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A MAXIMUM PRINCIPLE FOR ELLIPTIC AND PARABOLIC EQUATIONS WITH OBLIQUE DERIVATIVE BOUNDARY PROBLEMS*

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Abstract This paper prove a maximum principle for viscosity solutions of fully nonlinear, second order, uniformly elliptic and parabolic equations with oblique boundary value conditions.
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In this note, we prove an Aleksandrov-Backlman-Pucci type maximum principle when the boundary conditions consist of oblique derivative conditions and Dirichlet conditions. We will show that the maximum principle holds if a large portion of the boundary has Dirichlet boundary conditions.

This kind of estimates are important for the oblique derivative probiems. The reason is the following. If we blow up a solution of the oblique derivative problem near the boundary, the blow-up solution satisfies the above mixed boundary value problem. We will investigate this in a forthcoming paper.

1. The Maximum Principle, Elliptic Case

Let Ω be a bounded domain in R^n. Let $\partial\Omega = \partial_d\Omega \cup \partial_n\Omega$. Assume $\partial_d\Omega$ is closed and $\partial_n\Omega$ is open with respect to the relative topology of $\partial\Omega$.

Consider a fully nonlinear elliptic operator

$$F(D^2u, Du, u, x) = 0 \tag{1}$$

in Ω, with uniformly elliptic condition

$$\lambda|P| \le F(M+P, v, u, x) - F(M, v, u, x) \le \Lambda|P| \tag{2}$$

for any positive definite matrix P, where λ, Λ are fixed positive constants.

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We say u is a solution of (1), always according to [1], namely in the sense of viscosity solutions.

We will first consider a special class of operators

$$F(D^2u, Du, u, x) = g(x,t) \tag{3}$$

with the condition

$$F(0, P, u, x) \equiv 0 \tag{4}$$

We need some terminology. Let $\Gamma(\Omega)$ be the convex hull of Ω, namely $\Gamma(\Omega) = \cap\{D | D \supset \Omega, \text{ convex}\}$.

Generalized Gauss map for any domain D

$$G : \partial D \to 2^{S^{n-1}} \text{ (the subset of } S^{n-1})$$

$$x \to \{\theta : \quad \theta \text{ is an outer unit normal of } \partial D \text{ at } x\}$$

Let ds be the normalized surface measure on S^{n-1}.

$$ds(S^{n-1}) = 1$$

For A, a subset of R^n, let $A \triangleright \{0\}$ be the cone with vertex 0 generated by A.

Now, we want to consider the following problem.

$$\begin{cases} F(D^2u, Du, u, x) = g(x) & \\ \dfrac{\partial u}{\partial n} \geq 0 & \text{on } \partial_n \Omega \\ u \geq 0 & \text{on } \partial_d \Omega \end{cases} \tag{5}$$

Let C_n be the co-cone of $\partial_n \Omega$ as follows

$$C_n = \{v \in S^{n-1} | G(\partial_n \Omega) \cdot v < 0\}$$

Clearly $C_n \triangleright \{0\}$ is a convex set.

Theorem 1 *Let u be a continuous solution of (5). Assume $ds(C_n) \geq \alpha$ for some $\alpha > 0$. Then*

$$\sup_{\Omega} u^- \leq C \Big(\int_{\Gamma(u)=u} |g^-|^n \Big)^{\frac{1}{n}} \tag{6}$$

where $\Gamma(u)$ is the convex hull of u

$$\Gamma(u) = \sup\{v(x) | v \leq 0 \text{ on } \partial_d \Omega, \ v \leq u \text{ convex}\}$$

and the constant C depends only on λ, Lambda, n and the diameter of the domain Ω.

Lemma *$\Gamma(u)$ is C^{11}_{loc}.*

Proof We refer this to [1].

Proof of Theorem 1

Consider the map

$$\Phi: \ \Gamma(\Omega) \to R^n$$
$$x \to \nabla \Gamma u(x)$$

Clearly, as in [1],

$$|\Phi(\Gamma(\Omega))| \le \int |\det D^2 \Gamma u|$$
$$\le C\Big(\int_{\Gamma(u)=u} |g^-|^n\Big)$$

where $|A|$ is the Lebesgue measure for a set A.

We have to bound $|\Phi(\Gamma(\Omega))|$ from below by $\sup u^-$.

Let d be the diameter of Ω.

We claim that

$$(C_n \triangleright \{0\}) \cap B_{\frac{\sup u^-}{d}}(0) \subset \Phi(\Gamma(\Omega)) \tag{7}$$

For each $\nu \in (\Gamma(C_n \triangleright \{0\}) \cap B_{\frac{\sup u^-}{d}}(0))^\circ$, take a hyperplane P with gradient ν and below the graph of u. Clearly it is also below the graph of $\Gamma(u)$.

Now, consider the family of hyperplanes

$$P_t = P + t$$

for $t \in [0, +\infty)$.

There is clearly a critical t^* such that

$$\begin{cases} u \ge P + t^* \\ u(x_0) = P(x_0) + t^* \quad \text{for some } x_0 \end{cases}$$

We claim $x_0 \in \mathring{\Omega}$, therefore $\Phi(x_0) = \nu$.

