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On a Linear Partial Differential Equation of the Higher Order in Two Variables with Initial Condition Whose Coefficients are Real-valued Simple Step Functions

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Abstract. By using the method developed in the paper [*Georg. Inter. J. Sci. Tech.*, Volume 3, Issue 1 (2011), 107-129], it is obtained a representation in an explicit form of the weak solution of a linear partial differential equation of the higher order in two variables with initial condition whose coefficients are real-valued simple step functions.

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1 Introduction

In [1] has been obtained a representation in an explicit form of the solution of the linear partial differential equation of the higher order in two variables with initial condition whose coefficients were real-valued coefficients. The aim of the present manuscript is resolve an analogous problem for a linear partial differential equation of the higher order in two variables with initial condition whose coefficients are real-valued simple step functions.

The paper is organized as follows.

In Section 2, we consider some auxiliary notions and facts which come from works [1–3]. In Section 3, we get a representation in an explicit form of the weak solution of

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the partial differential equation of the higher order in two variables with initial condition whose coefficients are real-valued simple step functions.

2 Some auxiliary notions and results

Definition 2.1. *Fourier differential operator $(\mathcal{F})\frac{\partial}{\partial x}$ in R^∞ is defined as follows :*

$$(\mathcal{F})\frac{\partial}{\partial x}\begin{pmatrix}\frac{a_0}{2}\\ a_1\\ b_1\\ a_2\\ b_2\\ a_3\\ b_3\\ \vdots\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&0&0&\cdots\\ 0&0&\frac{1\pi}{l}&0&0&0&0&\cdots\\ 0&-\frac{1\pi}{l}&0&0&0&0&0&\cdots\\ 0&0&0&0&\frac{2\pi}{l}&0&0&\cdots\\ 0&0&0&-\frac{2\pi}{l}&0&0&0&\cdots\\ 0&0&0&0&0&0&\frac{3\pi}{l}&\ddots\\ 0&0&0&0&0&-\frac{3\pi}{l}&0&\ddots\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\ddots\end{pmatrix}\times\begin{pmatrix}\frac{a_0}{2}\\ a_1\\ b_1\\ a_2\\ b_2\\ a_3\\ b_2\\ \vdots\end{pmatrix}. \tag{2.1}$$

For $n\in\mathbb{N}$, let $FD^n[-l,l[$ be a vector space of all n-times differentiable functions on $[-l,l[$ such that for arbitrary $0\le k\le n-1$, a series obtained by a differentiation term by term of the Fourier series of $f^{(k)}$ pointwise converges to $f^{(k+1)}$ for all $x\in[-l,l[$.

Lemma 2.1. *Let $f\in FD^{(1)}[-l,l[$. Let G_M be an embedding of the $FD^{(1)}[-l,l[$ in to R^∞ which sends a function to a sequence of real numbers consisting from its Fourier coefficients. i.e., if*

$$f(x)=\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k\cos\left(\frac{k\pi x}{l}\right)+d_k\sin\left(\frac{k\pi x}{l}\right)\ (x\in[-l,l[),$$

then $G_F(f)=(\frac{c_0}{2},c_1,d_1,c_2,d_2,\dots)$. Then, for $f\in FD^{(1)}[-l,l[$, the following equality

$$\Big(G_F^{-1}\circ(\mathcal{F})\frac{\partial}{\partial x}\circ G_F\Big)(f)=\frac{\partial}{\partial x}(f) \tag{2.2}$$

holds.

Proof. Assume that for $f\in FD^{(1)}[-l,l[$, we have the following representation

$$f(x)=\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k\cos\Big(\frac{k\pi x}{l}\Big)+d_k\sin\Big(\frac{k\pi x}{l}\Big)\ (x\in[-l,l[).$$

By the definition of the class $FD^{(1)}[-l,l[$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(f)=\frac{\partial}{\partial x}\left(\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k\cos\Big(\frac{k\pi x}{l}\Big)+d_k\sin\Big(\frac{k\pi x}{l}\Big)\right)$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} c_k \frac{\partial}{\partial x}\left(\cos\left(\frac{k\pi x}{l}\right)\right) + d_k \frac{\partial}{\partial x}\left(\sin\left(\frac{k\pi x}{l}\right)\right) \\
&= \sum_{k=1}^{\infty} -c_k \frac{k\pi}{l}\sin\left(\frac{k\pi x}{l}\right) + d_k \frac{k\pi}{l}\cos\left(\frac{k\pi x}{l}\right) \\
&= \sum_{k=1}^{\infty} \frac{k\pi d_k}{l}\cos\left(\frac{k\pi x}{l}\right) - \frac{k\pi c_k}{l}\sin\left(\frac{k\pi x}{l}\right).
\end{aligned}
$$

