JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **30**, No. 1, pp. 76-83

doi: 10.4208/jpde.v30.n1.6
March 2017

A Regularity Criterion in Terms of Direction of Vorticity to the 3D Micropolar-fluid Equations

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Received 1 November 2016; Accepted 6 December 2016

Abstract. In this paper, we study the regularity of weak solutions to the 3D Micropolar-fluid equations. We show that the weak solutions actually is strong solution if the corresponding vorticity field $j=\nabla\times u$ satisfies certain condition in the high vorticity region.

AMS Subject Classifications: 35Q35, 76C99

Chinese Library Classifications: O175.24

Key Words: Microplar-fluid equations; regularity conditions; weak solutions.

1 Introduction

In this paper, we consider the Cauchy problem for the 3D Micropolar-fluid(MP) equations:

$$\begin{cases}\partial_t u+(u\cdot\nabla)u-\Delta u+\nabla p=\frac{1}{2}\nabla\times w,\\ \partial_t w-\Delta w-\nabla(\nabla\cdot w)+w+u\cdot\nabla w=\frac{1}{2}\nabla\times u,\\ \nabla\cdot u=0,\\ u(x,0)=u_0(x),\quad w(x,0)=w_0(x),\end{cases}\tag{1.1}$$

where $u(x,t)$, $w(x,t)$ are the unknown velocity field and micro-rotational velocity field; $p=p(x,t)$ is the unknown scalar pressure. While $u_0(x)$, $w_0(x)$ are the given initial data with $\nabla\cdot u_0=0$ in the sense of distribution.

Micropolar fluid system was firstly developed by Eringen [1,2]. It is a type of fluids which exhibits microrotational effects and microrotational inertia and can be viewed as

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a non-Newtonian fluid. It can describe many phenomena that appear in a large number of complex fluids such as the suspensions, animal blood, and liquid crystals which cannot be characterized appropriately by the Navier-Stokes system and that is important to the scientists working with the hydrodynamic-fluid problems and phenomena. The existences of weak and strong solutions for micropolar fluid equations were treated by Galdi and Rionero [3] and Yamaguchi [4] , respectively. The uniqueness of strong solutions to the micropolar flows and the magnetomicropolar flows either local for large data or global for small data is considered in [5,6] and references therein.

In the present paper, we consider the problem of the regularity of weak solutions to 3D MP equations. We observe that the equations include as a particular case the classical Navier-Stokes(NS) equations, which widely has been studied. There is a large literature on the problem of regularity of weak solutions to the 3D NS equations. It may be superuous to recall all results. To go directly to the main points of the present paper, we only review some known results which are closely related to our main result.

The study of conditions which involving the direction of voriticity and its physical-geometric interpretation, started with Constantin and Fefferman [7], who first derived some exact formulas and employed them in order to prove regularity in the whole space. They pointed out in the 3D case, NS equations exists a stretching mechanism for the vorticity magnitude which is non-linear and potentially capable of producing finite time singularities. They showed that the solution is smooth, if the direction of vorticity is sufficiently well behaved in the region of high vorticity magnitude. It is proved that if the direction field $\tilde{j}(x,t)$ of the vorticity $j(x,t)=\nabla\times u(x,t)$ satisfies that

$$|\tilde{j}(x+y,t)-\tilde{j}(x,t)|\le\rho|y|,\qquad t\in[0,T]. \tag{1.2}$$

for some constant $\rho>0$ when $|j(x,t)|>K$, $|j(x+y,t)|>K$ for some $K>0$; then the solution is smooth.

There are many improved and extended results to (1.2), we can see [9–13] etc. Especially, H Beirão da Veiga [9] proved that the following condition implies regularity:

Assumption H: For some $\beta\in[\frac{1}{2},1]$ and $g\in L^a(0,T;L^b)$, where

$$\frac{2}{a}+\frac{3}{b}=\beta-\frac{1}{2},\qquad a\in\left[\tfrac{4}{2\beta-1},\infty\right],$$

The weak solutions satisfy

$$|\tilde{j}(x+y,t)-\tilde{j}(x,t)|\le g(t,x)|y|^\beta,\qquad t\in[0,T], \tag{1.3}$$

when $|j(x,t)|>K,|j(x+y,t)|>K$ for some $K>0$.

