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Mortar Upwind Finite Volume Element Method with Crouzeix-Raviart Element for Parabolic Convection Diffusion Problems

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Abstract. In this paper, we study the semi-discrete mortar upwind finite volume element method with the Crouzeix-Raviart element for the parabolic convection diffusion problems. It is proved that the semi-discrete mortar upwind finite volume element approximations derived are convergent in the H^1- and L^2-norms.

Key words: Mortar upwind finite volume element method; Crouzeix-Raviart element; parabolic convection diffusion problems; error estimates.

AMS subject classifications: 65N55, 65N22, 65N30

1 Introduction

The mortar element method was first introduced by Bernardi, Maday and Patera in [2]. From then on, this method as a special nonconforming domain decomposition technique has aroused many researchers' attention because different types of discretizations can be employed in different parts of the computational domain. We refer to [2-5, 9, 10, 12, 18, 22] and the cited references there for details.

In the mortar element method, the computational domain is first decomposed into a polygonal partition. The meshes on different subdomains need not match across subdomain interfaces. The basic idea of this method is to replace the strong continuity condition on the interfaces between different subdomains by the so-called mortar condition. This condition guarantees the optimal discretization schemes, that is, the global discretization error is bounded by the sum of the optimal errors on different subdomains.

The finite volume element methods, also called the generalized difference methods in China, are popular in computational fluid mechanics due to their conservation properties of the original problems. In the past several decades, professors Li Ronghua *et al.* have systematically studied the finite volume element methods and obtained many important results. Interested readers are referred to the monographs [14, 15] for the general presentation of the finite volume element methods, and to [1, 6, 7, 11, 13, 16, 17, 19, 20, 23] and the references therein for details.

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Recently, Ewing, Lazarov, Lin and Lin [9] have considered mortar finite volume element approximations of the second-order self-adjoint elliptic problems. The discretization is based on the Petrov-Galerkin method with a solution space of continuous piecewise linear functions over each subdomain and a test space of piecewise constant functions. Bi and Li [3] have studied the mortar finite volume element method based on the mortar Crouzeix-Raviart finite element space and developed optimal order error estimates in the H^1- and L^2-norms.

In this paper, we construct and analyze the semi-discrete mortar upwind finite volume element method with the Crouzeix-Raviart element for parabolic convection diffusion problems. We use the mortar finite volume element method to discretize the diffusion term, and mortar upwind difference schemes to discretize the convection term, and establish error estimates in the H^1- and L^2-norms.

The remainder of this paper is organized as follows. In Section 2 we describe the parabolic convection diffusion problems, give the triangulation $\mathcal{T}_h$ of the computational domain Ω and the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$. Section 3 presents the semi-discrete mortar upwind finite volume element method for the parabolic convection diffusion problems. In Section 4, we get the error estimates in H^1- and L^2-norms.

In this paper, the notation of Sobolev spaces and associated norms and semi-norms are the same as those in Ciarlet [8], and C denotes the positive constant independent of the mesh parameter and the number of the subdomians, and may be different at different occurrences.

2 Notation and preliminaries

Consider the following parabolic convection diffusion problem on a bounded polygonal domain $\Omega \subset \mathcal{R}^2$:

$$\begin{cases} u_t - \nabla\cdot(A(x)\nabla u) + \nabla\cdot(b(x)u) = f, & x\in\Omega,\quad 0<t\le T,\\ u(x,t)=0, & x\in\partial\Omega,\quad 0<t\le T,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1}$$

We assume that $A=(a_{ij}(x))_{i,j=1}^2$ is a symmetric and uniformly positive definite matrix in Ω, $a_{ij}\in W^{1,\infty}(\overline{\Omega}),\ 1\le i,j\le 2,\ b(x)\in(W^{1,\infty}(\overline{\Omega}))^2$. In this paper, in order to get the existence and uniqueness of the approximation solution in Section 3, we further assume that $\nabla\cdot b\ge 0$.

In this paper, we consider a geometrically conforming version of the mortar upwind finite volume element method, i.e., Ω is divided into non-overlapping polygonal subdomains Ω_i, $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$, with $\overline{\Omega}_i\cap\overline{\Omega}_j$ being an empty set or a vertex or an edge for $i\ne j$.

