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Asymptotic Behavior of the Solution to a 3-D Simplified Energy-Transport Model for Semiconductors

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Abstract. The well-posedness of smooth solution to a 3-D simplified Energy-Transport model is discussed in this paper. We prove the local existence, uniqueness, and asymptotic behavior of solution to the equations with hybrid cross-diffusion. The smooth solution convergences to a stationary solution with an exponential rate as time tends to infinity when the initial date is a small perturbation of the stationary solution.

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1 Introduction

Energy-Transport model was first proposed by Stratton [1] and latter derived from the semiconductor Boltzmann equation by Ben Abdallah et al. [2]. The strong coupling and temperature gradients make it difficult to analyze the energy-transport model. Therefore, we consider in this paper a simplified energy-transport model which still includes temperature gradients with weakly coupling of the energy equation.

The simplified Energy-Transport model, achieved by Jüngel et al. in [3], consists of a drift-diffusion-type equation for the electron density $n(x,t)$, a nonlinear heat equation for the electron temperature $\theta(x,t)$, and the Poisson equation for the electric potential $V(x,t)$:

$$\partial_t n - \mathrm{div}(\nabla(n\theta) - n\nabla V) = 0, \tag{1.1}$$

$$\mathrm{div}(\kappa(n)\nabla\theta) = \frac{n}{\tau}(\theta - \theta_L(x)), \tag{1.2}$$

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$$\lambda^2 \triangle V = n - C(x). \tag{1.3}$$

Here, $\kappa(n)$ is the thermal conductivity, we suppose that $\kappa(n)=n$, $\theta_L(x)$ is the lattice temperature, and $C(x)$ is the doping profile characterizing the device under consideration. The energy relaxation time $\tau>0$ and the Debye length $\lambda>0$ are scaled physical parameters. Without lose of generality, we suppose that $\tau=\theta_L(x)=\lambda=1$, and set $E(x,t)=\nabla V(x,t)$. Then the model (1.1)-(1.3) can be changed into the following model for the electron density $n(x,t)$, the electron temperature $\theta(x,t)$ and the electric field $E(x,t)$:

$$\partial_t n - \mathrm{div} j = 0, \quad j = (\nabla(n\theta) - nE), \tag{1.4}$$

$$\mathrm{div}(n\nabla\theta) = n(\theta - 1), \tag{1.5}$$

$$\mathrm{div} E = n - C(x). \tag{1.6}$$

Eqs. (1.4)-(1.6) hold in the bounded main $\Omega \subset R^3$, with the initial boundary condition

$$n(x,0) = n_0(x), \tag{1.7}$$

$$j\cdot\vec{n}|_{\partial\Omega} = 0, \quad \nabla\theta\cdot\vec{n}|_{\partial\Omega} = 0, \quad E\cdot\vec{n}|_{\partial\Omega} = 0, \tag{1.8}$$

where $\vec{n}$ denotes the exterior unit normal vector on $\partial\Omega$, and the initial datum $n_0(x)$ satisfies the following condition

$$\int_\Omega n_0(x) - C(x)\mathrm{d}x = 0. \tag{1.9}$$

Before we exposit our results, we review the energy-transport model in the literature. The common form for energy-transport model [4] is

$$\begin{aligned} &\partial_t n - \frac{1}{q}\mathrm{div} J_n = 0, \\ &\partial_t U(n,\theta) - \mathrm{div} J_w = -J_n \cdot \nabla V + W(n,\theta), \\ &\lambda^2 \triangle V = n - C(x), \end{aligned}$$

with

$$\begin{aligned} J_n &= L_{11}\left(\frac{\nabla n}{n} - \frac{q\nabla V}{k_B\theta}\right) + \left(\frac{L_{12}}{k_B\theta} - \frac{3}{2}L_{11}\right)\frac{\nabla\theta}{\theta}, \\ qJ_w &= L_{21}\left(\frac{\nabla n}{n} - \frac{q\nabla V}{k_B\theta}\right) + \left(\frac{L_{22}}{k_B\theta} - \frac{3}{2}L_{21}\right)\frac{\nabla\theta}{\theta}, \end{aligned}$$

where $U(x,\theta)$ is the density of the internal energy, $W(n,\theta)$ is the energy relaxation term satisfying $W(n,\theta)(\theta-\theta_L(x)) \le 0$,

$$W(n,\theta) = -\frac{n(\theta - \theta_L(x))}{\tau_\beta}, \quad \tau_\beta = \frac{\pi^{\frac{5}{2}}\theta^{\frac{1}{2}-\beta}}{\sqrt{8}\Gamma(\beta+2)s_0},$$

where s_0 is a constant, J_n, J_w are the carrier flux density and energy flux density or heat flux, L the diffusion matrices, q the elementary charge, k_B the Boltzmann constant.

$$L=(L_{ij})=\mu_0\Gamma(2-\beta)nk_B\theta^{\frac{1}{2}-\beta}\begin{pmatrix}1 & (2-\beta)k_B\theta\\(2-\beta)k_B\theta & (3-\beta)(2-\beta)k_B^2\theta^2\end{pmatrix}$$

where μ_0 comes from the electron elastic scattering rate and Γ denotes the Gamma function with $\Gamma(\frac{1}{2})=\sqrt{\pi}$ and $\Gamma(x+1)=x\Gamma(x)$ for $x>0$.

When the energy band is parabolic, the relation U is given as $U=\frac{3}{2}n\theta$, which approximated by Boltzmann statistics. In general, we put $\beta=\frac{1}{2},0$, and $-\frac{1}{2}$. $\beta=\frac{1}{2}$ is first employed by Chen et al. in [5]. For the Chen model, Y. Li and L. Chen [6] have study the asymptotic behavior of global smooth solution to the initial boundary problem in 1-D space. $\beta=0$ used by Lyumkis et al. in [7], the Lyumkis model is a typical energy transport model in application. In [8], the existence and uniqueness of $(W_p^{2,1}(Q_\tau))^2\times L_q(0,\tau;W_q^1(\Omega))$ solution to Lyumkis model is discussed for $N+2<p\le q<\infty$ and $1\le N\le 3$. The global existence and asymptotic behavior of smooth solutions to the initial-boundary value problem for the 1-D Lyumkis energy transport model in semiconductor science was studied in [9]. Topical choice for $\beta=-\frac{1}{2}$ comes from the diffusion approximation of the hydrodynamic semiconductor model. Y. Li [10] studied the global existence and the large time behavior of smooth solutions to the initial boundary value problem for a degenerate compressible energy transport model. A simplified transient energy-transport system for semiconductors subject to mixed Dirichlet-Neumann boundary conditions was analyzed in [3]. Under the assumption that the thermal conductivity $\kappa(n,\theta)=n$, it proved the global-in-time existence of bounded weak solutions. In [11], J.W. Dong and Q.C. Ju proved the existence and uniqueness of stationary solutions to the energy-transport model for semiconductor in one space dimension, where the thermal conductivity $\kappa(n,\theta)=n\theta$. With the rapid development of science and technology, more and more semiconductor devices of nanoscale structure will come into use. K. Wang and S. Wang [12] studied the limit of vanishing Debye length in a bipolar drift-diffusion model for semiconductors with physical contactinsulating boundary conditions in one-dimensional case. The existence of global-in-time weak solution to a quantum energy-transport model for semiconductors is proved in [13]. J.W. Dong and S.H. Cheng [14, 15] have studied the classical solution to stationary one dimensional quantum energy-transport model with the $k(n,\theta)=n$ and $k(n,\theta)=n\theta$ respectively.

The main purpose of this article is to study the local existence, uniqueness, and asymptotic behavior of the solution to the 3-D simplified energy-transport model (1.4)-(1.6) when the initial data is around a stationary solution to the corresponding linear drift-diffusion model.