In [13], He and Xin gave a stronger condition than (1.3) to 3D MHD equations.It is worthy to emphasize that extend (1.3) to MHD equations is difficult because of the strong coupled terms of velocity and magnetic field. More precisely, they gave the following regularity condition :

$$|j(x+y,t)-j(x,t)|\leq c|j(x+y,t)||y|^{\frac{1}{2}}, \qquad t\in[0,T]. \tag{1.4}$$

The conditions (1.4) was extended to generalized MHD [14]. In this paper, we improve the (1.4) to establish a condition which involving the direction of the vorticity of velocity to MP equations. More precisely, our major assumption about the vorticity of velocity is

Assumption A: For some $\beta\in[\frac{1}{2},1]$ and $g\in L^a(0,T;L^b)$, where

$$\frac{2}{a}+\frac{3}{b}=\beta-\frac{1}{2}, \qquad a\in\left[\tfrac{4}{2\beta-1},\infty\right].$$

The weak solutions satisfy

$$|j(x+y,t)-j(x,t)|\leq g(t,x)|j(x+y,t)||y|^{\beta}, \qquad t\in[0,T], \tag{1.5}$$

when $|j(x,t)|>K$, $|j(x+y,t)|>K$ for some $K>0$.

Under this assumption, we can show the following a priori estimate

Theorem 1.1. *Let $(u_0,w_0)\in H^1(R^3)$. Assume that (u,w) is a smooth solution of MP equations on some interval $[0,T)$. Then if the assumption A holds on $[0,T)$, one has*

$$j(x,t),\nabla w\in L^\infty(0,T;L^2(R^3))\cap L^2(0,T;H^1(R^3)). \tag{1.6}$$

Remark 1.1. After we have the above a priori estimate, we can easily give a proof about the weak-strong uniqueness follows from the standard continuation principle.

2 Preliminaries

By the Biot-Savart law, the velocity field can be expressed in terms of its vorticity

$$u(x,t)=-\frac{1}{4\pi}\int_{R^3}\nabla\left(\frac{1}{|y|}\right)\times j(x+y,t)\mathrm{d}y. \tag{2.1}$$

The following two integral equations were obtained in [7]:

$$j(x,t)=\frac{1}{4\pi}P.V.\int_{R^3}\sigma(\hat{y})j(x+y,t)\frac{\mathrm{d}y}{|y|^3},$$
$$S(x,t)=\frac{1}{2}[\nabla u+(\nabla u)^T]=\frac{3}{4\pi}P.V.\int_{R^3}M(\hat{y},j(x+y,t))\frac{\mathrm{d}y}{|y|^3},$$

where the matrixes

$$\sigma(\hat{y})=3(\hat{y}\otimes\hat{y})-I, \qquad M(\hat{y},j)=\frac{1}{2}[\hat{y}\otimes(\hat{y}\times j)+(\hat{y}\times j)\times\hat{y}],$$

with $\hat{y}=y/|y|$, I is the identity matrix and the tensor product simply denotes the matrix $(a\otimes b)_{ij}=a_ib_j$. Moreover, the matrix σ is symmetric, traceless and has zero mean on the unit sphere. M is also a symmetric traceless matrix; which mean on the unit sphere is zero when the second variable j is held fixed and M is viewed as a function of $\hat{y}$ alone.

Let K be the number in assumption **A**, we split $j(x,t)$ and $S(x,t)$as

$$j(x,t)=\chi\Big(\frac{|j(x,t)|}{K}\Big)j(x,t)+\Big(1-\chi\Big(\frac{|w(x,t)|}{K}\Big)\Big)j(x,t)=j_1(x,t)+j_2(x,t), \tag{2.2}$$

$$S(x,t)=\chi\Big(\frac{|j(x,t)|}{K}\Big)S(x,t)+\Big(1-\chi\Big(\frac{|j(x,t)|}{K}\Big)\Big)S(x,t)=S_1(x,t)+S_2(x,t), \tag{2.3}$$

where the smooth bump function $\chi(\lambda)\in[0,1]$, is identically equal to 1 for $0\le\lambda\le 1$ and identically equal to 0 for $\lambda\ge 2$ or $\lambda\le -1$.

By the Calderón-Zygmund inequality, we have

$$\|j_i(x,t)\|_p\le\|j(x,t)\|_p, \tag{2.4}$$

$$\|S_i(x,t)\|_p\le\|S(x,t)\|_p, \tag{2.5}$$

$$\|S_i(t)\|_p\le c\|j_i(x,t)\|_p, \tag{2.6}$$

for any $i=1,2$ and $1<p<\infty$.