Each subdomain Ω_i is triangulated to produce a regular mesh $\mathcal{T}_h^i$ with the mesh parameter h_i, where h_i is the largest diameter of the elements in $\mathcal{T}_h^i$. The triangulations of subdomains generally do not align at the subdomain interfaces. Let Γ_{ij} denote the open straight line segment which is common to $\overline{\Omega}_i$ and $\overline{\Omega}_j$ and let Γ denote the union of all interfaces between the subdomains, i.e., $\Gamma=\cup\partial\Omega_i\backslash\partial\Omega$. We assume that the endpoints of each interface in Γ are vertices of $\mathcal{T}_h^i$ and $\mathcal{T}_h^j$. Let $\mathcal{T}_h$ denote the global mesh $\cup_i\mathcal{T}_h^i$ with $h=\max\limits_{1\le i\le N} h_i$.

Since the triangulations on two adjacent subdomains are independent, the interface $\overline{\Gamma}_{ij} = \overline{\Omega}_i\cap\overline{\Omega}_j$ is provided with two different and independent 1-D meshes, which are denoted by $\mathcal{T}_h^i(\Gamma_{ij})$ and $\mathcal{T}_h^j(\Gamma_{ij})$, respectively. We define one of the sides of Γ_{ij} as a mortar one, the other as a non-mortar one, denoted by γ_i and δ_j, respectively. Let $\Omega_{M(\Gamma_{ij})}$ denote the mortar domain of Γ_{ij} and $\Omega_{NM(\Gamma_{ij})}$ the non-mortar domain of Γ_{ij}. Define $u_{\gamma_i}^M$ and $u_{\delta_j}^{NM}$ to be the traces of $u|_{\Omega_{M(\Gamma_{ij})}}$ and $u|_{\Omega_{NM(\Gamma_{ij})}}$ on Γ_{ij}, respectively. Define CR nodal points as the midpoints of the edges of elements in $\mathcal{T}_h$. The sets of CR nodal points belonging to $\overline{\Omega}_i$, $\partial\Omega_i$, $\partial\Omega$, γ_i and δ_j are denoted by $\Omega_{i,h}^{CR}$, $\partial\Omega_{i,h}^{CR}$, $\partial\Omega_h^{CR}$, γ_i^{CR} and δ_j^{CR}, respectively.

In order to define the mortar Crouzeix-Raviart finite element space, we first define the finite element functions locally and introduce the space

$$\widetilde{V}_h(\Omega_i) = \{v : v|_K \text{ is linear for all } K \in \mathcal{T}_h^i, v \text{ is continuous at } \Omega_{i,h}^{CR}\backslash\partial\Omega_{i,h}^{CR} \text{ and } v = 0 \text{ at } \partial\Omega_{i,h}^{CR} \cap \partial\Omega_h^{CR}\},$$

with $||v||_{1,h,\Omega_i} = (\sum_{K\in\mathcal{T}_h^i} ||v||^2_{H^1(K)})^{1/2}$ and $|v|_{1,h,\Omega_i} = (\sum_{K\in\mathcal{T}_h^i} |v|^2_{H^1(K)})^{1/2}$.

We can now introduce the global space $\widetilde{V}_h = \prod_{i=1}^N \widetilde{V}_h(\Omega_i)$ with
the norm $||v||_{1,h} =(\sum_{i=1}^N ||v||^2_{1,h,\Omega_i})^{1/2}$ and the semi-norm $|v|_{1,h} = (\sum_{i=1}^N |v|^2_{1,h,\Omega_i})^{1/2}$.
Let $M(\delta_j)$ be the subspace of the space $L^2(\Gamma_{ij})$:

$$M(\delta_j) = \{v : v \in L^2(\Gamma_{ij}), v \text{ is piecewise constant on } \mathcal{T}_h^j(\delta_j)\}.$$

For each non-mortar side $\delta_j = \Gamma_{ij} \in \Gamma$, we define the L^2 orthogonal projection $Q^{\delta_j} : L^2(\Gamma_{ij}) \to M(\delta_j)$ by

$$(Q^{\delta_j} u, \psi)_{0,\delta_j} = (u, \psi)_{0,\delta_j}, \quad \forall\psi \in M(\delta_j),$$

where $(\cdot, \cdot)_{0,\delta_j}$ denotes the usual L^2-inner product on the space $L^2(\delta_j)$ and $||\cdot||_{0,\delta_j}$ is the induced norm from the L^2-inner product on the space $L^2(\delta_j)$.