We consider the smooth solution of (1.4)-(1.6) around a typical stationary solution $(\mathcal{N},1,\mathcal{E})$. The corresponding stationary problem is

$$\triangle\mathcal{N}-\mathrm{div}(\mathcal{N}\mathcal{E})=0, \tag{1.10}$$

$$\mathrm{div}\mathcal{E}=\mathcal{N}-C(x), \tag{1.11}$$

with the boundary condition

$$[\nabla\mathcal{N}-\mathcal{N}\mathcal{E}]\cdot\overrightarrow{n}|_{\partial\Omega}=0, \quad \mathcal{E}\cdot\overrightarrow{n}|_{\partial\Omega}=0. \tag{1.12}$$

The isothermal stationary problem (1.10)-(1.12) was studied in [16] and it obtained the following theorem.

Theorem 1.1. *Assume that* $0<\underline{C}\le\mathcal{C}(x)\le\overline{C}$ *and* $\mathcal{C}(x)\in L^\infty(\Omega)$*, then the problem* (1.10)-(1.12) *has a solution* $(\mathcal{N},\mathcal{E})$*, for which the following estimates hold:*

$$0<\underline{C}\le\mathcal{N}(x)\le\overline{C}, \qquad x\in\Omega, \tag{1.13}$$

$$\underline{c}\le\mathcal{E}(x)\le\overline{c}, \qquad x\in\Omega, \tag{1.14}$$

$$|\mathcal{E}(x)|,\ |\mathrm{div}\mathcal{E}(x)|,\ |\nabla\mathcal{N}(x)|,\ |\triangle\mathcal{N}(x)|\le a_0(\overline{C}-\underline{C}), \quad x\in\Omega, \tag{1.15}$$

where a_0 *is a positive constant and* $\underline{c}$*,* $\overline{c}$ *are constants.*

Our main theorems on the local existence and exponential decay for the smooth solution of (1.4)-(1.9) are as follows.

Theorem 1.2. *Assume that* $C(x)\in L^\infty(\Omega)$*,* $n_0(x)\in H^2(\Omega)$ *and* $n_0(x)\ge 2\underline{D}$*,* $\underline{D}$ *is a positive constant. Then there exists a* $T>0$*, such that the problem* (1.4)-(1.9) *has a unique smooth solution* $(n(x,t),\theta(x,t),E(x,t))$*, satisfying*

$$n(x,t)\in L^\infty([0,T),H^2(\Omega)); \quad (\theta(x,t),E(x,t))\in L^\infty([0,T),H^3(\Omega)).$$

Theorem 1.3. *Suppose* $0<\underline{C}\le C(x)\le\overline{C}$*,* $n_0(x)\in H^2(\Omega)$ *and* $n_0(x)\ge 2\underline{D}$*,* $\underline{D}$ *is a positive constant. There exists a positive* δ_0 *such that if* $\|n_0(x)-\mathcal{N}(x)\|_{H^2}\le\delta_0$*, then, the problem* (1.4)-(1.9) *has a unique smooth solution* $(n(x,t),\theta(x,t),E(x,t))\in\Omega\times[0,T)$*, satisfying*

$$\|E(\cdot,t)-\mathcal{E}(\cdot)\|_{H^3}+\|n(\cdot,t)-\mathcal{N}(\cdot)\|_{H^2}+\|\theta(\cdot,t)-1\|_{H^3}\le C_0\|n_0(x)\|_{H^2}\exp(-\alpha t).$$

for some positive constants C_0 *and* α*.*

The idea of proof is organized as follows. In Section 2 we focus on the local existence, uniqueness of the smooth solution to the system (1.4)-(1.9). Section 3 is devoted to the asymptotic behavior of the smooth solution to the system (1.4)-(1.9).

2 Local existence of the solution

In this section, we will prove the local existence of the solution with the help of Banach Fixed Point Theorem and Gagliardo-Nirenberg inequalities.

2.1 Linearize equation

First for any fixed $n(x,t)$, we can obtain a unique $E(x,t)$ by

$$\mathrm{div}E = n - C(x), \tag{2.1}$$
$$E|_{\partial\Omega} = 0, \tag{2.2}$$

and we can obtain a unique $\theta(x,t)$ by

$$-\mathrm{div}(n\nabla\theta) = n(1-\theta), \tag{2.3}$$
$$(\nabla\theta\cdot\overrightarrow{n})|_{\partial\Omega} = 0, \tag{2.4}$$

then solve the following system for u

$$u_t - \mathrm{div}(\theta\nabla u) + u(1-\theta) + E\cdot\nabla n + \mathrm{div}Eu = 0. \tag{2.5}$$

2.2 Existence of solution

In order to prove the local existence of the solution, we set the positive constant $M_0 = \|n_0\|^2_{H^2}$ and define the space $\mathcal{S}$.

$$\mathcal{S} := \{n(x,t) | \sup_{0\le t\le T}(\|n\|^2_{H^2}) + \|n_t\|^2_{L^2}) \le M,\ M \ge M_0,\ n \ge \underline{D}\}, \tag{2.6}$$

where $\underline{D}$ is a positive constant, and the metric $|||n(x,t)|||$ defined by :

$$|||n(x,t)||| = \sup_{0\le t\le T}\|n\|^2_{L^2} + \int_0^T \|n\|^2_{H^1}\mathrm{d}t. \tag{2.7}$$

We define the map $\mathcal{F}: n\in\mathcal{S}\to u$ by (2.1)-(2.5). Thus, we prove that there exists a $T>0$ such that $\mathcal{F}$ maps $\mathcal{S}$ into itself and $\mathcal{F}$ is contractive with metric (2.7).

Lemma 2.1. *Assume that $C(x)\in L^\infty(\Omega)$, $n_0(x)\in H^2(\Omega)$ with $n_0(x)\ge 2D$, then exists a $T>0$ such that $\mathcal{F}$ maps into itself.*

Proof. In order to obtain our result, we only need to prove $u\in\mathcal{S}$, for any given $n\in\mathcal{S}$. By Sobolev embedding theorem

$$\sup_{0\le t\le T}|n| \le M.$$

By (2.1), we have for all $t\in[0,T]$, $E\in L^\infty(0,T;H^3(\Omega))$ and $\mathrm{div}E_t\in L^\infty(0,T;L^2(\Omega))$.

We prove the lemma in several steps.

Step 1: Estimate of θ. Eq. (2.3) can be rewritten as

$$\mathrm{div}(n\nabla\theta) = n(\theta-1). \tag{2.8}$$

Multiplying (2.8) by θ and integrating it over Ω, noting the boundary conditions (1.8), we have

$$\int_\Omega n|\nabla\theta|^2 \mathrm{d}x + \int_\Omega n\theta^2 \mathrm{d}x = \int_\Omega n\theta \mathrm{d}x.$$

With the help of Young inequality and $\underline{n} \le n \le \overline{n}$, we have

$$\underline{n}\int_\Omega |\nabla\theta|^2 \mathrm{d}x + \frac{\underline{n}}{2}\int_\Omega \theta^2 \mathrm{d}x \le \frac{1}{2\underline{n}}\int_\Omega n^2 \mathrm{d}x. \tag{2.9}$$

By the above estimate, we can draw the conclusion that $\theta, \nabla\theta \in L^\infty(0,T;L^2(\Omega))$.

