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Non-local Problem for the Loaded Integral-differential Equation in Double-connected Domain

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Abstract. In this work an existence and uniqueness of solution of the non-local boundary value problem for the loaded elliptic-hyperbolic type equation with integral-differential operations in double-connected domain have been investigated. The uniqueness of solution is proved by the method of integral energy using an extremum principle for the mixed type equations, and the existence is proved by the method of integral equations.

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1 Introduction and formulation of a problem

We note that with intensive research on problem of optimal control of the agroeconomical system, regulating the label of ground waters and soil moisture, it has become necessary to investigate a new class of equations called "LOADED EQUATIONS". For the first time it was given the most general definition of a Loaded equations and various Loaded equations are classified in detail by A.M.Nakhushev [1]. After this work very interesting results on the theory of boundary value problems for the loaded equations of parabolic, parabolic-hyperbolic and elliptic-hyperbolic types were published, for example, see [2–4] and [5]. In this direction, we studied some local and non-local problems for the loaded second and third order elliptic-hyperbolic type equations in double-connected domains

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(see [6–9]). Notice that non-local problems for the loaded integral-differential elliptic-hyperbolic type equations in double-connected domains have not been investigated. In the given paper, for the equation:

$$u_{xx}+\mathrm{sgn}y u_{yy}+\frac{1-\mathrm{sgn}y}{2}\sum_{k=1}^{n}R_k(x,0)=0 \tag{1.1}$$

with operators [10]:

$$R_k(x,y)=R_k\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)=\begin{cases} p_k(\xi)D^{\alpha_k}_{\xi 1}u\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right), & \text{at } q\le x\le 1,\\ r_k(\eta)D^{\beta_k}_{-1\eta}u\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right), & \text{at } -1\le x\le -q,\end{cases} \tag{1.2}$$

$$D^{\alpha_k}_{ax}f(x)=\frac{1}{\Gamma(-\alpha_k)}\int_a^x (x-t)^{-1-\alpha_k}f(t)\mathrm{d}t,\qquad -1<\alpha_k<0, \tag{1.3}$$

$$D^{\beta_k}_{xa}f(x)=\frac{1}{\Gamma(-\alpha_k)}\int_x^a (t-x)^{-1-\beta_k}f(t)\mathrm{d}t,\qquad -1<\beta_k<0, \tag{1.4}$$

we will investigate the uniqueness and the existence of solution of the non-local problem. Let's, Ω, be double connected domain, bounded with two lines:

$$\sigma_1: x^2+y^2=1;\ \ \sigma_2: x^2+y^2=q^2,\ y>0,$$

and characteristics:

$$A_jC_1: x+(-1)^j y=(-1)^{j+1};\ B_jC_2: x+(-1)^j y=(-1)^{j+1}\cdot q;\ (0<q<1),\ (j=1,2)$$

of the Eq. (1.1) at $y<0$, where $x+y=\xi$, $x-y=\eta$; $A_1(1;0)$, $A_2(-1;0)$, $B_1(q;0)$, $B_2(-q;0)$, $C_1(0;-1)$, $C_2(0;-q)$.

Introduce designations: $\theta_1(x)=\frac{x+1}{2}+i\cdot\frac{x-1}{2}$, $\theta_2(x)=\frac{x-1}{2}-i\cdot\frac{x+1}{2}$, $(i^2=-1)$;

$$\Omega_0=\Omega\cap(y>0),\qquad \Delta_1=\Omega\cap(x+y>q)\cap(y<0),\qquad \Delta_2=\Omega\cap(y-x>q)\cap(y<0);$$
$$D_1=\Omega\cap(-q<x+y<q)\cap(x>0),\qquad D_2=\Omega\cap(-q<y-x<q)\cap(x<0);$$
$$D_3=\Omega\cap(-1<x+y<-q)\cap(-1<y-x<-q),\qquad D_0=\Omega_0\cup\Delta_1\cup\Delta_2;$$
$$I_{2+j}=\left\{x: 0<(-1)^{j-1}x<q\right\},\qquad I_j=\left\{x:\frac{q+1}{2}<(-1)^{j-1}x<1\right\},\qquad (j=1,2).$$

In the domain Ω the following problem is investigated.

Problem I. To find a function $u(x,y)$ satisfies the following properties:

1) $u(x,y)\in C(\bar{\Omega})$;

2) $u(x,y)$ is a regular solution of the Eq. (1.1) in the domain of $\Omega\backslash(y-x=\pm q)\backslash(x+y=\pm q)$, besides, $u_y\in C(A_1B_1\cup A_2B_2)$ and $u_y(x,0)$ can tend to infinity an order of less unit at $x\to\pm q$, and finite at $x\to\pm 1$;

3) on the line of changing type satisfy gluing condition:

$$u_y(x,-0)=u_y(x,+0), \qquad (x,0)\in A_1B_1\cup A_2B_2;$$

4) $u(x,y)$ satisfies boundary conditions:

$$u(x,y)\Big|_{\sigma_j}=\varphi_j(x,y), \qquad (x,y)\in\overline{\sigma_j}; \tag{1.5}$$

$$u(x,y)\Big|_{B_jC_2}=g_j(x), \qquad x\in\overline{I_{2+j}}; \tag{1.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}u(\theta_1(x))=a_1(x)u_y(x,0)+b_1(x)u_x(x,0)+c_1(x)u(x,0)+d_1(x), \qquad x\in(q,1); \tag{1.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}u(\theta_2(x))=a_2(x)u_y(x,0)+b_2(x)u_x(x,0)+c_2(x)u(x,0)+d_2(x), \qquad x\in(-1,-q); \tag{1.8}$$

where $\varphi_j(x,y)$, $g_j(x)$, $a_j(x)$, $b_j(x)$, $c_j(x)$, are given functions, such that: $g_1(0)=g_2(0)$, $g_2(-q)=\varphi_2(-q,0)$, $g_1(q)=\varphi_2(q,0)$, $(j=1,2)$.