By the definition of the composition of mappings, we have

$$
\left(G_F^{-1} \circ (\mathcal{F})\frac{\partial}{\partial x} \circ G_F\right)(f) = G_F^{-1}\left((\mathcal{F})\frac{\partial}{\partial x}((G_F(f)))\right) = G_F^{-1}\left((\mathcal{F})\frac{\partial}{\partial x}\begin{pmatrix} \frac{c_0}{2} \\ c_1 \\ d_1 \\ c_2 \\ d_2 \\ c_3 \\ d_3 \\ \vdots \end{pmatrix}\right)
$$

$$
= G_F^{-1}\left(\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & \frac{1\pi}{l} & 0 & 0 & 0 & 0 & \dots \\
0 & -\frac{1\pi}{l} & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & \frac{2\pi}{l} & 0 & 0 & \dots \\
0 & 0 & 0 & -\frac{2\pi}{l} & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3\pi}{l} & \ddots \\
0 & 0 & 0 & 0 & 0 & -\frac{3\pi}{l} & 0 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix} \times \begin{pmatrix} \frac{c_0}{2} \\ c_1 \\ d_1 \\ c_2 \\ d_2 \\ c_3 \\ d_3 \\ \vdots \end{pmatrix}\right)
$$

$$
= G_F^{-1}\left(\begin{pmatrix} 0 \\ \frac{1\pi d_1}{l} \\ -\frac{1\pi c_1}{l} \\ \frac{2\pi d_2}{l} \\ -\frac{2\pi c_2}{l} \\ \frac{3\pi d_3}{l} \\ -\frac{3\pi c_3}{l} \\ \vdots \end{pmatrix}\right) = \sum_{k=1}^{\infty} \frac{k\pi d_k}{l}\cos\left(\frac{k\pi x}{l}\right) - \frac{k\pi c_k}{l}\sin\left(\frac{k\pi x}{l}\right).
$$

□

By the scheme used in the proof of Lemma 2.1, we can get the validity of the following assertion.

Lemma 2.2. *Let G_M be an embedding of the $FD^n[-l,l[$ in to R^∞ which sends a function to a sequence of real numbers consisting from its Fourier coefficients.*

Then, for $f\in FD^{(n)}[-l,l[$ and $A_k\in R(0\le k\le n)$, the following equality

$$\Big(G_F^{-1}\circ\Big(\sum_{k=0}^{n}A_k((\mathcal{F})\frac{\partial}{\partial x})^k\Big)\circ G_F\Big)(f)=\sum_{k=0}^{n}A_k\frac{\partial^k}{\partial x^k}(f) \tag{2.3}$$

holds, where A_k are real numbers for $0\le k\le n$.

Example 2.1. [2] If A is the real matrix

$$\begin{pmatrix}\sigma & \omega\\ -\omega & \sigma\end{pmatrix}, \tag{2.4}$$

then

$$e^{tA}=e^{\sigma t}\begin{pmatrix}\cos(\omega t) & \sin(\omega t)\\ -\sin(\omega t) & \cos(\omega t)\end{pmatrix}. \tag{2.5}$$

Lemma 2.3. *For $m\ge 1$, let us consider a linear autonomous nonhomogeneous ordinary differential equations of the first order*

$$\frac{\mathrm{d}}{\mathrm{d}t}((a_k)_{k\in\mathbb{N}})=\Big(\sum_{n=0}^{2m}A_n\Big((\mathcal{F})\frac{\partial}{\partial x}\Big)^n\Big)\times((a_k)_{k\in\mathbb{N}})+(f_k)_{k\in\mathbb{N}} \tag{2.6}$$

with initial condition

$$(a_k(0))_{k\in\mathbb{N}}=(C_k)_{k\in\mathbb{N}}, \tag{2.7}$$

where

(i) $(C_k)_{k\in\mathbb{N}}\in\mathbf{R}^\infty$;

(ii) $f=(f_k)_{k\in\mathbb{N}}$ *is the sequence of continuous functions of a parameter t on R.*

For each $k\ge 1$, we put

$$\sigma_k=\sum_{n=0}^{m}(-1)^n A_{2n}\Big(\frac{k\pi}{l}\Big)^{2n}, \tag{2.8}$$

$$\omega_k=\sum_{n=0}^{m-1}(-1)^n A_{2n+1}\Big(\frac{k\pi}{l}\Big)^{2n+1}. \tag{2.9}$$