Similarly we can define $M(\gamma_i)$ and Q^{γ_i}. From the definitions of Q^{γ_i} and Q^{δ_j}, and the trace theorem, we obtain the following result.

Lemma 2.1. *Assume that $s \in \partial\Omega_i$ is a side of Ω_i (s may be a mortar or a nonmortar side). For any $\psi \in \widetilde{V}_h(\Omega_i)$ and $u \in H^1(\Omega_i)$, we have*

$$||\psi - Q^s\psi||_{0,s} \le Ch_i^{\frac{1}{2}}|\psi|_{1,h,\Omega_i}, \qquad ||u - Q^s u||_{0,s} \le Ch_i^{\frac{1}{2}}|u|_{1,\Omega_i}.$$

We define the discrete space V_h:

$$V_h = \{v \in \widetilde{V}_h, \quad Q^{\delta_j}(v|_{\gamma_i}) = Q^{\delta_j}(v|_{\delta_j}), \quad \forall\gamma_i = \delta_j \in \Gamma\}.$$

The condition on Γ is called the mortar condition. This mortar condition was constructed by Marcinkowski in [18]. We note that the mortar condition is not only dependent on the degrees of freedom on the interface but also on the degrees of freedom near the interface; see [18] for details.

Next, we give a basis of V_h. Let $\{\tilde{\phi}_i | i = 1, \cdots, \widetilde{Z}\}$ be the nodal basis of $\widetilde{V}_h$. The basis of V_h consists of the functions of the form

$$\phi_i = \tilde{\phi}_i + \sum_{\delta_j\in\Gamma} \mathcal{E}_{\delta_j}(\tilde{\phi}_i), \tag{2}$$

where the operator $\mathcal{E}_{\delta_j} : \widetilde{V}_h \to \widetilde{V}_h$ is defined by

$$\mathcal{E}_{\delta_j} v(m_e) = \begin{cases} Q^{\delta_j}(v^M_{\gamma_i} - v^{NM}_{\delta_j})(m_e), & m_e \in \delta_j^{CR}, \\ 0, & \text{otherwise.} \end{cases}$$

For any $\tilde{v} \in \widetilde{V}_h$, let

$$v = \tilde{v} + \sum_{\delta_j\in\Gamma} \mathcal{E}_{\delta_j}\tilde{v}. \tag{3}$$

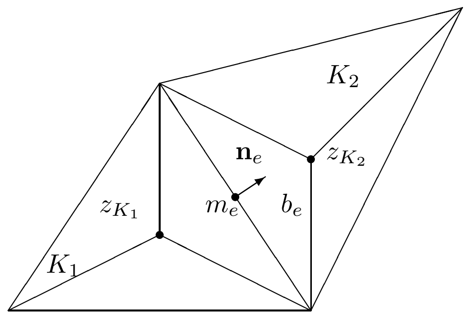

Figure 1: A box b_e corresponding to side e.

We can check that $v \in V_h$; see [22] for details.

It is easy to check that at all nodes which are in the interior of each non-mortar side $\delta_j \in \Gamma$ the ϕ_i are equal to zero. Apart from ϕ_i corresponding to those nodes on the non-mortar side, it is not difficult to check that these ϕ_i defined by (2) form a basis of V_h. Denote by Z the number of nonzero ϕ_i. We now re-index $\{\tilde{\phi}_i | i = 1, \cdots, \widetilde{Z}\}$ in such a way that every nonzero ϕ_i is in $\{\phi_i | i = 1, \cdots, Z\}$; see [22] for details.

In order to construct the mortar upwind finite volume element method, for a given triangulation $\mathcal{T}_h$, we build a dual mesh $\mathcal{T}_h^*$ based upon $\mathcal{T}_h$ whose elements are called the control volumes.

Given a triangle $K \in \mathcal{T}_h^i$, we denote the set of its edges by $E(K)$ and set $E_{h,i} = \cup_{K \in \mathcal{T}_h^i} E(K)$. Let $E_{h,i}^{\text{in}}$ be the set of the interior sides of the triangulation $\mathcal{T}_h^i$. Let m_e denote the midpoint of a side e, $e \in E(K), K \in \mathcal{T}_h$.