2 The uniqueness of solution of the Problem I

Known that the Eq. (1.1) at $y\le 0$ on the characteristics coordinate $\xi=x+y$ and $\eta=x-y$ has a form:

$$u_{\xi\eta}=\frac{1}{4}\sum_{k=1}^{n}R_k(\xi,0). \tag{2.1}$$

Notice, that solution the Cauchy problem for the Eq. (1.1) in the domain of Δ_1 with conditions $u(x,0)=\tau_1(x)$, $x\in\overline{A_1B_1}$ and $u_y(x,0)=\nu_1(x)$, $x\in A_1B_1$, with the account RiemannŁ function $R(\xi,\eta;\xi_0,\eta_0)=1$ of the Eq. (2.1), looks like:

$$u(x,y)=\frac{\tau_1(x+y)+\tau_1(x-y)}{2}-\frac{1}{2}\int_{x+y}^{x-y}\nu_1(t)\mathrm{d}t+\frac{1}{4}\int_{x+y}^{x-y}\mathrm{d}\xi\int_{\xi}^{x-y}\sum_{k=1}^{n}p_k(\xi)D_{\xi 1}^{\alpha_k}\tau_1(\xi)\mathrm{d}\eta \tag{2.2}$$

From here, considering

$$u[\theta_1(x)]=\frac{\tau_1(x)+\tau_1(1)}{2}-\frac{1}{2}\int_x^1\nu_1(t)\mathrm{d}t+\frac{1}{4}\sum_{k=1}^{n}\int_x^1(1-\xi)p_k(\xi)D_{\xi 1}^{\alpha_k}\tau_1(\xi)\mathrm{d}\xi$$

and by virtue (1.7) we will get:

$$\frac{1}{2}\tau_1'(x)+\frac{1}{2}\nu_1(x)+\frac{x-1}{4}\sum_{k=1}^{n}p_k(x)D_{x1}^{\alpha_k}\tau_1(x)=a_1(x)\nu_1(x)+b_1(x)\tau_1'(x)+c_1(x)\tau_1(x)+d_1(x)$$

$$\begin{aligned}(2a_1(x)-1)\nu_1(x)=&(1-2b_1(x))\tau_1'(x)-\frac{1-x}{2}\sum_{k=1}^{n}p_k(x)D_{x1}^{\alpha_k}\tau_1(x)\\&-2c_1(x)\tau_1(x)-2d_1(x).\end{aligned} \tag{2.3}$$

Precisely also, from the solution

$$u(x,y)=\frac{\tau_1(x+y)+\tau_1(x-y)}{2}-\frac{1}{2}\int_{x+y}^{x-y}\nu_1(t)\mathrm{d}t$$
$$+\frac{1}{4}\int_{x+y}^{x-y}\mathrm{d}\eta\int_{x+y}^{\eta}\sum_{k=1}^{n}r_k(\eta)D_{-1\eta}^{\beta_k}\tau_2(\eta)\mathrm{d}\xi; \tag{2.4}$$

of the Cauchy problem for the Eq. (1.1) in the domain of Δ_2 with conditions

$$u(x,0)=\tau_2(x),\ x\in\overline{A_2B_2};\qquad u_y(x,0)=\nu_2(x),\ x\in A_2B_2$$

and on the base of (1.8) we will obtain:

$$u[\theta_2(x)]=\frac{\tau_2(-1)+\tau_2(x)}{2}-\frac{1}{2}\int_{-1}^{x}\nu_2(t)\mathrm{d}t+\frac{1}{4}\sum_{k=1}^{n}\int_{-1}^{x}(1+\eta)r_k(\eta)D_{-1\eta}^{\beta_k}\tau_2(\eta)\mathrm{d}\eta,$$

$$\frac{1}{2}\tau_2'(x)-\frac{1}{2}\nu_2(x)+\frac{x+1}{4}\sum_{k=1}^{n}r_k(x)D_{-1x}^{\beta_k}\tau_2(x)=a_2(x)\nu_2(x)+b_2(x)\tau_2'(x)+c_2(x)\tau_2(x)+d_2(x),$$

$$(2a_2(x)+1)\nu_2(x)=(1-2b_2(x))\tau_2'(x)+\frac{x+1}{2}\sum_{k=1}^{n}r_k(x)D_{-1x}^{\beta_k}\tau_2(x)$$
$$-2c_2(x)\tau_2(x)-2d_2(x). \tag{2.5}$$

Theorem 2.1. *If satisfies conditions*

$$a_j(x)+(-1)^j>0,\quad c_j(x)\geq 0,\quad \left(\frac{1-2b_j(x)}{2a_j(x)+(-1)^j}\right)'\geq 0,\quad (j=1,2), \tag{2.6}$$

$$p_k(q)\leq 0,\ r_k(q)\geq 0,\ \left(\frac{(1-x)p_k(x)}{2a_1(x)-1}\right)'\leq 0,\ \left(\frac{(1+x)r_k(x)}{2a_2(x)+1}\right)'\geq 0,\quad (k=1,2,...,n), \tag{2.7}$$