Then the solution of (2.6)-(2.7) *is given by*

$$(a_k(t))_{k\in\mathbb{N}}=e^{t\Big(\sum_{n=0}^{2m}A_n\big((\mathcal{F})\frac{\partial}{\partial x}\big)^n\Big)}\times(C_k)_{k\in\mathbb{N}}+\int_0^t e^{(\tau-t)\Big(\sum_{n=0}^{2m}A_n\big((\mathcal{F})\frac{\partial}{\partial x}\big)^n\Big)}\times f(\tau)\mathrm{d}\tau, \tag{2.10}$$

where $\exp\big(t(\sum_{n=0}^{2m} A_n\big((\mathcal{F})\frac{\partial}{\partial x}\big)^n)\big)$ *denotes an exponent of the matrix* $t(\sum_{n=0}^{2m} A_n\big((\mathcal{F})\frac{\partial}{\partial x}\big)^n)$ *and it exactly coincides with an infinite-dimensional* (1,2,2,...)*-cellular matrix* $D(t)$ *with cells* $(D_k(t))_{k\in\mathbb{N}}$ *for which* $D_0(t)=(e^{tA_0})$ *and*

$$D_k(t)=e^{\sigma_k t}\begin{pmatrix}\cos(\omega_k t) & \sin(\omega_k t)\\ -\sin(\omega_k t) & \cos(\omega_k t)\end{pmatrix}, \tag{2.11}$$

where for $k\geq 1$, σ_k *and* ω_k *are defined by* (2.8)-(2.9)*, respectively.*

Proof. We know that if we have a linear autonomous inhomogeneous ordinary differential equations of the first order

$$\frac{\mathrm{d}}{\mathrm{d}t}((a_k)_{k\in\mathbb{N}})=E\times((a_k)_{k\in N})+(f_k)_{k\in\mathbb{N}} \tag{2.12}$$

with initial condition

$$(a_k(0))_{k\in\mathbb{N}}=(C_k)_{k\in\mathbb{N}}, \tag{2.13}$$

where

(i) $(C_k)_{k\in\mathbb{N}}\in\mathbf{R}^\infty$;

(ii) $(f_k))_{k\in\mathbb{N}}$ is the sequence of continuous functions of parameter t on R;

(iii) E is an infinite dimensional (1,2,2,...)-cellular matrix with cells $(E_k)_{k\in\mathbb{N}}$.

Then the solution of (2.6)-(2.7) is given by (cf. [2], §6, Section 1)

$$(a_k(t))_{k\in\mathbb{N}}=e^{tE}\times(C_k)_{k\in\mathbb{N}}+\int_0^t e^{(\tau-t)E}\times f(\tau)\mathrm{d}\tau, \tag{2.14}$$

where e^{tE} and $e^{(\tau-t)E}$ denote exponents of matrices tE and $(\tau-t)E$, respectively.

Note that $t\sum_{n=0}^{2m} A_n\Big((\mathcal{F})\frac{\partial}{\partial x}\Big)^n$ is an infinite-dimensional (1,2,2,...)-cellular matrix with cells $(tE_k)_{k\in\mathbb{N}}$ such that $tE_0=(tA_0)$ and

$$tE_k=\begin{pmatrix}t\sigma_k & t\omega_k\\ -t\omega_k & t\sigma_k\end{pmatrix} \tag{2.15}$$

for $k\geq 1$. Under notations (2.8)-(2.9), by using Example 2.1 we get that for $t\in R$, e^{tE} exactly coincides with an infinite-dimensional (1,2,2,...) -cellular matrix $D(t)$ with cells $(D_k(t))_{k\in\mathbb{N}}$ for which $D_0(t)=(e^{tA_0})$ and

$$D_k(t)=e^{\sigma_k t}\begin{pmatrix}\cos(\omega_k t) & \sin(\omega_k t)\\ -\sin(\omega_k t) & \cos(\omega_k t)\end{pmatrix}. \tag{2.16}$$

Note that, for $0\leq\tau\leq t$, the matrix $e^{(\tau-t)E}$ exactly coincides with an infinite-dimensional (1,2,2,...)-cellular matrix $D(\tau-t)$. □

The following proposition is a simple consequence of Lemma 2.3.