3 Proof of theorem

By applying that curl operator both sides of the first equation and $\partial_i(i=1,2,3)$ to the second equations of (1.1), we have

$$\begin{cases}\partial_t j-\Delta j+(u\cdot\nabla)j-(j\cdot\nabla)u=\frac{1}{2}(\nabla\times)^2 w,\\ \partial_t\partial_i w-\Delta\partial_i w-\nabla(\nabla\cdot\partial_i w)+\partial_i w+u\cdot\nabla\partial_i w+\partial_i u\cdot\nabla w-\frac{1}{2}\partial_i j=0.\end{cases} \tag{3.1}$$

Multiply the first equation by $j(x,t)$ and the second by $\partial_i w$ of (3.1) respectively, and take integral on the whole space, then we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|j\|_2^2+\|\partial_i w\|_2^2)+\|\nabla j\|_2^2+\|\nabla\partial_i w\|_2^2+\|\nabla\cdot\partial_i w\|_2^2+\|\partial_i w\|_2^2\\ =&\int_{R^3} j\cdot\nabla u\cdot j\mathrm{d}x+\frac{1}{2}\int_{R^3}(\nabla\times)^2 w\cdot j\mathrm{d}x+\frac{1}{2}\int_{R^3}\partial_i j\cdot\partial_i w\mathrm{d}x-\int_{R^3}\partial_i u\cdot\nabla w\cdot\partial_i w\mathrm{d}x\\ =:&I_1+I_2+I_3+I_4.\end{aligned} \tag{3.2}$$

The first term at the right-hand side can be written as

$$I_1=\int_{R^3} S(x,t)j(x,t)\cdot j(x,t)\mathrm{d}x$$

$$
\begin{aligned}
=&\sum_{i=1}^{2}\int_{R^3}\left(\sum_{k=1}^{2}(S_i(x,t)j_1(x,t))\cdot j_k(x,t)+S_i(x,t)j_2(x,t)\cdot j_1(x,t)\right)\mathrm{d}x\\
&\quad+\int_{R^3}S_1(x,t)j_2(x,t))\cdot j_2(x,t)\mathrm{d}x+\int_{R^3}S_2(x,t)j_2(x,t))\cdot j_2(x,t)\mathrm{d}x\\
=:&I_1^{(1)}+I_1^{(2)}+I_1^{(3)}.
\end{aligned}\tag{3.3}
$$

By using the inequalities (2.4)-(2.6) with $q=2$, we have

$$
\left|\int_{R^3}S_i(x,t)j_k(x,t)\cdot j_l(x,t)\mathrm{d}x\right|\le CK\|j(\cdot t)\|_2^2,\tag{3.4}
$$

when $(k,l)\neq(2,2)$; so we have

$$
|I_1^{(1)}|\le CK\|j(\cdot t)\|_2^2.\tag{3.5}
$$

Let us recall the following inequalities

$$
\begin{aligned}
&\|j(\cdot t)\|_4\le C\|j(\cdot t)\|_2^{\frac{1}{4}}\|\nabla j(\cdot t)\|_2^{\frac{3}{4}},\\
&\|j_1(\cdot t)\|_4\le C\|j_1(\cdot t)\|_\infty^{\frac{1}{2}}\|j_1(\cdot t)\|_2^{\frac{1}{2}}\le CK^{\frac{1}{2}}\|j(\cdot t)\|_2^{\frac{1}{2}}.
\end{aligned}
$$

Then we have

$$
|I_1^{(2)}|\le C\|j_1(\cdot t)\|_4\|j(\cdot t)\|_4\|j(\cdot t)\|_2\le CK^{\frac{4}{5}}\|j(\cdot t)\|_2^{\frac{14}{5}}+\frac{1}{6}\|\nabla j(\cdot t)\|_2^2.\tag{3.6}
$$

It follows from the definition of $S(x,t)$

$$
\begin{aligned}
|S_2(x,t)|=&\left|\frac{3}{4\pi}P.V.\int_{R^3}M(\hat{y},j_2(x+y,t))\frac{\mathrm{d}y}{|y|^3}\right|\\
\le&\left|\frac{3}{4\pi}P.V.\int_{|y|\ge\rho}M(\hat{y},j_2(x+y,t))\frac{\mathrm{d}y}{|y|^3}\right|+\left|\frac{3}{4\pi}P.V.\int_{|y|\ge\rho}M(\hat{y},j_2(x+y,t))\frac{\mathrm{d}y}{|y|^3}\right|\\
\le&\left|\frac{3}{4\pi}P.V.\int_{|y|\ge\rho}M(\hat{y},j_2(x+y,t))\frac{\mathrm{d}y}{|y|^3}\right|\\
&\quad+\left|\frac{3}{4\pi}P.V.\int_{|y|\ge\rho}M(\hat{y},j_2(x+y,t)-j_2(x,t))\frac{\mathrm{d}y}{|y|^3}\right|\\
\le&C\Big(\rho^{-\beta}+g(x,t)\Big)H(x,t),
\end{aligned}\tag{3.7}
$$