*then, the solution $u(x,y)$ of the **Problem I** is unique.*

Proof. Known, that as, a function $u(x,y)$ is a solution of the Eq. (1.1) in the domain of Ω_0, hence an identity:

$$0\equiv\iint\limits_{\omega_0}u(u_{xx}+u_{yy})\mathrm{d}x\mathrm{d}y=\int_{-1}^{-q}uu_y\mathrm{d}x+\int_{q}^{1}uu_y\mathrm{d}x-\iint\limits_{\Omega_0}\left(u_x^2+u_x^2\right)\mathrm{d}x\mathrm{d}y$$

i.e.

$$\int_{-1}^{-q}\tau_2(x)\nu_2(x)\mathrm{d}x+\int_{q}^{1}\tau_1(x)\nu_1(x)\mathrm{d}x-\iint\limits_{\Omega_0}(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y\equiv 0 \tag{2.8}$$

is valid. Further, by virtue (2.3) and (2.5) from (2.8) at $d_j(x)\equiv 0$ we obtain:

$$\int_{q}^{1}\frac{(1-2b_1(x))\tau'{}_1(x)\tau_1(x)}{2a_1(x)-1}\mathrm{d}x+\int_{-1}^{-q}\frac{(1-2b_2(x))\tau'{}_2(x)\tau_2(x)}{2a_2(x)+1}\mathrm{d}x$$

$$-\sum_{k=1}^{n}\frac{1}{2\Gamma(-\alpha_k)}\left[\int_q^1\frac{(1-x)p_k(x)}{2a_1(x)-1}\mathrm{d}x\int_x^1\frac{\tau_1(x)\tau_1(t)}{(t-x)^{1+\alpha_k}}\mathrm{d}t-\int_{-1}^{-q}\frac{(1+x)r_k(x)}{2a_2(x)+1}\mathrm{d}x\int_{-1}^{x}\frac{\tau_2(x)\tau_2(t)}{(x-t)^{1+\alpha_k}}\mathrm{d}t\right]$$
$$-2\int_q^1\frac{c_1(x)\tau_1^2(x)}{2a_1(x)-1}\mathrm{d}x-2\int_{-1}^{-q}\frac{c_2(x)\tau_2^2(x)}{2a_2(x)+1}\mathrm{d}x-\iint\limits_{\Omega_0}(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y\equiv 0. \tag{2.9}$$

On a base to the formula [10]:

$$|x-t|^{-\gamma}=\frac{1}{\Gamma(\gamma)\cos\frac{\pi\gamma}{2}}\int_0^\infty z^{\gamma-1}\cos[z(x-t)]\mathrm{d}z,\qquad 0<\gamma<1, \tag{2.10}$$

and

$$-\sum_{k=1}^{n}\frac{\sin(\pi\alpha_k)}{2\pi\cos\left(\pi(1+\alpha_k)/2\right)}\int_q^1\frac{(1-x)p_k(x)\tau_1(x)}{2a_1(x)-1}\mathrm{d}x\int_x^1\tau_1(t)\mathrm{d}t\int_0^\infty z^{\alpha_k}\cos[z(t-x)]\mathrm{d}z$$
$$+\sum_{k=1}^{n}\frac{\sin(\pi\beta_k)}{2\pi\cos\left(\pi(1+\beta_k)/2\right)}\int_{-1}^{-q}\frac{(1+x)r_k(x)\tau_2(x)}{2a_2(x)+1}\mathrm{d}x\int_{-1}^{x}\tau_2(t)\mathrm{d}t\int_0^\infty z^{\beta_k}\cos[z(x-t)]\mathrm{d}z$$
$$=\sum_{k=1}^{n}\frac{\cos\left(\frac{\pi\alpha_k}{2}\right)}{\pi}\int_q^1\frac{(1-x)p_k(x)\tau_1(x)}{2a_1(x)-1}\mathrm{d}x\int_x^1\tau_1(t)\mathrm{d}t\int_0^\infty z^{\alpha_k}(\cos zt\,coxzx+\sin zt\sin zx)\mathrm{d}z$$
$$-\sum_{k=1}^{n}\frac{\cos\left(\frac{\pi\beta_k}{2}\right)}{\pi}\int_{-1}^{-q}\frac{(1+x)r_k(x)\tau_2(x)}{2a_2(x)+1}\mathrm{d}x\int_{-1}^{x}\tau_2(t)\mathrm{d}t\int_0^\infty z^{\beta_k}(\cos zt\,coxzx+\sin zt\sin zx)\mathrm{d}z$$
$$=\sum_{k=1}^{n}\frac{\cos\left(\frac{\pi\alpha_k}{2}\right)}{2\pi}\int_0^\infty z^{\alpha_k}\mathrm{d}z\int_q^1\frac{(x-1)p_k(x)}{2a_1(x)-1}\mathrm{d}\left[\left(\int_x^1\tau_1(t)\cos zt\mathrm{d}t\right)^2+\left(\int_x^1\tau_1(t)\sin zt\mathrm{d}t\right)^2\right]$$
$$-\sum_{k=1}^{n}\frac{\cos\left(\frac{\pi\beta_k}{2}\right)}{2\pi}\int_0^\infty z^{\beta_k}\mathrm{d}z\int_{-1}^{-q}\frac{(1+x)r_k(x)}{2a_2(x)+1}\mathrm{d}\left[\left(\int_{-1}^{x}\tau_2(t)\cos zt\mathrm{d}t\right)^2+\left(\int_{-1}^{x}\tau_2(t)\sin zt\mathrm{d}t\right)^2\right].$$