In each subdomain Ω_i, we construct the dual partition of $\mathcal{T}_h^i$ in the same way as in [3, 7]. Choose an interior point z_K of $K \in \mathcal{T}_h^i$ and connect it with line segments to the vertices of the element K. Thus we partition K into three subtriangles, $K_e, e \in E(K)$. With each side $e \in E_{h,i}^{\text{in}}$, we construct a control volume b_e consisting of the two subtriangles which have e as a common edge, (see Fig. 1). For each side $e \subset \gamma_i$ or $e \subset \delta_j$, the control volume consisting of the subtriangle which has e as a side, is denoted by $b_e^{\gamma_i}$ or $b_e^{\delta_j}$ respectively, (see Fig. 2). Moreover, we also associate a corresponding boundary control volume b_e with each side $e \subset \partial\Omega$. Thus we finally obtain a group of control volumes covering the domain Ω, which is called the dual partition $\mathcal{T}_h^*$ of the triangulation $\mathcal{T}_h$. For simplicity, we denote the control volume by b_e corresponding to the side e.

We shall use the construction of the control volumes in which the point z_K is the barycenter of the element K. This type of control volumes can be introduced for any triangulation $\mathcal{T}_h$ and leads to relatively simple calculations.

The test space which is associated with the dual mesh $\mathcal{T}_h^*$ is defined by

$$U_h = \{v \in L^2(\Omega) : v|_{b_e} \text{ is constant for all } b_e \in \mathcal{T}_h^* \text{ and } v|_{\partial\Omega} = 0\}.$$

Let $I_h^* : V_h \to U_h$ be the piecewise constant interpolation operator:

$$I_h^* v = \sum_{i=1}^{N} \sum_{e \in E_{h,i}} v(m_e) \varphi_{b_e}(x), \quad \forall v \in V_h,$$

where φ_{b_e} is the characteristic function of the control volume b_e.

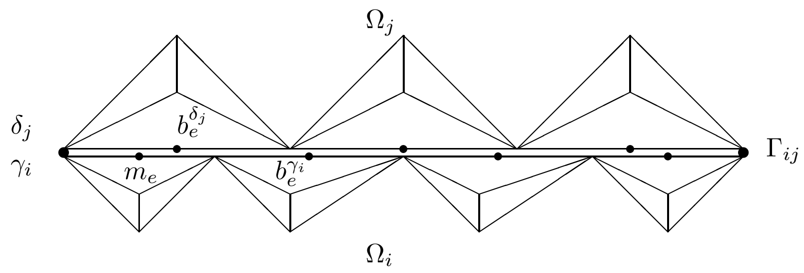

Figure 2: Non-matching meshes on the interface Γ_{ij}.

3 Mortar upwind finite volume element method

In this section, we construct the mortar upwind finite volume element method and give some lemmas which will be used in the convergence analysis.

We now write (1) into a weak form. Multiplying (1) by $I_h^* v_h \in U_h$, integrating it on Ω and applying Green's formula, we obtain

$$(u_t, I_h^* v_h) + a^{(2)}(u, I_h^* v_h) + a^{(1)}(u, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in V_h, \tag{4}$$

where here and elsewhere $(\cdot,\cdot)$ denotes $L^2(\Omega)$-inner product. The bilinear forms are defined by

$$a^{(2)}(u, I_h^* v_h) = -\sum_{i=1}^{N} \sum_{e \in E_{h,i}} v_h(m_e) \int_{\partial b_e} (A\nabla u)\cdot \mathbf{n} \mathrm{d}s, \tag{5}$$

$$a^{(1)}(u, I_h^* v_h) = \sum_{i=1}^{N} \sum_{e \in E_{h,i}} v_h(m_e) \int_{\partial b_e} (b\cdot \mathbf{n}) u \mathrm{d}s, \tag{6}$$

where $\mathbf{n}$ is the unit outer normal of ∂b_e.

In this paper, we will use the semi-discrete mortar finite volume element method to deal with the diffusion term and the mortar upwind schemes to the convection term.