Corollary 2.1. *For $m \geq 1$, let us consider a linear partial differential equation*

$$\frac{\partial}{\partial t}\Psi(t,x)=\sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}\Psi(t,x) \ \ ((t,x)\in[0,+\infty[\times[-l,l[) \tag{2.17}$$

with initial condition

$$\Psi(0,x)=\frac{c_0}{2}+\sum_{k=1}^{\infty} c_k \cos\Big(\frac{k\pi x}{l}\Big)+d_k \sin\Big(\frac{k\pi x}{l}\Big)\in FD^{(0)}[-l,l[. \tag{2.18}$$

If $(\frac{c_0}{2},c_1,d_1,c_2,d_2,\dots)$ is such a sequence of real numbers that a series $\Psi(t,x)$ defined by

$$\begin{aligned}\Psi(t,x)=&\frac{e^{tA_0}c_0}{2}+\sum_{k=1}^{\infty} e^{\sigma_k t}\Big((c_k\cos(\omega_k t)+d_k\sin(\omega_k t))\cos\Big(\frac{k\pi x}{l}\Big)\\ &+(d_k\cos(\omega_k t)-c_k\sin(\omega_k t))\sin(\frac{k\pi x}{l})\Big)\end{aligned} \tag{2.19}$$

belongs to the class $FD^{(2m)}[-l,l[$ as a series of a variable x for all $t \geq 0$, and is differentiable term by term as a series of a variable t for all $x \in [-l,l[$, then Ψ is a solution of (2.17)-(2.18).

3 Solution of a linear partial differential equation of the higher order in two variables with initial condition when coefficients are real-valued simple step functions

Let $0=t_0<\cdots<t_I=T$ and $-l=x_0<\cdots<x_J=l$. Suppose that

$$A_n(t,x)=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1} A_n^{(i,j)}\times \chi_{[t_i,t_{i+1}[\times[x_j,x_{j+1}[}(t,x),$$

where $A_n^{(i,j)}$ are given real numbers for $0\leq k\leq n, 0\leq i<I, 0\leq j<J$.

For $m \geq 1$, let us consider a partial differential equation

$$\frac{\partial}{\partial t}\Psi(t,x)=\sum_{n=0}^{2m} A_n(t,x) \frac{\partial^n}{\partial x^n}\Psi(t,x) \ \ ((t,x)\in[0,T[\times[-l,l[) \tag{3.1}$$

with initial condition

$$\Psi(0,x)=\frac{c_0}{2}+\sum_{k=1}^{\infty} c_k \cos\Big(\frac{k\pi x}{l}\Big)+d_k \sin\Big(\frac{k\pi x}{l}\Big)\in FD^{(0)}[-l,l[. \tag{3.2}$$

Definition 3.1. *We say that* $\Psi(t,x)$ *is a weak solution of* (3.1)-(3.2) *if the following conditions hold:*

(i) $\Psi(t,x)$ *satisfies* (3.1) *for each* $(t,x)\in[0,T[\times[-l,l[$ *for which* $t\neq t_i(0\leq i\leq I)$ *or* $x\neq x_j(0\leq j\leq J)$*;*

(ii) $\Psi(t,x)$ *satisfies* (3.2)*;*

(iii) for each fixed $x\in[-l,l[$*, the function* $\Psi(t,x)$ *is continuous with respect to* $t\in[0,T[$*, and for each* $t\in[0,T[$ *the function* $\Psi(t,x)$ *is continuous with respect to* x *on* $[-l,l[$ *except points* $\{x_j:0\leq j\leq J-1\}$.

First, let fix j and consider a partial differential equation

$$\frac{\partial}{\partial t}\Psi_{(0,j)}(t,x)=\sum_{n=0}^{2m}A_n^{(0,j)}\frac{\partial^n}{\partial x^n}\Psi_{(0,j)}(t,x)\ \ ((t,x)\in[0,+\infty[\times[-l,l[) \tag{(0,j)(PDE)}$$

with initial condition

$$\begin{aligned}\Psi_{(0,j)}(t_0,x)&=\frac{c_0}{2}+\sum_{k=1}^{\infty}c_k\cos\Big(\frac{k\pi x}{l}\Big)+d_k\sin\Big(\frac{k\pi x}{l}\Big)\\&=\frac{c_0^{(0,j)}}{2}+\sum_{k=1}^{\infty}c_k^{(0,j)}\cos\Big(\frac{k\pi x}{l}\Big)+d_k^{(0,j)}\sin\Big(\frac{k\pi x}{l}\Big)\in FD^{(0)}[-l,l[,\end{aligned} \tag{(0,j)(IC)}$$