Integrating integrals on x by parts, obtain

$$\sum_{k=1}^{n}\frac{(1-q)p_k(q)\cos\left(\frac{\pi\alpha_k}{2}\right)}{2\pi(2a_1(q)-1)}\int_0^\infty z^{\alpha_k}\left[\left(\int_q^1\tau_1(t)\cos zt\mathrm{d}t\right)^2+\left(\int_q^1\tau_1(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}z$$
$$+\sum_{k=1}^{n}\frac{\cos\left(\frac{\pi\alpha_k}{2}\right)}{2\pi}\int_0^\infty z^{\alpha_k}\mathrm{d}z\int_q^1 A'_k(x)\left[\left(\int_x^1\tau_1(t)\cos zt\mathrm{d}t\right)^2+\left(\int_x^1\tau_1(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}x$$
$$-\sum_{k=1}^{n}\frac{(1-q)r_k(-q)\cos\left(\frac{\pi\beta_k}{2}\right)}{2\pi(2a_2(-q)+1)}\int_0^\infty z^{\beta_k}\left[\left(\int_{-1}^{-q}\tau_2(t)\cos zt\mathrm{d}t\right)^2+\left(\int_{-1}^{-q}\tau_2(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}z$$
$$-\sum_{k=1}^{n}\frac{\cos\left(\frac{\pi\beta_k}{2}\right)}{2\pi}\int_0^\infty z^{\beta_k}\mathrm{d}z\int_{-1}^{-q}B'_k(x)\left[\left(\int_{-1}^{x}\tau_2(t)\cos zt\mathrm{d}t\right)^2+\left(\int_{-1}^{x}\tau_2(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}x. \tag{2.11}$$

where $A_k(x)=\frac{(1-x)p_k(x)}{2a_1(x)-1}$, $B_k(x)=\frac{(1+x)r_k(x)}{2a_2(x)+1}$. Further, considering

$$\int_q^1 \frac{(1-2b_1(x))\tau'_1(x)\tau_1(x)}{2a_1(x)-1}\mathrm{d}x+\int_{-1}^{-q}\frac{(1-2b_2(x))\tau'_2(x)\tau_2(x)}{2a_2(x)+1}\mathrm{d}x$$

$$=\frac{(1-2b_1(1))\tau_1^2(1)}{2(2a_1(1)-1)}-\frac{(1-2b_1(q))\tau_1^2(q)}{2(2a_1(q)-1)}-\frac{1}{2}\int_q^1\left(\frac{1-2b_1(x)}{2a_1(x)-1}\right)'\tau_1^2(x)\mathrm{d}x$$

$$+\frac{(1-2b_2(-q))\tau_2^2(-q)}{2(2a_2(-q)+1)}-\frac{(1-2b_2(-1))\tau_2^2(-1)}{2(2a_2(-1)+1)}-\frac{1}{2}\int_{-1}^{-q}\left(\frac{1-2b_2(x)}{2a_2(x)+1}\right)'\tau_2^2(x)\mathrm{d}x,$$

and by virtue (2.11) from (2.10) at $\varphi_j(x,y)\equiv 0\,(j=1,2)$ we will have:

$$\int_q^1\left(\frac{1-2b_1(x)}{2a_1(x)-1}\right)'\tau_1^2(x)\mathrm{d}x+\int_{-1}^{-q}\left(\frac{1-2b_2(x)}{2a_2(x)+1}\right)'\tau_2^2(x)\mathrm{d}x+\int_q^1\frac{4c_1(x)\tau_1^2(x)}{2a_1(x)-1}\mathrm{d}x$$
$$+\int_{-1}^{-q}\frac{4c_2(x)\tau_2^2(x)}{2a_2(x)+1}\mathrm{d}x$$
$$-\sum_{k=1}^n\frac{(1-q)p_k(q)\cos\left(\frac{\pi\alpha_k}{2}\right)}{\pi(2a_1(q)-1)}\int_0^\infty z^{\alpha_k}\left[\left(\int_q^1\tau_1(t)\cos zt\mathrm{d}t\right)^2+\left(\int_q^1\tau_1(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}z$$
$$-\sum_{k=1}^n\frac{\cos\left(\frac{\pi\alpha_k}{2}\right)}{\pi}\int_0^\infty z^{\alpha_k}\mathrm{d}z\int_q^1 A'_k(x)\left[\left(\int_x^1\tau_1(t)\cos zt\mathrm{d}t\right)^2+\left(\int_x^1\tau_1(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}x$$
$$+\sum_{k=1}^n\frac{(1-q)r_k(-q)\cos\left(\frac{\pi\beta_k}{2}\right)}{\pi(2a_2(-q)+1)}\int_0^\infty z^{\beta_k}\left[\left(\int_{-1}^{-q}\tau_2(t)\cos zt\mathrm{d}t\right)^2+\left(\int_{-1}^{-q}\tau_2(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}z$$
$$+\sum_{k=1}^n\frac{\cos\left(\frac{\pi\beta_k}{2}\right)}{\pi}\int_0^\infty z^{\beta_k}\mathrm{d}z\int_{-1}^{-q} B'_k(x)\left[\left(\int_{-1}^{x}\tau_2(t)\cos zt\mathrm{d}t\right)^2+\left(\int_{-1}^{x}\tau_2(t)\sin zt\mathrm{d}t\right)^2\right]\mathrm{d}x$$
$$+2\iint\limits_{\Omega_0}(u_x^2+u_y^2)\mathrm{d}x\mathrm{d}y. \tag{2.12}$$