For this purpose, we first introduce some notation which will be used in the mortar upwind schemes. Set $\Lambda_k = \{l : m_l \text{ is a neighboring CR node of } m_k\}$. For the adjoint sides e_k and e_l, let $\gamma_{kl} = \partial b_{e_k} \cap \partial b_{e_l}$. Define

$$\beta_{kl} = \int_{\gamma_{kl}} b\cdot \mathbf{n} \mathrm{d}s,$$

where $\mathbf{n}$ denotes the unit outer normal direction of γ_{kl} (viewing γ_{kl} as a part of the boundary of b_{e_k}). Then we can divide ∂b_{e_k} into a *flow-in* part and a *flow-out* part according to the sign of β_{kl}:

$$\begin{cases} (\partial b_{e_k})_- = \cup_{\{\beta_{kl}\le 0, l\in\Lambda_k\}} \gamma_{kl}, & \text{(Flow in)}; \\ (\partial b_{e_k})_+ = \cup_{\{\beta_{kl}> 0, l\in\Lambda_k\}} \gamma_{kl}, & \text{(Flow out)}. \end{cases}$$

The following facts are obvious:

$$\beta_{kl} + \beta_{lk} = 0, \quad |\beta_{kl}| \le C||b||_\infty |\gamma_{kl}|,$$

where $||b||_\infty$ denotes the L^∞-norm of b and $|\gamma_{kl}|$ denotes the length of γ_{kl}.

The semi-discrete mortar upwind finite volume element method is to find $u_h \in V_h$ such that

$$\begin{aligned} &(u_{h,t}, I_h^* v_h) + a_h^{(2)}(u_h, I_h^* v_h) + a_h^{(1)}(u_h, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in V_h, t > 0, \\ &u_h(x, 0) = u_{0h}(x), \quad x \in \Omega, \end{aligned} \tag{7}$$

where

$$\begin{aligned} a_h^{(2)}(u_h, I_h^* v_h) &= -\sum_{i=1}^{N} \sum_{e \in E_{h,i}^{\text{in}}} v_h(m_e) \int_{\partial b_e} (A\nabla u_h) \cdot \mathbf{n} \mathrm{d}s - \\ &\sum_{\gamma_i = \delta_j \in \Gamma} \left(\sum_{e \subset \gamma_i} v_h(m_e) \int_{\partial b_e \setminus \gamma_i} (A\nabla u_h) \cdot \mathbf{n} \mathrm{d}s + \sum_{e \subset \delta_j} v_h(m_e) \int_{\partial b_e \setminus \delta_j} (A\nabla u_h) \cdot \mathbf{n} \mathrm{d}s \right). \\ a_h^{(1)}(u_h, I_h^* v_h) &= \sum_{i=1}^{N} \sum_{e_k \in E_{h,i}} v_h(m_{e_k}) \sum_{l \in \Lambda_k} \{\beta_{kl}^+ u_h(m_{e_k}) - \beta_{kl}^- u_h(m_{e_l})\}. \end{aligned} \tag{8}$$

where $\beta_{kl}^+ = \max(\beta_{kl}, 0), \beta_{kl}^- = \max(-\beta_{kl}, 0)$.

The function u_{0h} is a certain approximation of $u_0(x)$ on V_h. In this paper, we choose u_{0h} as the interpolation function of $u_0(x)$ in V_h.

We note that, for the side $e \subset \gamma_i$ or $e \subset \delta_j$, the line integrals in the bilinear forms $a^{(2)}(\cdot, \cdot)$ in (5) and $a^{(1)}(\cdot, \cdot)$ in (6) are defined on the whole boundary of b_e, while the line integral in the bilinear form $a_h^{(2)}(\cdot, \cdot)$ and the upwind values in $a_h^{(1)}(\cdot, \cdot)$ are defined only on the part of the boundary of b_e, i.e., $\partial b_e \setminus \gamma_i$ or $\partial b_e \setminus \delta_j$.