Multiplying (2.8) by $\triangle\theta$ and integrating it over Ω, we have

$$\int_\Omega n(\triangle\theta)^2 \mathrm{d}x = \int_\Omega n\theta\triangle\theta \mathrm{d}x - \int_\Omega \nabla n\cdot\nabla\theta\triangle\theta \mathrm{d}x - \int_\Omega n\triangle\theta \mathrm{d}x.$$

With the help of Young inequality and $\underline{n} \le n \le \overline{n}$, we have

$$\begin{aligned}\underline{n}\int_\Omega (\triangle\theta)^2 \mathrm{d}x \le & 3\epsilon_1 \int_\Omega (\triangle\theta)^2 \mathrm{d}x + \frac{1}{4\epsilon_1}\int_\Omega (n\theta)^2 \mathrm{d}x + \frac{1}{4\epsilon_1}\int_\Omega n^2 \mathrm{d}x \\ & + \frac{1}{4\epsilon_1}\int_\Omega (\nabla n\cdot\nabla\theta)^2 \mathrm{d}x. \end{aligned} \tag{2.10}$$

For $\frac{1}{4\epsilon_1}\int_\Omega (\nabla n\cdot\nabla\theta)^2 \mathrm{d}x$, we have the estimate by using the Young inequality and Gagliardo-Nirenberg inequality as follows

$$\begin{aligned} & \frac{1}{4\epsilon_1}\int_\Omega (\nabla n\cdot\nabla\theta)^2 \mathrm{d}x \\ \le & \frac{1}{4\epsilon_1}\int_\Omega |\nabla n|^2|\nabla\theta|^2 \mathrm{d}x \\ \le & \frac{m(\epsilon_2)}{4\epsilon_1}\int_\Omega (|\nabla n|^2)^{\frac{5}{2}} \mathrm{d}x + \frac{\epsilon_2}{4\epsilon_1}\int_\Omega (|\nabla\theta|^2)^{\frac{5}{3}} \mathrm{d}x \\ = & \frac{m(\epsilon_2)}{4\epsilon_1}\int_\Omega |\nabla n|^5 \mathrm{d}x + \frac{\epsilon_2}{4\epsilon_1}\int_\Omega |\nabla\theta|^{\frac{10}{3}} \mathrm{d}x \\ \le & \frac{cm(\epsilon_2)}{4\epsilon_1}\left(\int_\Omega |\nabla n|^2 \mathrm{d}x\right)^{\frac{1}{4}}\left(\int_\Omega (\triangle n)^2 \mathrm{d}x\right)^{\frac{9}{4}} + \frac{c\epsilon_2}{4\epsilon_1}\left(\int_\Omega |\nabla\theta|^2 \mathrm{d}x\right)^{\frac{2}{3}}\int_\Omega (\triangle\theta)^2 \mathrm{d}x \\ \le & \frac{c(M)m(\epsilon_2)}{4\epsilon_1} + \frac{c(M)\epsilon_2}{4\epsilon_1}\int_\Omega (\triangle\theta)^2 \mathrm{d}x. \end{aligned} \tag{2.11}$$

where $c(M)$ is a constant depending on M, $m(\epsilon_2)$ is a constant depending on ϵ_2. We choose $3\epsilon_1 = \frac{\underline{n}}{4}$, $\frac{c(M)\epsilon_2}{4\epsilon_1} = \frac{\underline{n}}{4}$. Therefore inequality (2.10) becomes

$$\frac{\underline{n}}{4}\int_\Omega (\triangle\theta)^2 \mathrm{d}x \le \frac{3}{\underline{n}}\int_\Omega (n\theta)^2 \mathrm{d}x + \frac{3}{\underline{n}}\int_\Omega n^2 \mathrm{d}x + \frac{c(M)m(\epsilon_2)}{4\epsilon_1}. \tag{2.12}$$

Hence $\triangle\theta \in L^\infty(0,T;L^2(\Omega))$.

Differentiating (2.8) with respect to x and multiplying it by $\nabla\triangle\theta$, and integrating over Ω, using the Young inequality and Gagliardo-Nirenberg inequality, we get

$$\begin{aligned}\underline{n}\int_\Omega(\nabla\triangle\theta)^2\mathrm{d}x \le & 5\epsilon_3\int_\Omega(\nabla\triangle\theta)^2\mathrm{d}x+\frac{1}{4\epsilon_3}\int_\Omega|\theta\nabla n|^2\mathrm{d}x+\frac{1}{4\epsilon_3}\int_\Omega|n\nabla\theta|^2\mathrm{d}x\\ &+\frac{1}{4\epsilon_3}\int_\Omega|\nabla(\nabla n\cdot\nabla\theta)|^2\mathrm{d}x+cm(\epsilon_3)\Big(\int_\Omega|\nabla n|^2\mathrm{d}x\Big)^{\frac14}\Big(\int_\Omega(\triangle n)^2\mathrm{d}x\Big)^{\frac94}\\ &+\frac{1}{4\epsilon_3}\int_\Omega|\nabla n|^2\mathrm{d}x+c\epsilon_3\Big(\int_\Omega(\triangle\theta)^2\mathrm{d}x\Big)^{\frac23}\int_\Omega(\nabla\triangle\theta)^2\mathrm{d}x. \qquad (2.13)\end{aligned}$$

Hence $\nabla\triangle\theta \in L^\infty(0,T;L^2(\Omega))$.

Differentiating (2.8) with respect to t and multiplying it by θ_t, and integrating over Ω, using the Young inequality, Sobolev embedding theorem, and integration by parts whenever necessary, we get

$$\begin{aligned}(\underline{n}-\epsilon_4)\int_\Omega|\nabla\theta_t|^2\mathrm{d}x+(\underline{n}-2\epsilon_4)\int_\Omega\theta_t^2\mathrm{d}x \le & \frac{1}{4\epsilon_4}\int_\Omega n_t^2\mathrm{d}x+\frac{1}{4\epsilon_4}\int_\Omega(n_t\theta)^2\mathrm{d}x+\frac{1}{4\epsilon_4}\int_\Omega|n_t\nabla\theta|^2\mathrm{d}x\\ \le & M. \qquad (2.14)\end{aligned}$$

Hence $\theta_t,\nabla\theta_t \in L^\infty(0,T;L^2(\Omega))$.

Combined (2.9) with (2.11), (2.13), and (2.14), yields that $\theta\in L^\infty(0,T;H^3(\Omega)),\theta_t\in L^\infty(0,T;H^1(\Omega))$.

Step 2: Estimate of u.

Multiplying (2.5) by u and integrating it over Ω, using the Young inequality and integration by parts whenever necessary, we get

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}x+\int_\Omega\theta|\nabla u|^2\mathrm{d}x+\int_\Omega u^2\mathrm{d}x\le&\int_\Omega\theta u^2\mathrm{d}x+\epsilon_5\int_\Omega u^2\mathrm{d}x\\ &+\frac{1}{4\epsilon_5}\int_\Omega(E\cdot\nabla n)^2\mathrm{d}x+\int_\Omega u^2\mathrm{div}E\mathrm{d}x.\end{aligned}$$

Since $\theta\in L^\infty(0,T;H^3(\Omega)),E\in L^\infty(0,T;H^3(\Omega))$,

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}x+\int_\Omega\theta|\nabla u|^2\mathrm{d}x+\int_\Omega u^2\mathrm{d}x\le M_1\int_\Omega u^2\mathrm{d}x+K_1\int_\Omega|\nabla n|^2\mathrm{d}x.$$

where M_1 depends on θ, $\mathrm{div}E$ and ϵ_5, K_1 depends on E and ϵ_5. By using Gronwall inequality and choosing T small enough, we have

$$\int_\Omega u^2\mathrm{d}x\le K_1MT\exp(M_1T):=A_1\le M. \qquad (2.15)$$

Hence $u\in L^\infty(0,T;L^2(\Omega))$.

Multiplying (2.5) by $-\triangle u$ and integrating it over Ω, using the Young inequality, Sobolev embedding theorem, and integration by parts whenever necessary, similar to the above, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega \theta(\triangle u)^2 \mathrm{d}x \le M_2 \int_\Omega |\nabla u|^2 \mathrm{d}x \\ + K_2\left(\int_\Omega |\nabla n|^2 \mathrm{d}x + \int_\Omega u^2 \mathrm{d}x\right),$$

where M_2 depends on $\theta,\nabla\theta$ and ϵ_6, K_2 depends on $E,\mathrm{div}E$ and ϵ_6. From the Gronwall inequality, by choosing T small enough, we obtain

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \le K_2(M+A_1)T\exp(M_2T) := A_2 \le M. \tag{2.16}$$

Hence $\nabla u \in L^\infty(0,T;L^2(\Omega))$.