Thus, taking (2.6) and (2.7) into account from (2.12) we can conclude, that $\tau_1(x)\equiv\tau_2(x)\equiv 0$ and$u_x(x,y)\equiv u_y(x,y)\equiv 0$ i.e. $u(x,y)\equiv$const. Hence, based on $u(x,y)\in C\left(\overline{\Omega}_0\right)$ and boundary conditions (4_j) at $\varphi_j(x,y)\equiv 0\,(j=1,2)$ we will get $u(x,y)\equiv 0$ in $\overline{\Omega}_0$ [12]. Further, on the bases of functional relations (2.3) and (2.5), taking into account $\tau_1(x)\equiv\tau_2(x)\equiv 0$ we will have$\nu_1(x)\equiv\nu_2(x)\equiv 0$, hence, the solution of the Cauchy problem for the Eq. (1.1) in the domains Δ_j, identically equally to zero, i.e. $u(x,y)\equiv 0$ in the domains Δ_j, as owing to uniqueness of solution of the Goursat problem, we will get that $u(x,y)\equiv 0$ in the domains $D_j\,(j=1,2,3)$. Thus, we have received, that $u(x,y)\equiv 0$ on the domain $\overline{\Omega}$ see [6–8]. □

3 The existence of solution of the Problem I

Theorem 3.1. *If satisfies conditions* (2.6), (2.7) *and*

$$\varphi_j(x,y)=(xy)^\gamma\overline{\varphi_j}(x,y);\qquad \overline{\varphi_j}(x,y)\in C\left(\overline{\sigma_j}\right),\qquad 2<\gamma<3, \tag{3.1}$$

$$g_j(x)\in C\left(\overline{I_{2+j}}\right)\cap C^2\left(I_{2+j}\right),\qquad a_j(x),b_j(x),c_j(x),d_j(x)\in C\left(\overline{I_j}\right)\cap C^2\left(I_j\right); \tag{3.2}$$

$$p_k(x)\in C\left(\overline{A_1B_1}\right)\cap C^2\left(A_1B_1\right),\qquad q_k(x)\in C\left(\overline{A_2B_2}\right)\cap C^2\left(A_2B_2\right),\quad (k=\overline{1,n}), \tag{3.3}$$

are fulfilled then the solution of the investigating problem is exist.

Proof. Considering (1.3) from (2.3), (2.5) and (2.6) we will respectively get:

$$(2a_1(x)-1)\nu_1(x)=(1-2b_1(x))\tau_1'(x)+2c_1(x)\int_x^1\tau'_1(t)\mathrm{d}t-2c_1(x)\tau_1(1)$$
$$-\frac{1-x}{2}\sum_{k=1}^n\frac{p_k(x)}{\Gamma(1-\alpha_k)}\left[(1-x)^{-\alpha_k}\tau_1(1)-\int_x^1(t-x)^{-\alpha_k}\tau'_1(t)\mathrm{d}t\right]-2d_1(x).$$

$$(2a_2(x)+1)\nu_2(x)=(1-2b_2(x))\tau_2'(x)-2c_2(x)\int_{-1}^x\tau'_2(t)\mathrm{d}t-2c_2(x)\tau_2(-1)$$
$$+\frac{x+1}{2}\sum_{k=1}^n\frac{r_k(x)}{\Gamma(1-\beta_k)}\left[(x+1)^{-\beta_k}\tau_2(-1)+\int_{-1}^x(x-t)^{-\beta_k}\tau'_2(t)\mathrm{d}t\right]-2d_2(x).$$

Hence, on a base $u(x,y)\in C(\overline{\Omega})$ i.e. $\tau_1(1)=\varphi_1(1,0)$, $\tau_2(-1)=\varphi_1(-1,0)$ we will receive

$$(1-2b_1(x))\tau_1'(x)+\int_x^1\left[\frac{1-x}{2}\sum_{k=1}^n\frac{p_k(x)(t-x)^{-\alpha_k}}{\Gamma(1-\alpha_k)}+2c_1(x)\right]\tau'_1(t)\mathrm{d}t$$
$$=(2a_1(x)-1)\nu_1(x)+f_{21}(x), \tag{3.4}$$

$$(1-2b_2(x))\tau_2'(x)+\int_{-1}^x\left[\frac{x+1}{2}\sum_{k=1}^n\frac{r_k(x)(x-t)^{-\beta_k}}{\Gamma(1-\beta_k)}-2c_2(x)\right]\tau'_2(t)\mathrm{d}t$$
$$=(2a_2(x)+1)\nu_2(x)+f_{22}(x), \tag{3.5}$$

where

$$f_{21}(x)=\frac{1}{2}\sum_{k=1}^n\frac{p_k(x)}{\Gamma(1-\alpha_k)}(1-x)^{1-\alpha_k}\varphi_1(1,0)+2c_1(x)\varphi_1(1,0)+2d_1(x),$$

$$f_{22}(x)=2d_2(x)-\frac{1}{2}\sum_{k=1}^n\frac{(x+1)^{1-\beta_k}r_k(x)\varphi_1(-1,0)}{\Gamma(1-\beta_k)}-2c_2(x)\varphi_1(-1,0).$$