For the sake of the later analysis, we introduce for any v_h, $\omega_h \in V_h$, the bilinear form associated with the finite element method,

$$\widetilde{a}^{(2)}(v_h, \omega_h) = \sum_{K \in \mathcal{T}_h} \int_K A\nabla v_h \cdot \nabla \omega_h \mathrm{d}x. \tag{9}$$

The following result is proved by Bi and Li in [3]:

$$|a^{(2)}(v_h, I_h^* \omega_h) - \widetilde{a}^{(2)}(v_h, \omega_h)| \le C \sum_{i=1}^{N} h_i |v|_{1,h} |\omega_h|_{1,h}, \quad \forall v_h, \omega_h \in V_h. \tag{10}$$

The Poincare inequality for the space V_h,

$$||v||_{0,\Omega} \le C|v|_{1,h}, \quad \forall v \in V_h, \tag{11}$$

is proved in Lemma 3.7 in [3]. From the Poincare inequality, we know that the bilinear form $\widetilde{a}^{(2)}(v_h, v_h), \forall v_h \in V_h$ is positive definite. Then from (10) we see that there exist $h_0 > 0, \alpha > 0$ such that for $0 < h \le h_0$,

$$\alpha ||v_h||_{1,h}^2 \le a_h^{(2)}(v_h, I_h^* v_h). \quad \forall v_h \in V_h. \tag{12}$$

The following Lemma 3.1 was proved by Rui and Bi in [21].

Lemma 3.1. *For any given $v_h \in V_h$, we have that*

$$\begin{aligned} a^{(1)}(v_h, I_h^* v_h) &= \frac{1}{2} \sum_{\gamma_{kl}} (v_h(m_{e_k}) - v_h(m_{e_l}))^2 \int_{\gamma_{kl}} |\mathbf{b} \cdot \mathbf{n}| \mathrm{d}s \\ &+ \sum_{i=1}^{N} \sum_{e \in E_{h,i}} \int_{b_e} (\frac{1}{2} \nabla \cdot \mathbf{b}) v_h(m_e)^2 \mathrm{d}s. \end{aligned}$$

For the subsequent analysis, we define the following discrete norm on the space V_h,

$$|||v_h|||_0^2 = (v_h, I_h^* v_h), \quad \forall v_h \in V_h.$$

Lemma 3.2. *There exist two constants C_0, C_1 independent of h such that*

$$C_0||v_h||_{0,\Omega} \le |||v_h|||_0 \le C_1||v_h||_{0,\Omega}, \quad \forall v_h \in V_h. \tag{13}$$

Proof Let $K \in \mathcal{T}_h$ be an element with nodal points e_i, e_j and e_k. Since the point $z_K \in K$ is the barycenter of the element K, we obtain by simple calculations that

$$\begin{aligned}(v_h, I_h^* v_h)_{0,K} &= \frac{\text{meas}(K)}{27}(v_h(e_i), v_h(e_j), v_h(e_k))\begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix}\begin{pmatrix} v_h(e_i) \\ v_h(e_j) \\ v_h(e_k) \end{pmatrix} \\ &= \frac{\text{meas}(K)}{27}(v_h(e_i), v_h(e_j), v_h(e_k))H\begin{pmatrix} v_h(e_i) \\ v_h(e_j) \\ v_h(e_k) \end{pmatrix}.\end{aligned} \tag{14}$$

Since the triangulation $\mathcal{T}_h^i$ is regular, a scaling argument show that there exists a constant C independent of h and $K \in \mathcal{T}_h^i$ such that

$$C^{-1}||v_h||_{0,K}^2 \le \sum_{e \in E(K)} h_K^2 v_h^2(m_e) \le C||v_h||_{0,K}^2, \quad \forall K \in \mathcal{T}_h. \tag{15}$$

Note that the matrix H is symmetric and positive definite. Thus, combining (14) with (15) yields the desired result (13). This completes the proof. ■

By simple calculations, we have, for any $u_h, v_h \in V_h$,

$$\begin{aligned}(u_h, I_h^* v_h)_{0,K} &= \frac{\text{meas}(K)}{27}(u_h(e_i), u_h(e_j), u_h(e_k))\begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix}\begin{pmatrix} v_h(e_i) \\ v_h(e_j) \\ v_h(e_k) \end{pmatrix} \\ &= \frac{\text{meas}(K)}{27}(v_h(e_i), v_h(e_j), v_h(e_k))\begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix}\begin{pmatrix} u_h(e_i) \\ u_h(e_j) \\ u_h(e_k) \end{pmatrix} \\ &= (v_h, I_h^* u_h)_{0,K}.\end{aligned}$$

Then, summing over all elements of the triangulation $\mathcal{T}_h$, we get

$$(u_h, I_h^* v_h) = (v_h, I_h^* u_h), \quad \forall u_h, v_h \in V_h. \tag{16}$$