Differentiating (2.5) with respect to t and multiplying it by u_t, and integrating over Ω, using the Young inequality, Gagliardo-Nirenberg inequality, Sobolev embedding theorem, and integration by parts whenever necessary, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t^2 \mathrm{d}x + \int_\Omega u_t^2 \mathrm{d}x + \int_\Omega \theta|\nabla u_t|^2 \mathrm{d}x \le M_3\int_\Omega u_t^2 \mathrm{d}x + \int_\Omega (u^2+|\nabla u|^2)\mathrm{d}x \\ + K_3(\int_\Omega (|n_t|^2+|\nabla n|^2)\mathrm{d}x,$$

where M_3 depends on $\theta,\mathrm{div}E$ and ϵ_7, K_3 depends on E,E_t and ϵ_7. By using Gronwall inequality, by choosing T small enough, we have

$$\int_\Omega u_t^2 \mathrm{d}x \le (K_3T(M+A_2))\exp(M_3T) := A_3 \le M. \tag{2.17}$$

Therefore $u_t \in L^\infty(0,T;L^2(\Omega))$.

Multiplying (2.5) by $\triangle u$ and integrating it over Ω, using the Young inequality, Sobolev embedding theorem, and Gagliardo-Nirenberg inequality, we get

$$\int_\Omega \theta(\triangle u)^2 \mathrm{d}x \le 6\epsilon_8 \int_\Omega (\triangle u)^2 \mathrm{d}x + \frac{1}{4\epsilon_8}\int_\Omega [u_t^2+u^2+u^2\theta^2+(E\cdot\nabla n)^2+(\mathrm{div}Eu)]\mathrm{d}x \\ + \frac{c(M)m(\epsilon_9)}{4\epsilon_8} + \frac{c(M)\epsilon_9}{4\epsilon_8}\int_\Omega (\triangle u)^2 \mathrm{d}x. \tag{2.18}$$

Hence $\triangle u \in L^\infty(0,T;L^2(\Omega))$.

Combining (2.15) with (2.16)- (2.18) yields that

$$\sup_{0\le t\le T}(\|u\|_{H^2}^2 + \|u_t\|_{L^2}^2) \le K_4 \le M. \qquad \square$$

Lemma 2.2. *Assume that* $C(x)\in L^\infty(\Omega)$, $n_0(x)\in H^2(\Omega)$ *with* $n_0(x)\geq 2D$, *then exists a* $T>0$ *such that the map* $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ *is a contraction mapping with metric* (2.7).

Proof. For given $n_1(x,t)$ and $n_2(x,t)$, suppose (u_1,θ_1,E_1) and (u_2,θ_2,E_2) are the solutions to the Eqs. (2.19)-(2.21) respectively.

$$u_t-\mathrm{div}(\nabla(u\theta)-uE)=0, \tag{2.19}$$

$$-\mathrm{div}(n\nabla\theta)=n(1-\theta), \tag{2.20}$$

$$\mathrm{div}E=n-C(x). \tag{2.21}$$

Let $\delta n=n_1-n_2,\delta u=u_1-u_2,\delta\theta=\theta_1-\theta_2,\delta E=E_1-E_2$, we have

$$(\delta u)_t-\mathrm{div}(\nabla(u_1\delta\theta+\delta u\theta_2)-u_1\delta E-\delta uE_2)=0, \tag{2.22}$$

$$-\mathrm{div}(n_1\nabla(\delta\theta)+\delta n\nabla\theta_2)=\delta n-n_1\delta\theta-\delta n\theta_2, \tag{2.23}$$

$$\mathrm{div}(\delta E)=\delta n, \tag{2.24}$$

with the initial boundary condition

$$\delta u(x,0)=0,\quad \delta\theta(x,0)=0, \tag{2.25}$$

$$\nabla(\delta u)\cdot\overrightarrow{n}|_{\partial\Omega}=0,\ \nabla(\delta\theta)\cdot\overrightarrow{n}|_{\partial\Omega}=0,\ \delta E\cdot\overrightarrow{n}|_{\partial\Omega}=0. \tag{2.26}$$

Since $n,u\in\mathcal{S}$, with the help of Sobolev embedding theorem, we obtain

$$\sup_{0\leq t\leq T}(u,E,\mathrm{div}E,\theta,\nabla\theta)\leq M.$$

We can use (2.24) and $(\delta E)(0,t)=0$ to get that

$$\int_\Omega|\delta E|^2\mathrm{d}x,\int_\Omega|\mathrm{div}\delta E|^2\mathrm{d}x\leq\int_\Omega(\delta n)^2\mathrm{d}x.$$

Multiplying (2.23) by $\delta\theta$, integrating it over Ω, by the boundary condition we obtain

$$\int_\Omega n_1(\delta\theta)^2\mathrm{d}x+\int_\Omega n_1|\nabla(\delta\theta)|^2\mathrm{d}x=\int_\Omega\delta n\delta\theta\mathrm{d}x-\int_\Omega\delta n\theta_2\delta\theta\mathrm{d}x+\int_\Omega\delta n\nabla(\delta\theta)\cdot\nabla\theta_2\mathrm{d}x. \tag{2.27}$$

Notice that $\theta\in L^\infty(0,T;H^3(\Omega))$. By using Young inequality, we have

$$\begin{aligned}\underline{n_1}\int_\Omega(\delta\theta)^2\mathrm{d}x+\underline{n_1}\int_\Omega|\nabla(\delta\theta)|^2\mathrm{d}x\leq&\epsilon_{10}\int_\Omega(\delta\theta)^2\mathrm{d}x+\frac{1}{4\epsilon_{10}}\int_\Omega(\delta n)^2\mathrm{d}x+\epsilon_{10}M\int_\Omega(\delta\theta)^2\mathrm{d}x\\&+\frac{M}{4\epsilon_{10}}\int_\Omega(\delta n)^2\mathrm{d}x+\epsilon_{10}M\int_\Omega|\nabla\delta\theta|^2\mathrm{d}x\\&+\frac{M}{4\epsilon_{10}}\int_\Omega(\delta n)^2\mathrm{d}x.\end{aligned}$$

Consequently, we obtain that

$$\int_\Omega (\delta\theta)^2 \mathrm{d}x + \int_\Omega |\nabla(\delta\theta)|^2 \mathrm{d}x \le \int_\Omega (\delta n)^2 \mathrm{d}x. \tag{2.28}$$

Multiplying (2.22) by δu, integrating it over Ω, by the boundary condition we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\delta u)^2 \mathrm{d}x + \int_\Omega \theta_2 |\nabla \delta u|^2 \mathrm{d}x = &-\int_\Omega u_1 \nabla\delta u \cdot \nabla\delta\theta \mathrm{d}x - \int_\Omega \delta\theta \nabla\delta u \cdot \nabla u_1 \mathrm{d}x \\ &-\int_\Omega \delta u \nabla\delta u \cdot \nabla\theta_2 \mathrm{d}x + \int_\Omega u_1 \nabla\delta u \cdot \delta E \mathrm{d}x \\ &+\int_\Omega \delta\theta \nabla\delta u \cdot E_2 \mathrm{d}x.\end{aligned}$$

By using Young inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\delta u)^2 \mathrm{d}x + \int_\Omega |\nabla\delta u|^2 \mathrm{d}x \le& K\int_\Omega |\delta u|^2 \mathrm{d}x + \epsilon_{11}\int_\Omega (|\delta E|^2 + |\nabla\delta\theta|^2 + (\delta\theta)^2)\mathrm{d}x \\ \le& K\int_\Omega |\delta u|^2 \mathrm{d}x + 3\epsilon_{11}\int_\Omega |\delta n|^2 \mathrm{d}x. \end{aligned} \tag{2.29}$$