Now we should consider two cases : $b_j(x)\neq\frac{1}{2}$ and $b_j(x)\equiv\frac{1}{2}$ Let's $b_j(x)\neq\frac{1}{2}$, then from (3.4) and (3.5), we will accordingly get

$$\tau_1'(x)+\int_x^1K_{11}(x,t)\tau'_1(t)\mathrm{d}t=\tilde{f}_{21}(x), \tag{3.6}$$

$$\tau_2'(x)+\int_{-1}^{x}K_{12}(x,t)\tau'_2(t)\mathrm{d}t=\tilde{f}_{22}(x), \tag{3.7}$$

where

$$K_{11}(x,t)=\frac{1}{1-2b_1(x)}\left[\frac{1-x}{2}\sum_{k=1}^{n}\frac{p_k(x)(t-x)^{-\alpha_k}}{\Gamma(1-\alpha_k)}+2c_1(x)\right], \qquad q\le x\le t\le 1;$$
$$K_{12}(x,t)=\frac{1}{1-2b_2(x)}\left[\frac{1+x}{2}\sum_{k=1}^{n}\frac{r_k(x)(x-t)^{-\beta_k}}{\Gamma(1-\beta_k)}-2c_2(x)\right], \qquad -1\le t\le x\le -q;$$
$$\tilde{f}_{2j}(x)=\frac{1}{1-2b_j(x)}\left[(2a_j(x)+(-1)^j)\nu_j(x)+f_{2j}(x)\right],$$

at that, by virtue (3.2), (3.3) and owing to class of the function $u_y(x,0)=\nu_j(x)$, $x\in A_jB_j$ we will obtain

$$\left|K_{1j}(x,t)\right|\le \mathrm{const}, \qquad \left|\tilde{f}_{2j}(x)\right|\le \left(x+(-1)^jq\right)^{-\varepsilon}\cdot \mathrm{const}, \quad 0<\varepsilon<1, \quad j=1,2. \tag{3.8}$$

Notice, that the integral equations (3.6) and (3.7) are Volterra integral equations of second kind. Solving these equations by the method of consecutive approach it is found:

$$\tau_1'(x)=\frac{2a_1(x)-1}{1-2b_1(x)}\nu_1(x)+\int_x^1\frac{2a_1(t)-1}{1-2b_1(t)}\Re_{11}(x,t)\nu_1(t)\mathrm{d}t+F_1(x), \tag{3.9}$$
$$\tau_2'(x)=\frac{2a_2(x)+1}{1-2b_2(x)}\nu_2(x)+\int_{-1}^x\frac{2a_2(t)+1}{1-2b_2(t)}\Re_{12}(x,t)\nu_2(t)\mathrm{d}t+F_2(x), \tag{3.10}$$

where $\Re_{1j}(x,t)$ are resolvents of the kernels $K_{1j}(x,t)$ and

$$F_j(x)=\frac{f_{2j}(x)}{1-2b_j(x)}-(j-1)\int_{-1}^{x}\frac{f_{22}(x)}{1-2b_2(x)}\cdot\Re_{12}(x,t)\mathrm{d}t$$
$$+(j-2)\int_x^1\frac{f_{21}(x)}{1-2b_1(x)}\cdot\Re_{11}(x,t)\mathrm{d}t, \tag{3.11}$$

besides, on a base (3.8) we have $\left|\Re_{1j}(x,t)\right|\le$ const, hence, from (3.11) take into account (3.2) and (3.3) we will receive

$$\left|F_j(x)\right|\le \mathrm{const}, \ (j=1,2). \tag{3.12}$$

Further, note that a solution of the problem N for the Eq. (1.1) in the domain of Ω_0 with boundary conditions (1.5) and $u_y(x,+0)=\nu_1^+(x)$, $(x,0)\in A_1B_1$, $u_y(x,+0)=\nu_2^+(x)$, $(x,0)\in A_2B_2$ is unique and represented on the form [7,8]:

$$u(x,y)=\int_{\sigma_1}\varphi_1(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,y)\mathrm{d}S-\int_{\sigma_2}\varphi_2(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,y)\mathrm{d}S$$

$$-\int_q^1 \nu_1^+(t)G(t,0;x,y)\mathrm{d}t+\int_{-1}^{-q} \nu_2^+(t)G(t,0;x,y)\mathrm{d}t, \tag{3.13}$$

where $G(\xi,\eta;x,y)$ is GreenŁ function of the problem N for the equation $u_{xx}+u_{yy}=0$ in the domain Ω_0 and it is represented as [7,8]:

$$G(\xi,\eta;x,y)=\frac{1}{2\pi}\ln\left|\frac{\theta_1\left(\frac{\ln v+\ln\overline{\mu}}{2\pi ir}\right)\theta_1\left(\frac{\ln\overline{v}+\ln\overline{\mu}}{2\pi ir}\right)}{\theta_1\left(\frac{\ln v-\ln\mu}{2\pi ir}\right)\theta_1\left(\frac{\ln\overline{v}-\ln\mu}{2\pi ir}\right)}\right|, \tag{3.14}$$

where $v=\xi+i\eta$, $\overline{v}=\xi-i\eta$, $\mu=x+iy$, $\overline{\mu}=x-iy$, $r=\frac{1}{\pi i}\ln q$, $i^2=-1$, $\theta_1(\xi)=\theta_1\left(\xi\middle|-\frac{1}{r}\right)$ is Theta function. From (3.13) at $y=0$, we will get a functional relation between $\tau_1^+(x)$ and$\nu_1^+(x)$ $\left(\tau_2^+(x)\text{ and }\nu_2^+(x)\right)$ on the piece A_1B_1 (A_2B_2) respectively, getting from the domain :

$$\begin{aligned}\tau_1^+(x)=&\int_{\sigma_1}\varphi_1(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,0)\mathrm{d}S-\int_{\sigma_2}\varphi_2(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,0)\mathrm{d}S\\&-\int_q^1 \nu_1^+(t)G(t,0;x,0)\mathrm{d}t+\int_{-1}^{-q}\nu_2^+(t)G(t,0;x,0)\mathrm{d}t,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\tau_2^+(x)=&\int_{\sigma_1}\varphi_1(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,0)\mathrm{d}S-\int_{\sigma_2}\varphi_2(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,0)\mathrm{d}S\\&-\int_q^1 \nu_1^+(t)G(t,0;x,0)\mathrm{d}t+\int_{-1}^{-q}\nu_2^+(t)G(t,0;x,0)\mathrm{d}t.\end{aligned} \tag{3.16}$$

