1.2) *and* (6.1)-(6.2), *respectively. Then for every* $t\in R$, $\omega\in\Omega$, *we can obtain that,*

$$\begin{aligned}&\|u_\varepsilon(t,\omega,(u_\varepsilon)_0)-\widetilde{u}(t,\widetilde{u}_0)\|^2+\|v_\varepsilon(t,\omega,(v_\varepsilon)_0)-\widetilde{v}(t,\widetilde{v}_0)\|^2\\ \le&\frac{1}{2}\gamma e^{-\nu t}(\|(u_\varepsilon)_0-\widetilde{u}_0-z(\omega)\|^2+\|(v_\varepsilon)_0-\widetilde{v}_0-z^*(\omega)\|^2)+\frac{1}{2\sigma}e^{-\nu t}\int_0^t e^{\nu s}p_1(\theta_s\omega)\mathrm{d}s. \end{aligned}\tag{6.4}$$

Proof. Let $n-\widetilde{u}=\widetilde{n},m-\widetilde{v}=\widetilde{m}$, and we let (3.13)-(3.14) and (6.1)-(6.2) subtract, we can obtain that,

$$\frac{\mathrm{d}\widetilde{n}}{\mathrm{d}t}-\mu\Delta\widetilde{n}+\lambda\widetilde{n}+\alpha\widetilde{m}=\mu\Delta z(\theta_t\omega)-\alpha z^*(\theta_t\omega), \tag{6.5}$$

$$\frac{\mathrm{d}\widetilde{m}}{\mathrm{d}t}+\delta\widetilde{m}-\beta\widetilde{n}=\beta z(\theta_t\omega)+(\lambda-\delta)z^*(\theta_t\omega). \tag{6.6}$$

Taking the inner product of both sides of (6.5) with $\beta\widetilde{n}$, similarity, taking the inner product of both sides of (6.6) with $\alpha\widetilde{m}$, then we add the two equation together, we can obtain that,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\beta\|\widetilde{n}\|^2+\alpha\|\widetilde{m}\|^2)+\mu\beta\|\nabla\widetilde{n}\|^2+\lambda\beta\|\widetilde{n}\|^2+\delta\alpha\|\widetilde{m}\|^2\\ =&\beta\mu(\Delta z(\theta_t\omega),\widetilde{n})-\alpha\beta(z^*(\theta_t\omega),\widetilde{n})+\alpha\beta(z(\theta_t\omega),\widetilde{m})+\alpha(\lambda-\delta)(z^*(\theta_t\omega),\widetilde{m}).\end{aligned} \tag{6.7}$$

Now, we start to estimate the above equation,

$$|\beta\mu(\Delta z(\theta_t\omega),\widetilde{n})|=|\beta\mu\int_{R^n}\Delta z(\theta_t\omega)\widetilde{n}\mathrm{d}x|\le\frac{1}{2}\beta\mu\|\nabla\widetilde{n}\|^2+\frac{1}{2}\beta\mu\|\nabla z(\theta_t\omega)\|^2. \tag{6.8}$$

$$|-\alpha\beta(z^*(\theta_t\omega),\widetilde{n})|=|\alpha\beta\int_{R^n}z^*(\theta_t\omega)\widetilde{n}\mathrm{d}x|\le\frac{\lambda\beta}{2}\|\widetilde{n}\|^2+\frac{1}{2\lambda}\beta\alpha^2\|z^*(\theta_t\omega)\|^2. \tag{6.9}$$

$$|\alpha\beta(z(\theta_t\omega),\widetilde{m})|\le\frac{1}{4}\alpha\delta\|\widetilde{m}\|^2+\frac{\alpha\beta^2}{\delta}\|z(\theta_t\omega)\|^2. \tag{6.10}$$

$$|\alpha(\lambda-\delta)(z^*(\theta_t\omega),\widetilde{m})|\le\frac{1}{4}\alpha\delta\|\widetilde{m}\|^2+\frac{1}{\delta}\alpha(\lambda-\delta)^2\|z^*(\theta_t\omega)\|^2. \tag{6.11}$$

By (6.7)-(6.11), we can get that,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\beta\|\widetilde{n}\|^2+\alpha\|\widetilde{m}\|^2)+\mu\beta\|\nabla\widetilde{n}\|^2+\lambda\beta\|\widetilde{n}\|^2+\delta\alpha\|\widetilde{m}\|^2\\ \le&\beta\mu\|\nabla z(\theta_t\omega)\|^2+\frac{\beta\alpha^2}{\lambda}\|z^*(\theta_t\omega)\|^2+\frac{2\alpha\beta^2}{\delta}\|z(\theta_t\omega)\|^2+\frac{2\alpha(\lambda-\delta)^2}{\delta}\|z^*(\theta_t\omega)\|^2.\end{aligned} \tag{6.12}$$