By using Gronwall inequality, we obtain

$$\int_\Omega (\delta u)^2 \mathrm{d}x \le \epsilon_{11}\exp(KT)\int_0^T\int_\Omega |\delta n|^2 \mathrm{d}x\mathrm{d}t. \tag{2.30}$$

Integrating (2.30) over $[0,T]$, we obtain

$$|||\delta u||| \le \epsilon_{11}(1 + KT\exp KT)|||\delta n|||. \tag{2.31}$$

Thus we are able to choose T and ϵ_{11} suitable small, such that $\epsilon_{11}(1+KT\exp KT) \le \frac{1}{2}$. Consequently, the map $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is contractive. □

Proof of Theorem 1.2. By Banach Fixed Point Theorem and with the help of Lemma 2.1 and Lemma 2.2, we can show that for a small $T>0$, there exists exactly one fixed point n with $n=\mathcal{F}(n)$ in $\mathcal{S}$, and the fixed point is the unique solution of (1.4)-(1.6).

3 Asymptotic behavior of smooth solution

In this section, we will study the asymptotic behavior of smooth solution by Gagliardo-Nirenberg inequality. Let (n,θ,E) be a solution to (1.4)-(1.6), and set $\rho = n - \mathcal{N}, \vartheta = \theta - \theta_L, \psi = E - \mathcal{E}$, where $\theta_L = 1$.

Lemma 3.1. *There exist positive constants $\delta > 0$ and $\alpha > 0$ such that for any $T > 0$, if*

$$\sup_{0\le t\le T} (\|\rho(x,t)\|_{H^2}) \le \delta, \tag{3.1}$$

and

$$\overline{C}-\underline{C}\leq\delta,$$

then

$$\|\rho(x,t)\|_{H^2}^2\leq C\|\rho(0,t)\|_{H^2}^2\exp(-\alpha t) \tag{3.2}$$

for any $t\in[0,T]$.

Proof. Since $\rho=n-\mathcal{N}$, $\vartheta=\theta-\theta_L$, $\psi=E-\mathcal{E}$,

$$n=\rho+\mathcal{N},\ \theta=\vartheta+1,\ E=\psi+\mathcal{E}.$$

Imbedding into (1.4)-(1.6), we have

$$\rho_t-\mathrm{div}(\nabla[(\rho+\mathcal{N})(\vartheta+1)]-(\rho+\mathcal{N})(\psi+\mathcal{E}))=0, \tag{3.3}$$

$$\mathrm{div}[(\rho+\mathcal{N})\cdot\nabla\vartheta]=(\rho+\mathcal{N})\vartheta, \tag{3.4}$$

$$\mathrm{div}\psi=\rho, \tag{3.5}$$

and the boundary condition

$$\nabla\rho\cdot\overrightarrow{n}|_{\partial\Omega}=0,\ \psi\cdot\overrightarrow{n}|_{\partial\Omega}=0,\ \nabla\vartheta\cdot\overrightarrow{n}|_{\partial\Omega}=0. \tag{3.6}$$

By using (3.1) and (3.5), combining Theorem 1.1, we obtain

$$\int_\Omega|\psi|^2\mathrm{d}x+\int_\Omega|\mathrm{div}\psi|^2\mathrm{d}x\leq\int_\Omega\rho^2\mathrm{d}x. \tag{3.7}$$

Multiplying (3.4) by ϑ and integrating it over Ω, by the boundary condition and integration by parts whenever necessary, we get

$$\int_\Omega\mathcal{N}|\nabla\vartheta|^2\mathrm{d}x+\int_\Omega\mathcal{N}\vartheta^2\mathrm{d}x=-\int_\Omega\rho\vartheta^2\mathrm{d}x-\int_\Omega\rho|\nabla\vartheta|^2\mathrm{d}x.$$

Using Young inequality, Theorem 1.1 and (3.1), we have

$$\int_\Omega\mathcal{N}|\nabla\vartheta|^2\mathrm{d}x+\int_\Omega\mathcal{N}\vartheta^2\mathrm{d}x\leq\mathcal{O}(\delta)\int_\Omega\rho^2\mathrm{d}x. \tag{3.8}$$

Multiplying (3.4) by $\triangle\vartheta$, integrating it over Ω, we get

$$\begin{aligned}\int_\Omega\mathcal{N}|\triangle\vartheta|^2\mathrm{d}x=&\int_\Omega\mathcal{N}\vartheta\triangle\vartheta\mathrm{d}x+\int_\Omega\rho\vartheta\triangle\vartheta\mathrm{d}x-\int_\Omega\rho|\triangle\vartheta|^2\mathrm{d}x-\int_\Omega\triangle\vartheta\nabla\vartheta\cdot\nabla\rho\mathrm{d}x\\&-\int_\Omega\triangle\vartheta\nabla\vartheta\cdot\nabla\mathcal{N}\mathrm{d}x.\end{aligned}$$

For $\int_\Omega\mathcal{N}\vartheta\triangle\vartheta\mathrm{d}x$, we have

$$\int_\Omega\mathcal{N}\vartheta\triangle\vartheta\mathrm{d}x\leq\epsilon\int_\Omega|\triangle\vartheta|^2\mathrm{d}x+\frac{1}{4\epsilon}\int_\Omega\mathcal{N}^2\vartheta^2\mathrm{d}x\leq\epsilon\int_\Omega|\triangle\vartheta|^2\mathrm{d}x+\frac{\overline{C}^2}{4\epsilon}\int_\Omega\vartheta^2\mathrm{d}x.$$

Similar to $\int_\Omega \mathcal{N}\vartheta\triangle\vartheta \mathrm{d}x$, we have

$$\int_\Omega \rho\vartheta\triangle\vartheta\mathrm{d}x \le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{\delta^2}{4\epsilon}\int_\Omega \vartheta^2\mathrm{d}x.$$

For $\int_\Omega \triangle\vartheta\nabla\vartheta\cdot\nabla\rho\mathrm{d}x$,

$$\begin{aligned}\int_\Omega \triangle\vartheta\nabla\vartheta\cdot\nabla\rho\mathrm{d}x &\le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |\nabla\vartheta\cdot\nabla\rho|^2\mathrm{d}x\\ &\le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |\nabla\vartheta|^2|\nabla\rho|^2\mathrm{d}x\\ &\le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{1}{4\epsilon}\Big(\int_\Omega |\nabla\vartheta|^3\mathrm{d}x\Big)^{\frac{2}{3}}\Big(\int_\Omega |\nabla\rho|^6\mathrm{d}x\Big)^{\frac{1}{3}}\\ &\le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |\nabla\vartheta|^2\mathrm{d}x\int_\Omega |\nabla\rho|^2\mathrm{d}x\\ &\le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{\delta}{4\epsilon}\int_\Omega |\nabla\vartheta|^2\mathrm{d}x.\end{aligned}$$

Here we have used Young inequality, Hölder inequality and $L^p \hookrightarrow L^q (p<q)$.