Differentiating equalities (3.15) and (3.16) by x, obtain

$$\tau_j'^+(x)=\int_{-1}^{-q}\nu_2^+(t)\frac{\partial G(t,0;x,0)}{\partial x}\mathrm{d}t-\int_q^1\nu_1^+(t)\frac{\partial G(t,0;x,0)}{\partial x}\mathrm{d}t+\tilde{F}_j(x), \tag{3.17}$$

where

$$\tilde{F}_j(x)=\int_{\sigma_1}\varphi_1(\xi,\eta)\frac{\partial}{\partial x}\left[\frac{\partial}{\partial n}G(\xi,\eta;x,0)\right]\mathrm{d}S-\int_{\sigma_2}\varphi_2(\xi,\eta)\frac{\partial}{\partial x}\left[\frac{\partial}{\partial n}G(\xi,\eta;x,0)\right]\mathrm{d}S$$

Having excluded $\tau_j'(t)$, $(j=1,2)$ from the relations (3.9), (3.10) and (3.17) we will get a system of integral equations:

$$\begin{cases}\dfrac{2a_1(x)-1}{1-2b_1(x)}\cdot\nu_1(x)+\displaystyle\int_x^1\frac{2a_1(t)-1}{1-2b_1(t)}\Re_{11}(x,t)\nu_1(t)\mathrm{d}t+\frac{1}{\pi}\int_q^1\nu_1(t)\tilde{K}_2(x,t)\mathrm{d}t\\ =\dfrac{1}{\pi}\displaystyle\int_{-1}^{-q}\nu_2(t)\tilde{K}_2(x,t)\mathrm{d}t+\tilde{F}_1(x)-F_1(x),\\ \dfrac{2a_2(x)+1}{1-2b_2(x)}\nu_2(x)+\displaystyle\int_{-1}^x\frac{2a_2(t)+1}{1-2b_2(t)}\Re_{12}(x,t)\nu_2(t)\mathrm{d}t-\frac{1}{\pi}\int_{-1}^{-q}\nu_2(t)\tilde{K}_2(x,t)\mathrm{d}t\\ =-\dfrac{1}{\pi}\displaystyle\int_q^1\nu_1(t)\tilde{K}_2(x,t)\mathrm{d}t+\tilde{F}_2(x)-F_2(x)\end{cases} \tag{3.18}$$

where

$$\tilde{K}_2(x,t)=\frac{2\ln|t|}{x\ln q}+\frac{1}{\pi}\left[\frac{1}{t-x}-\frac{t}{1-tx}+\sum_{n=1}^{\infty}\left(\frac{q^{2n}}{t-q^{2n}x}-\frac{q^{2n}t}{1-q^{2n}tx}-\frac{q^{-2n}t}{1-q^{-2n}tx}+\frac{q^{-2n}}{t-q^{2n}x}\right)\right].$$

Considering (2.6) the system (3.18) we will rewrite on the form:

$$\begin{cases} \nu_1(x)+\int_q^1 K_{21}(x,t)\nu_1(t)\mathrm{d}t=\dfrac{1-2b_1(x)}{\pi(2a_1(x)-1)}\cdot\int_{-1}^{-q}\nu_2(t)\tilde{K}_2(x,t)\mathrm{d}t+F_1^*(x),\\ \nu_2(x)+\int_{-1}^{-q} K_{22}(x,t)\nu_2(t)\mathrm{d}t=-\dfrac{1-2b_2(x)}{\pi(2a_2(x)+1)}\int_q^1\nu_1(t)\tilde{K}_2(x,t)\mathrm{d}t+F_2^*(x),\end{cases}$$

where

$$K_{21}(x,t)=\begin{cases}\dfrac{2a_1(t)-1}{1-2b_1(t)}\cdot\dfrac{1-2b_1(x)}{2a_1(x)-1}\cdot\Re_{11}(x,t)+\dfrac{1-2b_1(x)}{2a_1(x)-1}\cdot\tilde{K}_2(x,t), & x\le t\le 1,\\ \dfrac{1-2b_1(x)}{2a_1(x)-1}\cdot\tilde{K}_2(x,t), \quad 0\le t\le x,\end{cases}$$

$$F_1^*(x)=\frac{1-2b_1(x)}{2a_1(x)-1}\cdot\big(\tilde{F}_1(x)-F_1(x)\big),\qquad F_2^*(x)=\frac{1-2b_1(x)}{2a_1(x)+1}\cdot\big(\tilde{F}_2(x)-F_2(x)\big). \tag{3.19}$$

Notice that the functions $\tilde{F}_j(x)$ was investigated in the research works [7, 8] and for this functions takes place $|\tilde{F}_j(x)|\le\text{const}(x+(-1)^j q)^{\gamma-3}\,(2<\gamma<3)$. Hence, taking into account (2.7), (3.1)-(3.3) and (3.12) from (3.19), we will conclude that $F_1^*(x)\in C^2(q,1)$ $\big(F_2^*(x)\in C^2(-1,-q)\big)$ and $F_j^*(x)$ can tend to infinity an order of less one at$x\to(-1)^{j-1}q$, and at $x\to(-1)^{j-1}$ $(j=1,2)$ it is limited.