By Corollary 2.1, under some restrictions on $(\frac{c_0}{2},c_1,d_1,c_2,d_2,\ldots)$, a series $\Psi_{(0,j)}(t,x)$ defined by

$$\begin{aligned}\Psi_{(0,j)}(t,x)=&\frac{e^{tA_0^{(0,j)}}c_0^{(0,j)}}{2}+\sum_{k=1}^{\infty}e^{\sigma_k^{(0,j)}t}\Big((c_k^{(0,j)}\cos(\omega_k^{(0,j)}t)+d_k^{(0,j)}\sin(\omega_k^{(0,j)}t))\cos\Big(\frac{k\pi x}{l}\Big)\\&+(d_k^{(0,j)}\cos(\omega_k^{(0,j)}t)-c_k^{(0,j)}\sin(\omega_k^{(0,j)}t))\sin\Big(\frac{k\pi x}{l}\Big)\Big)\end{aligned} \tag{3.3}$$

is a solution of (0,j)(PDE)-(0,j)(IC).

Now let consider a partial differential equation

$$\frac{\partial}{\partial t}\Psi_{(1,j)}(t,x)=\sum_{n=0}^{2m}A_n^{(1,j)}\frac{\partial^n}{\partial x^n}\Psi_{(1,j)}(t,x)\ \ ((t,x)\in[0,+\infty[\times[-l,l[) \tag{(1,j)(PDE)}$$

with initial condition

$$\Psi_{(1,j)}(t_1,x)=\Psi_{(0,j)}(t_1,x). \tag{(1,j)(IC)}$$

We will try to present the solution of the (1,j)(PDE) by the following form

$$\Psi_{(1,j)}(t,x)=\frac{e^{tA_0^{(1,j)}}c_0^{(1,j)}}{2}+\sum_{k=1}^{\infty}e^{\sigma_k^{(1,j)}t}\Big((c_k^{(1,j)}\cos(\omega_k^{(1,j)}t)+d_k^{(1,j)}\sin(\omega_k^{(1,j)}t))\cos\Big(\frac{k\pi x}{l}\Big)$$

$$+(d_k^{(1,j)}\cos(\omega_k^{(1,j)}t)-c_k^{(1,j)}\sin(\omega_k^{(1,j)}t))\sin(\frac{k\pi x}{l})\Big). \tag{3.4}$$

In order to get validity of the condition (1,j)(IC), we consider the following infinite system of equations:

$$\frac{e^{t_1A_0^{(1,j)}}c_0^{(1,j)}}{2}=\frac{e^{t_1A_0^{(0,j)}}c_0^{(0,j)}}{2}, \tag{3.5}$$

$$\begin{aligned}&e^{\sigma_k^{(1,j)}t_1}(c_k^{(1,j)}\cos(\omega_k^{(1,j)}t_1)+d_k^{(1,j)}\sin(\omega_k^{(1,j)}t_1))\\ =&e^{\sigma_k^{(0,j)}t_1}(c_k^{(0,j)}\cos(\omega_k^{(0,j)}t_1)+d_k^{(0,j)}\sin(\omega_k^{(0,j)}t_1))(k\in\mathbb{N}),\end{aligned} \tag{3.6}$$

$$\begin{aligned}&e^{\sigma_k^{(1,j)}t_1}(d_k^{(1,j)}\cos(\omega_k^{(1,j)}t_1)-c_k^{(1,j)}\sin(\omega_k^{(1,j)}t_1))\\ =&e^{\sigma_k^{(0,j)}t_1}(d_k^{(0,j)}\cos(\omega_k^{(0,j)}t_1)-c_k^{(0,j)}\sin(\omega_k^{(0,j)}t_1))(k\in\mathbb{N}).\end{aligned} \tag{3.7}$$

We have

$$c_0^{(1,j)}=e^{t_1(A_0^{(0,j)}-A_0^{(1,j)})}c_0^{(0,j)}. \tag{3.8}$$

For $k\in\mathbb{N}$ we can rewrite Eqs. (3.6)-(3.7) as follows:

$$\begin{aligned}c_k^{(1,j)}\cos(\omega_k^{(1,j)}t_1)+d_k^{(1,j)}\sin(\omega_k^{(1,j)}t_1)=&e^{(\sigma_k^{(0,j)}-\sigma_k^{(1,j)})t_1}(c_k^{(0,j)}\cos(\omega_k^{(0,j)}t_1)\\ &+d_k^{(0,j)}\sin(\omega_k^{(0,j)}t_1)),\end{aligned} \tag{3.9}$$

$$\begin{aligned}-c_k^{(1,j)}\sin(\omega_k^{(1,j)}t_1)+d_k^{(1,j)}\cos(\omega_k^{(1,j)}t_1)=&e^{(\sigma_k^{(0,j)}-\sigma_k^{(1,j)})t_1}(d_k^{(0,j)}\cos(\omega_k^{(0,j)}t_1)\\ &-c_k^{(0,j)}\sin(\omega_k^{(0,j)}t_1)).\end{aligned} \tag{3.10}$$