Similar to $\int_\Omega \triangle\vartheta\nabla\vartheta\cdot\nabla\rho\mathrm{d}x$, we have

$$\begin{aligned}\int_\Omega \triangle\vartheta\nabla\vartheta\cdot\nabla\mathcal{N}\mathrm{d}x &\le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |\nabla\vartheta|^2\mathrm{d}x\int_\Omega |\nabla\mathcal{N}|^2\mathrm{d}x\\ &\le \epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{C}{4\epsilon}\int_\Omega |\nabla\vartheta|^2\mathrm{d}x,\end{aligned}$$

here C depends on Ω and $\overline{C}$. Therefore, we obtain

$$\begin{aligned}\int_\Omega \mathcal{N}|\triangle\vartheta|^2\mathrm{d}x \le & 4\epsilon\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \delta\int_\Omega |\triangle\vartheta|^2\mathrm{d}x + \frac{\overline{C}^2}{4\epsilon}\int_\Omega \vartheta^2\mathrm{d}x + \frac{\delta^2}{4\epsilon}\int_\Omega \vartheta^2\mathrm{d}x\\ &+ \frac{\delta}{4\epsilon}\int_\Omega |\nabla\vartheta|^2\mathrm{d}x + \frac{C}{4\epsilon}\int_\Omega |\nabla\vartheta|^2\mathrm{d}x.\end{aligned}$$

By using (3.8),

$$\int_\Omega |\triangle\vartheta|^2\mathrm{d}x \le \mathcal{O}(\delta)\int_\Omega \rho^2\mathrm{d}x. \tag{3.9}$$

Differentiating (3.4) with respect to x and multiplying it by $\nabla\triangle\vartheta$, and integrating over Ω,

$$\begin{aligned}\int_\Omega \mathcal{N}|\nabla\triangle\vartheta|^2\mathrm{d}x =& -\int_\Omega \rho|\nabla\triangle\vartheta|^2\mathrm{d}x - \int_\Omega \triangle\vartheta\nabla(\rho+\mathcal{N})\cdot\nabla\triangle\vartheta\mathrm{d}x\\ &- \int_\Omega \nabla(\nabla(\rho+\mathcal{N})\cdot\nabla\vartheta)\cdot\nabla\triangle\vartheta\mathrm{d}x + \int_\Omega \nabla((\rho+\mathcal{N})\vartheta)\cdot\nabla\triangle\vartheta\mathrm{d}x.\end{aligned}$$

For $\int_\Omega \triangle\vartheta\nabla(\rho+\mathcal{N})\cdot\nabla\triangle\vartheta dx$, using Young inequality and Hölder inequality,

$$\begin{aligned}\int_\Omega \triangle\vartheta\nabla(\rho+\mathcal{N})\cdot\nabla\triangle\vartheta dx &\le \epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{1}{4\epsilon}\int_\Omega |\triangle\vartheta|^2|\nabla(\rho+\mathcal{N})|^2 dx\\ &\le \epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{1}{4\epsilon}\left(\int_\Omega |\triangle\vartheta|^3 dx\right)^{\frac{2}{3}}\left(\int_\Omega |\nabla(\rho+\mathcal{N})|^6 dx\right)^{\frac{1}{3}}\\ &\le \epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{1}{4\epsilon}\int_\Omega |\triangle\vartheta|^2 dx\int_\Omega |\nabla(\rho+\mathcal{N})|^2 dx\\ &\le \epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{C}{4\epsilon}\int_\Omega |\triangle\vartheta|^2 dx.\end{aligned}$$

Here C depends on Ω, $|\nabla\mathcal{N}|$ and δ.

For $-\int_\Omega \nabla(\nabla(\rho+\mathcal{N})\cdot\nabla\vartheta)\cdot\nabla\triangle\vartheta dx$, we have

$$\begin{aligned}&-\int_\Omega \nabla(\nabla(\rho+\mathcal{N})\cdot\nabla\vartheta)\cdot\nabla\triangle\vartheta dx\\ \le&\epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{1}{4\epsilon}\int_\Omega |\nabla(\nabla(\rho+\mathcal{N})\cdot\nabla\vartheta)|^2 dx\\ \le&\epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{1}{4\epsilon}\left(\int_\Omega |\triangle(\rho+\mathcal{N})|^2|\nabla\vartheta|^2 dx+\int_\Omega |\nabla(\rho+\mathcal{N})|^2|\triangle\vartheta|^2 dx\right)\\ \le&\epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{1}{4\epsilon}\left(\int_\Omega |\triangle(\rho+\mathcal{N})|^2 dx\int_\Omega |\nabla\vartheta|^2 dx+\int_\Omega |\nabla(\rho+\mathcal{N})|^2 dx\int_\Omega |\triangle\vartheta|^2 dx\right)\\ \le&\epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{C}{4\epsilon}\left(\int_\Omega |\nabla\vartheta|^2 dx+\int_\Omega |\triangle\vartheta|^2 dx\right).\end{aligned}$$

Here C depends on Ω, $|\nabla\mathcal{N}|$, $|\triangle\mathcal{N}|$ and δ. Similar to $\int_\Omega \triangle\vartheta\nabla(\rho+\mathcal{N})\cdot\nabla\triangle\vartheta dx$, we have

$$\begin{aligned}\int_\Omega \nabla((\rho+\mathcal{N})\vartheta)\cdot\nabla\triangle\vartheta dx &\le \epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{1}{4\epsilon}\int_\Omega |\nabla((\rho+\mathcal{N})\vartheta)|^2 dx\\ &\le \epsilon\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{C}{4\epsilon}\left(\int_\Omega \vartheta^2 dx+\int_\Omega |\nabla\vartheta|^2 dx\right).\end{aligned}$$

Therefore, we obtain

$$\int_\Omega \mathcal{N}|\nabla\triangle\vartheta|^2 dx\le(3\epsilon+\delta)\int_\Omega |\nabla\triangle\vartheta|^2 dx+\frac{C}{4\epsilon}\left(\int_\Omega \vartheta^2 dx+\int_\Omega |\nabla\vartheta|^2 dx+\int_\Omega |\triangle\vartheta|^2 dx\right).$$

By using (3.8) and (3.9), we have

$$\int_\Omega |\nabla\triangle\vartheta|^2 dx\le\mathcal{O}(\delta)\int_\Omega \rho^2 dx. \tag{3.10}$$

Together (3.10) with (3.8), (3.9), we observe that

$$\int_\Omega \vartheta^2+|\nabla\vartheta|^2+|\triangle\vartheta|^2+|\nabla\triangle\vartheta|^2 dx\le\mathcal{O}(\delta)\int_\Omega \rho^2 dx. \tag{3.11}$$

So $\|\vartheta\|_{L^\infty}<\delta$.

Multiplying (3.3) by ρ, integrating it over Ω, by the boundary condition and integration by parts whenever necessary, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho^2\mathrm{d}x+\int_\Omega|\nabla\rho|^2\mathrm{d}x+\int_\Omega\mathcal{N}\rho^2\mathrm{d}x\\=&\int_\Omega\rho\nabla\rho\cdot(\psi+\mathcal{E}-\nabla\vartheta)\mathrm{d}x-\int_\Omega\vartheta|\nabla\rho|^2\mathrm{d}x-\int_\Omega\rho\nabla\mathcal{N}\cdot\psi\mathrm{d}x-\int_\Omega\vartheta\nabla\rho\cdot\nabla\mathcal{N}\mathrm{d}x-\int_\Omega\mathcal{N}\nabla\rho\cdot\nabla\vartheta\mathrm{d}x.\end{aligned}$$

For $\int_\Omega\vartheta|\nabla\rho|^2\mathrm{d}x$,

$$\int_\Omega\vartheta|\nabla\rho|^2\mathrm{d}x\le\delta\int_\Omega|\nabla\rho|^2\mathrm{d}x.$$

For $\int_\Omega\nabla\rho\cdot\nabla\mathcal{N}\vartheta\mathrm{d}x$, since $|\nabla\mathcal{N}|\le a_0(\overline{C}-\underline{C})\le a_0\delta$,

$$\begin{aligned}\int_\Omega\nabla\rho\cdot\nabla\mathcal{N}\vartheta\mathrm{d}x\le&\epsilon\int_\Omega|\nabla\rho|^2\mathrm{d}x+\frac{1}{4\epsilon}\int_\Omega|\nabla\mathcal{N}\vartheta|^2\mathrm{d}x\\\le&\epsilon\int_\Omega|\nabla\rho|^2\mathrm{d}x+\frac{a_0^2(\overline{C}-\underline{C})^2}{4\epsilon}\int_\Omega\vartheta^2\mathrm{d}x.\end{aligned}$$