Thus, the system of integral equation (3.18) reduce to the Fredholm integral equations of the second kind, by the known method of the Karleman-Vekua [13], just as in works [7, 8]. Note, that the unique solvability of the Fredholm integral equations of the second kind follows from the uniqueness of solution of the Problem I and from the theory integral equations. Solving the system of integral equations (3.18), we will found $\nu_1(x)$ and $\nu_2(x)$ [7], further owing to account (3.9), (3.10) and $\tau_1(q)=\varphi_2(q,0)$, $\tau_2(-q)=\varphi_2(-q,0)$ from (3.15) and (3.16) we will find $\tau_j(x)$, $(j=1,2)$.

Hence, after founding $\tau_1(x)$ and $\nu_1(x)$, $\tau_2(x)$ and $\nu_2(x)$, the solution of the **Problem I** can be restored in the domain Ω_0 as the solution of the **Problem N** (3.13), and in domains Δ_j $(j=1,2)$ as the solution of the Cauchy problem. The solution of the **Problem I** in domains of D_j (j=1,2), we can restore as a solution of the Goursat problem with conditions (1.6) and, where $h_j(x)$ (j=1,2) are traces of solution of the Cauchy problems in domains Δ_j $(j=1,2)$, on the line $y-(-1)^j x=q$, and reciprocally in domain D_3 as the solution of the Goursat problem with conditions (j=1,2) where $\tilde{h}_j(t)$ (j=1,2) are traces of solution of the Goursat problems in domains D_j $(j=1,2)$.

The theorem is proved on the case of $b_j(x)\neq\frac{1}{2}$.

Let's $b_j(x)=\frac{1}{2}$, then from (2.3) and (2.5,) we will accordingly get

$$2c_1(x)\tau_1(x)+\frac{1-x}{2}\int_x^1\sum_{k=1}^n\frac{p_k(x)}{\Gamma(-\alpha_k)}(t-x)^{-\alpha_k-1}\tau_1(t)\mathrm{d}t=(1-2a_1(x))\nu_1(x)-2d_1(x), \quad (3.20)$$

$$2c_2(x)\tau_2(x)-\frac{1+x}{2}\int_{-1}^x\sum_{k=1}^n\frac{r_k(x)}{\Gamma(-\beta_k)}(x-t)^{-\beta_k-1}\tau_2(t)\mathrm{d}t=-(1+2a_2(x))\nu_2(x)-2d_2(x). \quad (3.21)$$

Note that the integral equations (3.20) and (3.21) at $c_j(x)\neq 0$ are Volterra integral equations of second kind.

Solving these equations by the method of consecutive approach it is found:

$$\tau_1(x)=\frac{1-2a_1(x)}{2c_1(x)}\cdot\nu_1(x)+\int_x^1\frac{1-2a_1(t)}{2c_1(t)}\tilde{\Re}_{11}(x,t)\nu_1(t)\mathrm{d}t+M_1(x), \quad (3.22)$$

$$\tau_2(x)=-\frac{2a_2(x)+1}{2c_2(x)}\nu_2(x)-\int_{-1}^x\frac{2a_2(t)+1}{2c_2(t)}\tilde{\Re}_{12}(x,t)\nu_2(t)\mathrm{d}t+M_2(x), \quad (3.23)$$

where $\tilde{\Re}_{1j}(x,t)$ are resolvents of kernel $\tilde{K}_{1j}(x,t)$ of the integral equations (3.20) and (3.21), where

$$\tilde{K}_{11}(x,t)=\frac{1-x}{4c_1(x)}\sum_{k=1}^n\frac{p_k(x)(t-x)^{-\alpha_k-1}}{\Gamma(-\alpha_k)}, \qquad q\le x\le t\le 1,$$

$$\tilde{K}_{12}(x,t)=-\frac{1+x}{4c_2(x)}\sum_{k=1}^n\frac{r_k(x)(x-t)^{-\beta_k-1}}{\Gamma(-\beta_k)}, \qquad -1\le t\le x\le -q$$

$$M_1(x)=-\int_x^1\frac{d_1(t)}{c_1(t)}\tilde{\Re}_{11}(x,t)\mathrm{d}t-\frac{d_1(x)}{c_1(x)}, \qquad M_2(x)=-\int_{-1}^x\frac{d_2(t)}{c_2(t)}\tilde{\Re}_{12}(x,t)\mathrm{d}t-\frac{d_2(x)}{c_2(x)},$$

besides, $\left|\tilde{K}_{1j}(x,t)\right|\le|x-t|^{-\varepsilon}$const, $0<\varepsilon<1$; $\left|M_j(x)\right|\le$const,$(j=1,2)$.

Further, this problem can be investigated similarly for the case of $b_j(x)\neq\frac{1}{2}$.

Now we will assume, that $c_j(x)=0$ then the integral equations (3.20) and (3.21) are Volterra integral equations of the first kind. Known that such equations can be solved leading them to the Volterra integral equations of the second kind. As a final result in this case we can formulate the following Lemma:

Lemma 3.1. *If satisfies conditions* (2.7), (3.1)-(3.3) *and*

$$2a_j(1)+(-1)^j>0, \quad d_j\Big(-(-1)^j\Big)=0, \quad c_j(x)\equiv 0, \quad 1-2b_j(x)\equiv 0, \quad (j=1,2),$$

then the solution of the investigating problem is exist. □

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