We prove it by contradiction. There are two cases

1) $\quad x_0 \in \partial_d \Omega$, $\qquad$ 2) $\quad x_0 \in \partial_n \Omega$

If it were case 1), then $t^* + P(x_0) = u(x_0) \ge 0$. Hence,

$$\begin{aligned} u(x) &\ge P(x) + t^* \\ &\ge -\mathrm{dis}(x, (P + t^*) \cap R^n) \cdot |\nabla P| \\ &> -d \cdot \frac{\sup u}{d} = -\sup u^- \end{aligned}$$

which is a contradiction.

If it were case 2), then

$$\frac{\partial u}{\partial n} \le \frac{\partial P}{\partial n} = n \cdot \nu < 0 \tag{8}$$

which is a contradiction to the boundary condition.

From the claim,

$$\omega_n \alpha \Big(\frac{\sup u^-}{d}\Big)^n \le |\Phi((C_n \triangleright \{0\}) \cap B_{(\frac{\sup u^-}{d})})|$$

$$\le |\Gamma(\Omega)|$$

$$\le C \int_{u=\Gamma(u)} |g^-|^n \tag{9}$$

2. The Parabolic Case

Consider the parabolic equation

$$u_t - F(D^2u, Du, u, x, t) = g(x, t) \tag{10}$$

in a bounded domain $\Omega \subset \boldsymbol{R}^{n+1}$. We follow the terminology as in [4].

We introduce a few more terminologies.

We say Ω is convex iff $\Omega \cap \{t = \text{const}\}$ is convex and $\Omega \cap \{t = t_1\} \subset \Omega \cap \{t = t_2\}$ for any $t_1 < t_2$ and $\Omega \cap \{t = t_2\} \ne \phi$

We consider the boundary value problem, with $\partial_p \Omega = \partial_{p,n}\Omega \cap \partial_{p,n}\Omega$,

$$\begin{cases} u_t - F(D^2u, Du, u, x, t) = g(x, t) & \\ \dfrac{\partial u}{\partial n} \ge 0 & \text{on } \partial_{p,n}\Omega \\ u \ge 0 & \text{on } \partial_{p,d}\Omega \end{cases} \tag{11}$$

Let C_n be the co-cone of $\partial_{p,n}\Omega$, as

$$C_n = \{\nu : \text{ for any } t, G(\partial_{p,n}\Omega \cap \{t\}) \cdot \nu < 0\}$$

As in [3], we consider the map for a function v, which is convex in x and decreasing on t,

$$\Phi_v : \Gamma(\Omega) \to \boldsymbol{R}^{n+1}$$

$$(x, t) \to (\bigtriangledown v,\ v - x \cdot \bigtriangledown v)$$

Clearly, we have

$$J\Phi_v = -v_t \det D^2 v$$

Theorem 2 *Let u be a solution of (11). Assume $0 \in \Omega \cap \{t = t_{\min}\}$, where $t_{\min}$ is the minimum t coordinate of the points in $\partial_p\Omega$. Assume $ds(C_n) \ge \alpha$ for some $\alpha > 0$. Then*

$$\sup_{\Omega} u^- \le C\Big(\int_{\Gamma(u)=u} |g^-|^{n+1}\Big)^{\frac{1}{n+1}}$$

where $\Gamma(u)$ is the convex hull of u, defined as

$$\Gamma(u) = \sup\{v | v \le 0 \text{ on } \partial_{p,d}\Omega,\ v \le u,\ v \text{ convex in } x,\ \text{decreasing in } t\}$$

and C depends on $\alpha, \lambda, \Lambda, n$ and the diameter of the domain d only.

The following lemma is established in [4].

Lemma *$\Gamma(u)$ is C^{11}_{loc} in $\Gamma(\Omega)$.*

Proof of Theorem 2 We follow as in [3] and [4]. Let $M = -u(x_0, t_0)$. We claim

$$D = \{(\xi, h) | \xi \in C_n, |\xi| \leq \frac{M}{d}, d|\xi| < h < M\} \subset \Phi_{\Gamma(u)}(\Gamma(\Omega)).$$

For $(\xi, h) \in D$, the hyperplane

$$P(x) = \xi x + h$$

is below $u(x,t)$ on $\partial_d \Omega$.

If we translate $P(y)$ along the positive t-direction, since $P(0) = h < M$, it would touch u at $(x_1, t_1) \in \Omega$. As in Section 1, we can show that $(x_1, t_1) \in \mathring{\Omega}$. Therefore

$$(x_1, t_1) \in \{\Gamma u = u\}$$

Moreover, we have

$$\Phi_{\Gamma(u)}(x_1, t_1) = (\xi, h)$$

The rest of the proof follows as in [3].

$$\int_{\Gamma(u)=u\leq 0} -(\Gamma u)_t \det D^2(\Gamma u) dx dt \geq \alpha C_n M^{n+1} d^{-n}$$

We use geometric-arithmatic mean inequality

$$\sup u^- \leq C\Big(\int_{u=\Gamma(u)} |g^-|^{n+1}\Big)^{\frac{1}{n+1}}$$

as in [4] and [3].

Remark 1 The solutions of (10) or (3) are the same as the function class $S(\lambda, \Lambda, g)$ in [1] or [4].

Remark 2 The similar estimates hold when we have other oblique boundary conditions. We only need to define the co-cone accordingly.

Remark 3 Similarly these estimates hold when g involves $\nabla u, u$ linearly. We refer this to [3].

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