Setting

$$\mathbb{A}=e^{(\sigma_k^{(0,j)}-\sigma_k^{(1,j)})t_1}(c_k^{(0,j)}\cos(\omega_k^{(0,j)}t_1)+d_k^{(0,j)}\sin(\omega_k^{(0,j)}t_1)) \tag{3.11}$$

and

$$\mathbb{B}=e^{(\sigma_k^{(0,j)}-\sigma_k^{(1,j)})t_1}(d_k^{(0,j)}\cos(\omega_k^{(0,j)}t_1)-c_k^{(0,j)}\sin(\omega_k^{(0,j)}t_1)), \tag{3.12}$$

for $k\in\mathbb{N}$, we obtain

$$c_k^{(1,j)}\cos(\omega_k^{(1,j)}t_1)+d_k^{(1,j)}\sin(\omega_k^{(1,j)}t_1)=\mathbb{A} \tag{3.13}$$

and

$$-c_k^{(1,j)}\sin(\omega_k^{(1,j)}t_1)+d_k^{(1,j)}\cos(\omega_k^{(1,j)}t_1)=\mathbb{B}. \tag{3.14}$$

It is obvious that the system of Eqs. (3.13)-(3.14) has the unique solution which can be done as follows:

$$c_k^{(1,j)}=\mathbb{A}\cos(\omega_k^{(1,j)}t_1)-\mathbb{B}\sin(\omega_k^{(1,j)}t_1) \tag{3.15}$$

and

$$d_k^{(1,j)} = \mathbb{B}\cos(\omega_k^{(1,j)} t_1) + \mathbb{A}\sin(\omega_k^{(1,j)} t_1) \tag{3.16}$$

for $k \in \mathbb{N}$.

By Corollary 2.1, under some restrictions on $(\frac{c_0^{(1,j)}}{2}, c_1^{(1,j)}, d_1^{(1,j)}, c_2^{(1,j)}, d_2^{(1,j)}, \dots)$, the series $\Psi_{(1,j)}(t,x)$ defined by (3.4) is the solution of (1,j)(PDE)-(1,j)(IC).

It is obvious that under nice restrictions on coefficients participated in (3.1) and (3.2), we can continue our procedure step by step. Correspondingly we can construct a sequence $(\Psi_{(s,j)})_{0 \le s \le I-1, 1 \le j \le J-1}$ such that $\Psi_{(s,j)}$ satisfies a linear partial differential equation

$$\frac{\partial}{\partial t}\Psi_{(s,j)}(t,x) = \sum_{n=0}^{2m} A_n^{(s,j)} \frac{\partial^n}{\partial x^n}\Psi_{(s,j)}(t,x) \ \ ((t,x) \in [0,+\infty[\times[-l,l[) \qquad \text{(s.j)(PDE)}$$

with initial condition

$$\Psi_{(s,j)}(t_s,x) = \Psi_{(s-1,j)}(t_s,x) = \frac{c_0^{(s,j)}}{2} + \sum_{k=1}^{\infty} c_k^{(s,j)} \cos\Big(\frac{k\pi x}{l}\Big) + d_k^{(s,j)} \sin\Big(\frac{k\pi x}{l}\Big). \qquad \text{(s,j)(IC)}$$

Theorem 3.1. *If for coefficients* $(\frac{c_0^{(i,j)}}{2}, c_1^{(i,j)}, d_1^{(i,j)}, c_2^{(i,j)}, d_2^{(i,j)}, \dots)(1 \le i \le I, 1 \le j \le J)$ *functions* $\Psi_{(i,j)}(t,x)$ *satisfy conditions of Corollary* 2.1, *then a function* $\Psi(t,x) : [0,T[\times[-l,l[\to R$ *defined by*

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \Psi_{(i,j)}(x,t) \times \chi_{[t_i,t_{i+1}[\times[x_j,x_{j+1}[}(t,x) \tag{3.17}$$

is a weak solution of (3.1) *and* (3.2).