For $\int_\Omega\rho\nabla\mathcal{N}\cdot\psi\mathrm{d}x$, similar to above, we have

$$\begin{aligned}\int_\Omega\rho\nabla\mathcal{N}\cdot\psi\mathrm{d}x\le&\epsilon\int_\Omega\rho^2\mathrm{d}x+\frac{1}{4\epsilon}\int_\Omega|\nabla\mathcal{N}\psi|^2\mathrm{d}x\\\le&\epsilon\int_\Omega\rho^2\mathrm{d}x+\frac{a_0^2(\overline{C}-\underline{C})^2}{4\epsilon}\int_\Omega\psi^2\mathrm{d}x.\end{aligned}$$

For $\int_\Omega\mathcal{N}\nabla\rho\cdot\nabla\vartheta\mathrm{d}x$, using Young inequality,

$$\begin{aligned}\int_\Omega\mathcal{N}\nabla\rho\cdot\nabla\vartheta\mathrm{d}x\le&\epsilon\int_\Omega|\nabla\rho|^2\mathrm{d}x+\frac{1}{4\epsilon}\int_\Omega\mathcal{N}^2|\nabla\vartheta|^2\mathrm{d}x\\\le&\epsilon\int_\Omega|\nabla\rho|^2\mathrm{d}x+\frac{\overline{C}^2}{4\epsilon}\int_\Omega|\nabla\vartheta|^2\mathrm{d}x.\end{aligned}$$

Since $\sup_{0\le t\le T}(\|\rho(x,t)\|_{H^2})\le\delta$, using Young inequality and Theorem 1.1, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho^2\mathrm{d}x+\int_\Omega|\nabla\rho|^2\mathrm{d}x+\int_\Omega\mathcal{N}\rho^2\mathrm{d}x\le&5\epsilon\int_\Omega|\nabla\rho|^2\mathrm{d}x+\mathcal{O}(\delta)\int_\Omega|\nabla\rho|^2\mathrm{d}x\\&+\frac{C}{4\epsilon}\int_\Omega|\psi|^2+\vartheta^2+|\nabla\vartheta|^2\mathrm{d}x.\end{aligned}$$

Here C depends on Ω ,$|\nabla\mathcal{N}|$ and $|\mathcal{E}|$. From (3.11) and (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho^2\mathrm{d}x+\int_\Omega|\nabla\rho|^2\mathrm{d}x+\int_\Omega\rho^2\mathrm{d}x\le\mathcal{O}(\delta)\int_\Omega\rho^2+|\nabla\rho|^2\mathrm{d}x. \tag{3.12}$$

Multiplying (3.3) by $-\triangle\rho$, integrating it over Ω, by the boundary condition and integration by parts whenever necessary, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\rho|^2\mathrm{d}x+\int_\Omega\mathcal{N}|\nabla\rho|^2\mathrm{d}x+\int_\Omega|\triangle\rho|^2\mathrm{d}x\\=&\int_\Omega\vartheta|\triangle\rho|^2\mathrm{d}x-\int_\Omega\rho\nabla\mathcal{N}\cdot\nabla\rho\mathrm{d}x+\int_\Omega\triangle\rho(2\nabla\vartheta\cdot\nabla\rho+\rho\triangle\vartheta+\vartheta\triangle\mathcal{N}+2\nabla\mathcal{N}\cdot\nabla\vartheta\\&\quad+\mathcal{N}\triangle\vartheta+\rho\mathrm{div}\mathcal{E}+\nabla\rho\cdot\mathcal{E}+\rho^2+\nabla\rho\cdot\psi+\psi\cdot\nabla\mathcal{N})\mathrm{d}x\end{aligned}$$

For $\int_\Omega\rho\nabla\mathcal{N}\cdot\nabla\rho\mathrm{d}x$, since $|\nabla\mathcal{N}|\le a_0(\overline{C}-\underline{C})\le a_0\delta$,

$$\begin{aligned}\int_\Omega\rho\nabla\mathcal{N}\cdot\nabla\rho\mathrm{d}x\le&\epsilon\int_\Omega|\nabla\rho|^2\mathrm{d}x+\frac{1}{4\epsilon}\int_\Omega\rho^2|\nabla\mathcal{N}|^2\mathrm{d}x\\\le&\epsilon\int_\Omega|\nabla\rho|^2\mathrm{d}x+\frac{a_0^2\delta^2}{4\epsilon}\int_\Omega|\nabla\vartheta|^2\mathrm{d}x.\end{aligned}$$

Similar to $\int_\Omega\rho\nabla\mathcal{N}\cdot\nabla\rho\mathrm{d}x$, we have

$$\begin{aligned}&\int_\Omega\triangle\rho\vartheta\triangle\mathcal{N}\mathrm{d}x\le\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{a_0^2\delta^2}{4\epsilon}\int_\Omega\vartheta^2\mathrm{d}x;\\&\int_\Omega\triangle\rho\nabla\vartheta\cdot\nabla\mathcal{N}\mathrm{d}x\le\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{a_0^2\delta^2}{4\epsilon}\int_\Omega|\nabla\vartheta|^2\mathrm{d}x;\\&\int_\Omega\triangle\rho\nabla\vartheta\cdot\psi\mathrm{d}x\le\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{a_0^2\delta^2}{4\epsilon}\int_\Omega|\psi|^2\mathrm{d}x;\\&\int_\Omega\triangle\rho\rho\mathrm{div}\mathcal{E}\mathrm{d}x\le\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{a_0^2\delta^2}{4\epsilon}\int_\Omega\rho^2\mathrm{d}x;\\&\int_\Omega\triangle\rho\nabla\rho\cdot\mathcal{E}\mathrm{d}x\le\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{a_0^2\delta^2}{4\epsilon}\int_\Omega|\nabla\rho|^2\mathrm{d}x.\end{aligned}$$

For $\int_\Omega\triangle\rho\nabla\vartheta\cdot\nabla\rho\mathrm{d}x$, we have

$$\begin{aligned}\int_\Omega\triangle\rho\nabla\vartheta\cdot\nabla\rho\mathrm{d}x\le&\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{1}{4\epsilon}\int_\Omega|\nabla\vartheta|^2|\nabla\rho|^2\mathrm{d}x,\\\le&\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{\delta^2}{4\epsilon}\int_\Omega|\nabla\rho|^2\mathrm{d}x.\end{aligned}$$

Similar to $\int_\Omega\triangle\rho\nabla\vartheta\cdot\nabla\rho\mathrm{d}x$,

$$\begin{aligned}&\int_\Omega\triangle\rho\rho\triangle\vartheta\mathrm{d}x\le\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{\delta^2}{4\epsilon}\int_\Omega|\triangle\vartheta|^2\mathrm{d}x;\\&\int_\Omega\triangle\rho\mathcal{N}\triangle\vartheta\mathrm{d}x\le\epsilon\int_\Omega|\triangle\rho|^2\mathrm{d}x+\frac{\overline{C}^2}{4\epsilon}\int_\Omega|\triangle\vartheta|^2\mathrm{d}x.\end{aligned}$$