where

$$
H(x,t)=\int_{R^3}|j_2(x+y)|\frac{\mathrm{d}y}{|y|^{3-\beta}}.
$$

So, we have the following iequality

$$
|I_1^{(3)}|\le C\rho^{-\beta}\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|j(\cdot t)\|^2_{\frac{12}{3-2\beta}}+C\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|j(\cdot t)\|_q^2\|g(\cdot t)\|_b.\tag{3.8}
$$

Applying interpolation and Young inequalities, we have

$$\begin{aligned}\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|j(\cdot t)\|^2_{\frac{12}{3+2\beta}} &\leq C\|j(\cdot t)\|_2\|j(\cdot t)\|_2^{2(1-\vartheta)}\|\nabla j(\cdot t)\|_2^{2\vartheta}\\ &\leq C\|j(\cdot t)\|_2^{\frac{1}{1-\vartheta}}\|j(\cdot t)\|_2^2+\frac{1}{12}\|\nabla j(\cdot t)\|_2^2. \end{aligned}\tag{3.9}$$

we should point out because $\beta\in[\frac{1}{2},1]$, then $\frac{1}{1-\vartheta}\leq 2$.

By the classical interpolation and Hardy-Littlewood-Sobolev inequalities, we have

$$\|j(\cdot t)\|_q^2\leq C\|j(\cdot t)\|_2^{2(1-\theta)}\|\nabla j(\cdot t)\|_2^{2\theta},\qquad \|H(\cdot t)\|_{\frac{6}{3-2\beta}}\leq C\|j(\cdot t)\|_2.$$

So, we have

$$\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|j(\cdot t)\|_q^2\|g(\cdot t)\|_b\leq C\|j(\cdot t)\|_2^{2+\frac{1}{1-\theta}}\|g(\cdot t)\|_b^{\frac{1}{1-\theta}}+\frac{1}{12}\|\nabla j(\cdot t)\|_2^2. \tag{3.10}$$

Note the parameters q,b,θ satisfy

$$\frac{2}{q}+\frac{1}{b}=\frac{1}{2}+\frac{\beta}{3};\qquad \theta=\frac{3}{2}-\frac{3}{q}, \tag{3.11}$$

and

$$\frac{1}{1-\theta}\leq 2\ \Rightarrow\ q\leq 3.$$

Applying Young inequality, we can get

$$|I_1^{(3)}|\leq C\|j(\cdot t)\|_2^2\left(\|j(\cdot t)\|_2^2+\|g(\cdot t)\|_b^{\frac{2q}{6-2q}}\right)+\frac{1}{6}\|\nabla j(\cdot t)\|_2^2. \tag{3.12}$$

In fact, set $a=\frac{2q}{6-2q}$, then we have $\frac{2}{a}+\frac{3}{b}=\beta-\frac{1}{2}$.

Then we can estimate I_1 as following

$$|I_1|\leq C\|j(\cdot t)\|_2^2\left(\|j(\cdot t)\|_2^2+\|g(\cdot t)\|_b^{\frac{2q}{6-2q}}\right)+\frac{1}{3}\|\nabla j(\cdot t)\|_2^2. \tag{3.13}$$

Now, we pay attention to the term I_2+I_3,

$$\begin{aligned}|I_2+I_3| &=\frac{1}{2}\left|\int_{R^3}(\nabla\times)^2 w(x,t)\cdot j(x,t)\mathrm{d}x+\int_{R^3}\partial_i j(x,t)\cdot\partial_i w(x,t)\mathrm{d}x\right|\\ &\leq C\|j(\cdot t)\|_2\|\nabla\partial_i w(\cdot t)\|_2+C\|\nabla j(\cdot t)\|_2\|\partial_i w(\cdot t)\|_2\\ &\leq C\left(\|j(\cdot t)\|_2^2+\|\partial_i w(\cdot t)\|_2^2\right)+\frac{1}{6}\left(|\nabla j(\cdot t)\|_2^2+\|\nabla\partial_i w(\cdot t)\|_2^2\right). \end{aligned}\tag{3.14}$$

where we used the inequality: $|(\nabla\times)^2 w|\leq|\nabla\partial_i w|$.