Example 3.1. Let consider a linear partial differential equation of the 22 order in two variables

$$\frac{\partial}{\partial t}\Psi(t,x) = A(t,x) \times \frac{\partial^2}{\partial x^2}\Psi(t,x) + B(t,x) \times \frac{\partial^{22}}{\partial x^{22}}\Psi(t,x) \ \ ((t,x) \in [0,2\pi[\times[0,\pi[) \tag{3.18}$$

with initial condition

$$\Psi(0,x) = \frac{0.015}{2} + 5\sin(x), \tag{3.19}$$

where

$$A(t,x) = 0.5 \times \chi_{[0,\pi[\times[0,\pi[}(t,x) + 0.55 \times \chi_{[\pi,2\pi[\times[0,\pi[}(t,x)$$

and

$$B(t,x) = 2 \times \chi_{[0,\pi[\times[0,\pi[}(t,x) + 2.5 \times \chi_{[\pi,2\pi[\times[0,\pi[}(t,x).$$

The programm in MatLab (cf. [4]) for a solution of (3.18) and (3.19), has the following form:

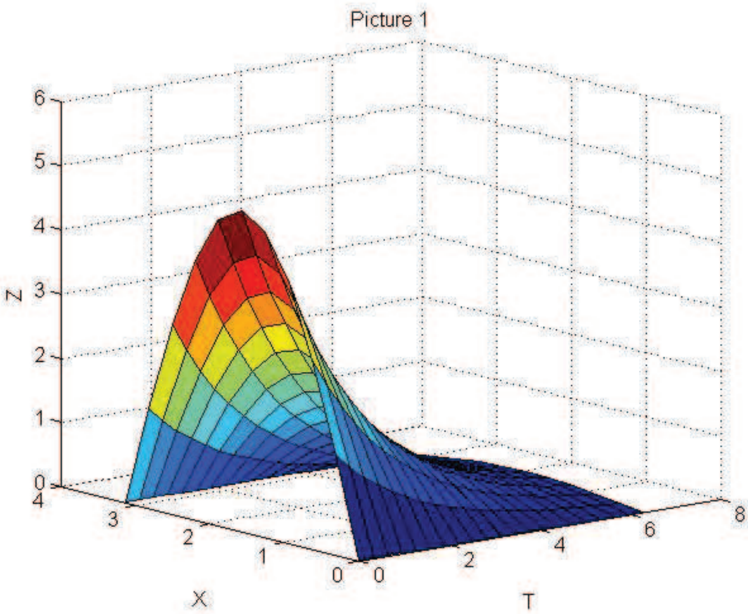

Figure 1: Graphic of the solution of the LPDE-(3.18) with IC-(3.19).

```
A1=[0,0.5,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2];
A2=[0,0.55,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2.5];
C1=[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0];
D1=[5,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0];
A10=0;A20=0;C10=0.015;
for k=1:20
S1(k)=A10;S2(k)=A20;
for n=1:10
S1(k)=S1(k)+(-1)^(n)*A1(2*n)*k^(2*n);
S2(k)=S2(k)+(-1)^(n)*A2(2*n)*k^(2*n);
end
end
for k=1:20
O1(k)=0;
O2(k)=0;
end
for k=1:20
for n=1:10
O1(k)=O1(k)+(-1)^n*A1(2*n+1)*k^(2*n+1);
O2(k)=O2(k)+(-1)^n*A2(2*n+1)*k^(2*n+1);
end
end
```

```
[T1,X1]=meshgrid(0:(pi/10):pi,0:(pi/10):pi);
Z1=0.5*C10*exp(T1.*A10);
for k=1:20
Z1=Z1+C1(k)*exp(T1*S1(k)).*cos(X1.*k).*cos(T1*O1(k))+D1(1)*exp(T1*S1(k)).*
cos(X1.*k).*sin(T1*O1(k))+
D1(k)*exp(T1*S1(k)).*sin(X1.*k).*cos(T1*O1(k))-C1(k)*exp(T1*S1(k)).*sin(X1.*
k).*sin(T1*O1(k));
end
C20=exp(pi*(A10-A20))*C10;
for k=1:20
A(k)=exp((S1(k)-S2(k))*pi)*(C1(k)*cos(O1(k)*pi)+D1(k)*sin(O1(k)*pi));
B(k)=exp((S1(k)-S2(k))*pi)*(D1(k)*cos(O1(k)*pi)-C1(k)*sin(O1(k)*pi));
end
for k=1:20
C2(k)=A(k)*cos(O2(k)*pi)-B(k)*sin(O2(k)*pi);
D2(k)=B(k)*cos(O2(k)*pi)+A(k)*sin(O2(k)*pi);
end
[T2,X2]=meshgrid(pi:(pi/10):(2*pi),0:(pi/10):pi);
Z2=0.5*C20*exp((T2)*A20);
for k=1:20
Z2=Z2+C2(k)*exp(T2*S2(k)).*cos(X2.*k).*cos(T2*O2(k))+D2(1)*exp(T2*S2(k)).*
cos(X2.*k).*sin(T2*O2(k))+
D2(k)*exp(T2*S2(k)).*sin(X2.*k).*cos(T2*O2(k))-C2(k)*exp(T2*S2(k)).*sin(X2.*
k).*sin(T2*O2(k));
end
surf(T1,X1,Z1)
hold on
surf(T2,X2,Z2)
hold off
```