For $\int_\Omega \triangle\rho\nabla\rho\cdot\psi \mathrm{d}x$, using Young inequality and $L^2 \hookrightarrow L^3$, we have

$$\begin{aligned}\int_\Omega \triangle\rho\nabla\rho\cdot\psi \mathrm{d}x \le& \epsilon\int_\Omega |\triangle\rho|^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |\nabla\rho|^2|\psi|^2\mathrm{d}x\\ \le& \epsilon\int_\Omega |\triangle\rho|^2\mathrm{d}x + \frac{1}{4\epsilon}\Big(\int_\Omega |\nabla\rho|^3\mathrm{d}x\Big)^{\frac{2}{3}}\Big(\int_\Omega |\psi|^6\mathrm{d}x\Big)^{\frac{1}{3}}\\ \le& \epsilon\int_\Omega |\triangle\rho|^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega |\nabla\rho|^2\mathrm{d}x\int_\Omega |\psi|^2\mathrm{d}x\\ \le& \epsilon\int_\Omega |\triangle\rho|^2\mathrm{d}x + \frac{\delta^2}{4\epsilon}\int_\Omega |\psi|^2\mathrm{d}x.\end{aligned}$$

So we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\rho|^2\mathrm{d}x + \int_\Omega |\nabla\rho|^2\mathrm{d}x + \int_\Omega |\triangle\rho|^2\mathrm{d}x\\ \le& \mathcal{O}(\delta)\int_\Omega |\triangle\rho|^2 + |\nabla\rho|^2 + \rho^2 + \vartheta^2 + |\nabla\vartheta|^2 + |\triangle\vartheta|^2 + |\psi|^2\mathrm{d}x\\ \le& \mathcal{O}(\delta)\int_\Omega |\triangle\rho|^2 + |\nabla\rho|^2 + \rho^2\mathrm{d}x. \qquad (3.13)\end{aligned}$$

Differentiating (3.4) with respect to t and multiplying it by ϑ_t, and integrating over Ω, by the boundary condition and integration by parts whenever necessary, we get

$$\int_\Omega \mathcal{N}\vartheta_t^2\mathrm{d}x + \int_\Omega \mathcal{N}|\nabla\vartheta_t|^2\mathrm{d}x = -\int_\Omega \rho\vartheta_t^2\mathrm{d}x - \int_\Omega \rho|\nabla\vartheta_t|^2\mathrm{d}x - \int_\Omega \rho_t\nabla\vartheta\cdot\nabla\vartheta_t\mathrm{d}x - \int_\Omega \rho_t\vartheta\vartheta_t\mathrm{d}x.$$

Since $\vartheta \le \delta$ and $|\nabla\vartheta| \le \delta$,

$$\begin{aligned}&\int_\Omega \rho_t\nabla\vartheta\cdot\nabla\vartheta_t\mathrm{d}x \le \epsilon\int_\Omega |\nabla\vartheta_t|^2\mathrm{d}x + \frac{\delta^2}{4\epsilon}\int_\Omega |\rho_t|^2\mathrm{d}x;\\ &\int_\Omega \rho_t\vartheta\vartheta_t\mathrm{d}x \le \epsilon\int_\Omega \vartheta_t^2\mathrm{d}x + \frac{\delta^2}{4\epsilon}\int_\Omega |\rho_t|^2\mathrm{d}x.\end{aligned}$$

Therefore, we obtain that

$$\int_\Omega \vartheta_t^2\mathrm{d}x + \int_\Omega |\nabla\vartheta_t|^2\mathrm{d}x \le \mathcal{O}(\delta)\int_\Omega \rho_t^2\mathrm{d}x. \qquad (3.14)$$

We rewrite Eq. (3.3) in the following form,

$$\rho_t - \mathrm{div}(\nabla\rho + \nabla\mathcal{N} + \vartheta\nabla\rho + \vartheta\nabla\mathcal{N} + \rho\nabla\vartheta + \mathcal{N}\nabla\vartheta - \rho\psi - \rho\mathcal{E} - \mathcal{N}\psi - \mathcal{N}\mathcal{E}) = 0. \qquad (3.15)$$

Differentiating (3.15) with respect to t and multiplying it by ρ_t, and integrating over Ω,

$$\begin{aligned}&\int_\Omega \rho_t\rho_{tt}\mathrm{d}x - \int_\Omega \rho_t\mathrm{div}(\nabla\rho + \nabla\mathcal{N} + \vartheta\nabla\rho + \vartheta\nabla\mathcal{N} + \rho\nabla\vartheta + \mathcal{N}\nabla\vartheta - \rho\psi - \rho\mathcal{E}\\ &\quad - \mathcal{N}\psi - \mathcal{N}\mathcal{E})_t\mathrm{d}x = 0.\end{aligned}$$

Using the boundary condition and integration by parts whenever necessary, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_t^2 \mathrm{d}x+\int_\Omega |\nabla\rho_t|^2\mathrm{d}x+\int_\Omega \mathcal{N}\rho_t^2\mathrm{d}x\\ =&\int_\Omega \rho_t\nabla\rho_t\cdot(\psi+\mathcal{E})\mathrm{d}x-\int_\Omega\rho_t\nabla\rho\cdot\psi_t\mathrm{d}x-\int_\Omega\rho\rho_t^2\mathrm{d}x-\int_\Omega\rho_t\nabla\mathcal{N}\cdot\psi_t\mathrm{d}x\\ &-\int_\Omega\vartheta_t\nabla\rho_t\cdot(\nabla\rho+\nabla\mathcal{N})\mathrm{d}x-\int_\Omega\rho_t\nabla\rho_t\cdot\nabla\vartheta\mathrm{d}x-\int_\Omega(\rho+\mathcal{N})\nabla\rho_t\cdot\nabla\vartheta_t\mathrm{d}x.\end{aligned}$$

Using Young inequality and Theorem 1.1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho_t^2\mathrm{d}x+\int_\Omega|\nabla\rho_t|^2\mathrm{d}x+\int_\Omega\rho_t^2\mathrm{d}x\le\mathcal{O}(\delta)\int_\Omega\rho_t^2+\rho^2+|\nabla\rho|^2\mathrm{d}x. \tag{3.16}$$

Multiplying (3.15) by $\triangle\rho$, integrating it over Ω, we get

$$\begin{aligned}\int_\Omega(\vartheta+1)|\triangle\rho|^2\mathrm{d}x=&\int_\Omega\rho_t\triangle\rho\mathrm{d}x+\int_\Omega\rho^2\triangle\rho\mathrm{d}x+\int_\Omega\rho(\mathcal{N}+\mathrm{div}\mathcal{E}-\triangle\vartheta)\triangle\rho\mathrm{d}x\\ &+\int_\Omega\triangle\rho\nabla\rho\cdot(\psi+\mathcal{E}-2\nabla\vartheta)\mathrm{d}x+\int_\Omega\triangle\rho\nabla\mathcal{N}\cdot\psi\mathrm{d}x\\ &+\int_\Omega(\mathcal{E}-2\nabla\vartheta)\cdot\nabla\mathcal{N}\triangle\rho\mathrm{d}x+\int_\Omega\mathcal{N}(\triangle\vartheta+\mathrm{div}\mathcal{E})\triangle\rho\mathrm{d}x\\ &-\int_\Omega(1+\vartheta)\triangle\mathcal{N}\triangle\rho\mathrm{d}x.\end{aligned}$$

Using Young inequality and Theorem 1.1, we obtain

$$\int_\Omega|\triangle\rho|^2\mathrm{d}x\le\mathcal{O}(1)\int_\Omega\rho_t^2+\rho^2+|\nabla\rho|^2\mathrm{d}x. \tag{3.17}$$

Combining (3.12)-(3.16) with (3.17) together, and choosing δ small enough, we obtain the following estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\rho^2+|\nabla\rho|^2+\rho_t^2)\mathrm{d}x+C_0\int_\Omega(\rho^2+|\nabla\rho|^2+\rho_t^2)\mathrm{d}x\le 0. \tag{3.18}$$

Thus by Gronwall inequality, we get

$$\|\rho(x,t)\|_{H^2}^2\le C_0(\|\rho(x,0)\|_{H^2}^2)\exp(-\alpha t).$$ □

Remark 3.1. By the standard argument, Theorem 1.3 is proved with the help of Theorem 1.2 and Lemma 3.1.

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