Finally, we estimate the term of I_4

$$\begin{aligned}|I_4|=&\left|\int_{R^3}\partial_i u(x,t)\cdot\nabla w(x,t)\cdot\partial_i w(x,t)\mathrm{d}x\right|\\ \leq& C\|j_1(\cdot t)\|_4\|\nabla w(\cdot t)\|_2^2+C\left|\int_{R^3}j_2(x,t)\cdot\nabla w(x,t)\cdot\partial_i w(x,t)\mathrm{d}x\right|\\ =&I_4^{(1)}+I_4^{(2)},\end{aligned}\tag{3.15}$$

where we used the following inequality

$$\|\nabla f\|_p\leq C\|\nabla\times f\|_p,\qquad \text{for any}\quad f\in W^{1,p}\quad \text{with}\quad \mathrm{div} f=0.$$

Similarly, we get

$$|I_4^{(1)}|\leq CK^{\frac{1}{2}}\|j(\cdot t)\|_2^{\frac{1}{2}}\|\nabla w\|_2^2.\tag{3.16}$$

From the expression of $j(x,t)$, we obtain that

$$\begin{aligned}|j_2(x)|=&\left|\int_{R^3}\sigma(\hat{y})j_2(x+y,t)\frac{\mathrm{d}y}{|y|^3}\right|\\ \leq&\left|\int_{|y|\leq\rho}\sigma(\hat{y})(j_2(x+y,t)-j_2(x,t))\frac{\mathrm{d}y}{|y|^3}\right|+\left|\int_{|y|\geq\rho}\sigma(\hat{y})j_2(x+y,t)\frac{\mathrm{d}y}{|y|^3}\right|\\ \leq& C(\rho^{-\beta}+g(x,t))\int_{R^3}|j_2(x+y,t)|\frac{\mathrm{d}y}{|y|^{3-\beta}}.\end{aligned}\tag{3.17}$$

Then the last term can be estimate as following

$$|I_4^{(2)}|\leq C\rho^{-\beta}\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|\partial_i w\|^2_{\frac{12}{3+2\beta}}+C\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|\partial_i w(\cdot t)\|_q^2\|g(\cdot t)\|_b.$$

Similar to the term $I_1^{(3)}$, we have

$$\begin{aligned}\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|\partial_i w\|^2_{\frac{12}{3+2\beta}}\leq& C\|j(\cdot t)\|_2\|\partial_i w\|_2^{2(1-\vartheta)}\|\nabla\partial_i w\|_2^{2\vartheta}\\ \leq& C\|j(\cdot t)\|_2^{\frac{1}{1-\vartheta}}\|\partial_i w\|_2^2+\frac{1}{6}\|\nabla\partial_i w\|_2^2,\end{aligned}\tag{3.18}$$

$$\begin{aligned}&\|H(\cdot t)\|_{\frac{6}{3-2\beta}}\|\partial_i w(\cdot t)\|_q^2\|g(\cdot t)\|_b\\ &\qquad\leq C\|\partial_i w(\cdot t)\|_2^2\Big(\|\partial_i w(\cdot t)\|_2^2+\|g(\cdot t)\|_b^a\Big)+\frac{1}{6}\|\nabla\partial_i w(\cdot t)\|_2^2.\end{aligned}$$

Then we have

$$|I_4|\leq C(\|\partial_i w(\cdot t)\|_2^2+\|j(\cdot t)\|_2^2+\|g(\cdot t)\|_b^a)\|\partial_i w(\cdot t)\|_2^2+\frac{1}{3}\|\nabla\partial_i w(\cdot t)\|_2^2.\tag{3.19}$$

Using (3.13), (3.14), (3.19) in (3.2), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|j\|_2^2+\|\partial_i w\|_2^2\right)+\|\nabla j\|_2^2+\|\nabla\partial_i w\|_2^2\\
\le& C\left(\|\partial_i w(\cdot t)\|_2^2+\|j(\cdot t)\|_2^2+\|g(\cdot t)\|_b^a\right)\left(\|\partial_i w(\cdot t)\|_2^2+\|j(\cdot t)\|_2^2\right).
\end{aligned} \tag{3.20}
$$

(1.6) is a straight consequence by applying the Gronwall inequality on (3.20).

This finishes the proof of theorem. □

Acknowledgments

The author thanks to the editior and the referee for their invaluable comments and suggestion which helped improve the paper greatly. This work supported by Anhui Education Bureau (AQKJ2014B009) and by the Doctor's funding of AnQing teacher's University (K05000130009).

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