Example 3.2. Let consider a linear partial differential equation of the 21 order in two variables

$$\begin{aligned}\frac{\partial}{\partial t}\Psi(t,x)=A(t,x)\Psi(t,x)+B(t,x)\times\frac{\partial^2}{\partial x^2}\Psi(t,x)\\+100\frac{\partial^3}{\partial x^3}\Psi(t,x)+2\frac{\partial^{21}}{\partial x^{21}}\Psi(t,x)\ ((t,x)\in[0,2\pi[\times[0,\pi[)\end{aligned}\tag{3.20}$$

with initial condition

$$\Psi(0,x)=\frac{0.015}{2}+100\sin(x),\tag{3.21}$$

where

$$A(t,x)=1\chi_{[0,\pi[\times[0,\pi[}(t,x)+0\chi_{[\pi,2\pi[\times[0,\pi[}(t,x)$$

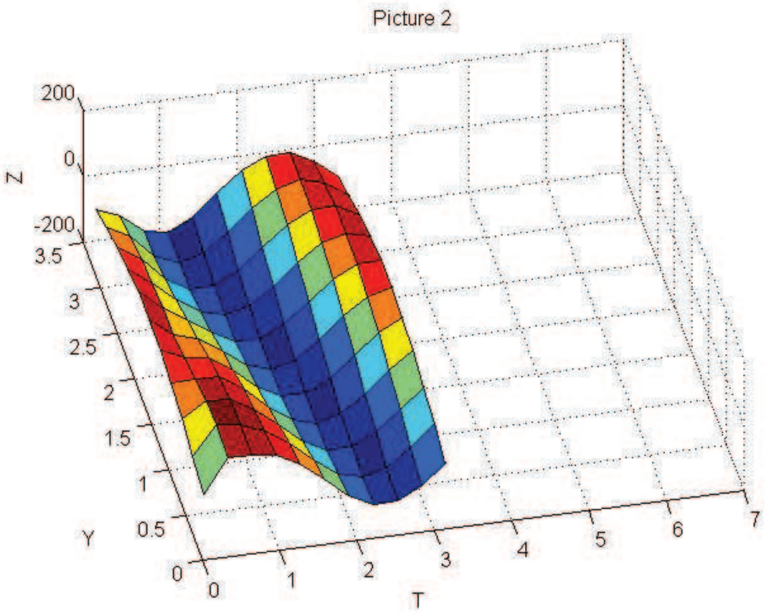

Figure 2: Graphic of the solution of the LPDE-(3.20) with IC-(3.21).

and

$$B(t,x)=\chi_{[0,\pi[\times[0,\pi[}(t,x)-\chi_{[\pi,2\pi[\times[0,\pi[}(t,x).$$

The graphical solution of (3.20)-(3.21) can be obtained by MatLab programm used in Example 3.1 for the following data:

$A1=[0,1,100,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0]$;

$A2=[0,-1,100,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,0]$;

$C1=[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$;

$D1=[100,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$;

$A10=1;A20=0;C10=0.15$;

We see that we have no graphic on the region $[\pi,2\pi[\times[0,\pi[$ which hints us that coefficients of the LPDE (3.20)-(3.21) on that region do not satisfy conditions of Theorem 3.1.

Remark 3.1. Notice that for each natural number $M>1$, one can easily modify the MatLab program described in Example 3.1 for obtaining the graphical solution of the linear partial differential equation (3.1)-(3.2) whose coefficients $(A_n(t,x))_{0\le n\le 2M}$ are real-valued simple step functions on $[0,T[\times[-l,l[$ and f is a trigonometric polynomial on $[-l,l[$.

Remark 3.2. The approach used for a solution of (3.1)-(3.2) can be used in such a case when coefficients $(A_n(t,x))_{0\le n\le 2M}$ are rather smooth continuous functions on $[0,T[\times[-l,l[$. If we will approximate $(A_n(t,x))_{0\le n\le 2M}$ by real-valued simple step functions, then it is natural to wait that under some "nice restrictions" on $(A_n(t,x))_{0 \le n \le 2M}$ the solutions obtained by Theorem 3.1, will give us a "good approximation" of the solution of the required linear partial differential equation of the higher order in two variables with corresponding initial conditions.

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