We let $\nu=\min\{\delta,\lambda\}$, $\sigma=\min\{\alpha,\beta\}$ and $\gamma=\max\{\alpha,\beta\}$, so we can from (6.12) get that,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma(\|\widetilde{n}\|^2+\|\widetilde{m}\|^2))+\nu\sigma(\|\widetilde{n}\|^2+\|\widetilde{m}\|^2)\le p_1(\theta_t\omega), \tag{6.13}$$

where

$$p_1(\theta_t\omega)=\beta\mu\|\nabla z(\theta_t\omega)\|^2+\frac{\beta\alpha^2}{\lambda}\|z^*(\theta_t\omega)\|^2+\frac{2\alpha\beta^2}{\delta}\|z(\theta_t\omega)\|^2+\frac{2\alpha(\lambda-\delta)^2}{\delta}\|z^*(\theta_t\omega)\|^2.$$

Applying Gronwall's lemma on (6.13), we can find that,

$$\|\widetilde{n}\|^2+\|\widetilde{m}\|^2\le\gamma e^{-\nu t}(\|\widetilde{n}_0\|^2+\|\widetilde{m}_0\|^2)+\frac{1}{\sigma}\int_0^t e^{\nu(s-t)}p_1(\theta_s\omega)\mathrm{d}s. \tag{6.14}$$

From (6.14), we can also get that,

$$\|n-\widetilde{u}\|^2+\|m-\widetilde{v}\|^2\le\gamma e^{-\nu t}(\|n_0-\widetilde{u}_0\|^2+\|m_0-\widetilde{v}_0\|^2)+\frac{1}{\sigma}\int_0^t e^{\nu(s-t)}p_1(\theta_s\omega)\mathrm{d}s. \tag{6.15}$$

Because $n(t,\omega,n_0)=u_\varepsilon(t,\omega,n_0-z(\omega))-z(\theta_t\omega)$, $m(t,\omega,m_0)=v_\varepsilon(t,\omega,m_0-z^*(\omega))-z^*(\theta_t\omega)$, so we can get that,

$$\|u_\varepsilon(t,\omega,(u_\varepsilon)_0)-\widetilde{u}(t,\widetilde{u}_0)-z(\theta_t\omega)\|^2+\|v_\varepsilon(t,\omega,(v_\varepsilon)_0)-\widetilde{v}(t,\widetilde{v}_0)-z^*(\theta_t\omega)\|^2$$

$$\leq \gamma e^{-\nu t}(\|(u_\varepsilon)_0-\widetilde{u}_0-z(\omega)\|^2+\|(v_\varepsilon)_0-\widetilde{v}_0-z^*(\omega)\|^2)+\frac{1}{\sigma}\int_0^t e^{\nu(s-t)}p_1(\theta_s\omega)\mathrm{d}s. \tag{6.16}$$

From above, we can also obtain that,

$$\begin{aligned}&\|u_\varepsilon(t,\omega,(u_\varepsilon)_0)-\widetilde{u}(t,\widetilde{u}_0)\|^2+\|v_\varepsilon(t,\omega,(v_\varepsilon)_0)-\widetilde{v}(t,\widetilde{v}_0)\|^2\\ \leq&\frac{1}{2}\gamma e^{-\nu t}(\|(u_\varepsilon)_0-\widetilde{u}_0-z(\omega)\|^2+\|(v_\varepsilon)_0-\widetilde{v}_0-z^*(\omega)\|^2)+\frac{1}{2\sigma}e^{-\nu t}\int_0^t e^{\nu s}p_1(\theta_s\omega)\mathrm{d}s.\end{aligned} \tag{6.17}$$

□

We are finally in a position to present the upper semicontinuity of pullback attractors.

Theorem 6.1. *Suppose ahead information is hold, $\mathcal{A}_\varepsilon(\omega)$ be the pullback attractor of ϕ_ε and $\mathcal{A}_0$ be the global attractor of ϕ_0. Foe all $\omega\in\Omega$, then we have*

$$\lim_{\varepsilon\mapsto 0}\mathrm{dist}(\mathcal{A}_\varepsilon(\omega),\mathcal{A}_0)=0.$$

Proof. We know that K_ε and K_0 be the families of subset of $L^2(R^n)\times L^2(R^n)$, then K_ε is the $\mathcal{D}_\varepsilon$-pullback absorbing set of ϕ_ε, and K_0 is the $\mathcal{D}_0$-bounded absorbing set of ϕ_0, we can obtain that, for every $\omega\in\Omega$,

$$\limsup_{\varepsilon\to 0}\|K_\varepsilon(\omega)\|=\|K_0\|. \tag{6.18}$$

If $\varepsilon_n\to 0$ and $(\widetilde{u_{0,n}},\widetilde{v_{o,n}})\to(\widetilde{u_0},\widetilde{v_o})$ in $L^2(R^n)\times L^2(R^n)$, then we find taht, for every $t\in R^+$ and $\omega\in\Omega$,

$$\phi_{\varepsilon_n}(t,\omega,(\widetilde{u_{0,n}},\widetilde{v_{o,n}}))\to\phi_0(t,(\widetilde{u_0},\widetilde{v_o})). \tag{6.19}$$

Based on (6.18), (6.19), and the set $\bigcup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon(\omega)$ of pullback random attractors is precompact in $L^2(R^n)\times L^2(R^n)$, so the theorem is proved. □

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