From (15), we know that the norm $||I_h^* \cdot ||_0 = (I_h^* \cdot, I_h^* \cdot)^{1/2}$ on the space V_h is equivalent to the usual L^2 norm $|| \cdot ||_0$ on the space V_h; that is, there exist two constants C_0, C_1 independent of h such that

$$C_0||v_h||_0 \le ||I_h^* v_h||_0 \le C_1||v_h||_0, \quad \forall v_h \in V_h. \tag{17}$$

In the same way as for the finite element method, the semi-discrete mortar finite volume element method (7) may be written as a system of ordinary differential equations. In fact, let $\{\phi_i, i = 1, \cdots, Z\}$ be the basis of V_h and let $\{\varphi_i, i = 1, \cdots, Z\}$ be the associated basis of U_h. Writing $u_h(t) = \sum_{i=1}^{Z} \mu_i(t)\phi_i(x)$, $u_{0h} = \sum_{i=1}^{Z} \beta_i \phi_i$, (7) then takes the form

$$\begin{cases} M\dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + (K^{(2)} + K^{(1)})\mathbf{u} = F, \\ \mathbf{u}(0) = \beta, \end{cases}$$

where the matrices and vectors is defined by

$$M = [m_{ij}] = [(\phi_i, \varphi_j)], \quad K^{(2)} = [k_{ij}^{(2)}] = [a_h^{(2)}(\phi_i, \varphi_i)],$$
$$K^{(1)} = [k_{ij}^{(1)}] = [a_h^{(1)}(\phi_i, \varphi_i)], \quad \mathbf{u} = [\mu_1(t), \cdots, \mu_Z(t)]^T,$$
$$F = [(f, \varphi_1), \cdots, (f, \varphi_Z)]^T, \quad \beta = [\beta_1, \cdots, \beta_Z]^T.$$

From Lemma 3.1 and (12), under the condition $\nabla \cdot b \geq 0$, we know that the matrix $K^{(2)} + K^{(1)}$ is positive definite; from Lemma 3.2 and (16) we know that the matrix M is symmetric and positive definite. The ordinary differential equation theory tells us the problem (7) has a unique solution $u_h \in V_h$ for any $f \in L^2(\Omega)$.

Taking $v_h = u_h$ in (7), we have, by (16),

$$\frac{1}{2}\frac{\partial}{\partial t}|||u_h|||_0^2 + a_h^{(2)}(u_h, I_h^* u_h) + a_h^{(1)}(u_h, I_h^* u_h) = (f, I_h^* u_h). \tag{18}$$

By virtue of (12) and Lemma 3.1, we obtain

$$\frac{\partial}{\partial t}|||u_h|||_0^2 + ||u_h||_{1,h}^2 \leq ||I_h^* u_h||_0||f||_0 \leq C||u_h||_0||f||_0 \leq C(||u_h||_0^2 + ||f||_0^2), \tag{19}$$

where the equivalence of $||I_h^* u_h||_0$ and $||u_h||_0$ has been used.

Noting that $|||u_h|||_0$ and $||u_h||_0$ are equivalent, viewing $\phi(t) = |||u_h|||_0$ as an unknown function, integrating the above inequality, and by means of the Gronwall inequality, we have

$$||u_h(t)||_0^2 + \int_0^t ||u_h||_{1,h}^2 \mathrm{d}t \leq C(||u_{0h}||_0^2 + \int_0^t ||f||_0^2 \mathrm{d}t), \quad 0 \leq t \leq T. \tag{20}$$

This means that the semi-discrete solution $u_h(t)$ is stable with respect to the initial value and the right-hand side term f.

By a scaling argument, we can prove the following Lemma 3.3.

Lemma 3.3. *There exists a constant C independent of h such that*

$$C^{-1}|v|_{1,h}^2 \leq \sum_{i=1}^N \sum_{K \in \mathcal{T}_h^i} \sum_{e_k, e_l \in E(K)} (v(m_{e_k}) - v(m_{e_l}))^2 \leq C|v|_{1,h}^2, \quad \forall v \in V_h.$$

The following Lemma 3.4 is proved in [7]. We state it in a form that will be used in our convergence proof.

Lemma 3.4. *There exists a constant C independent of h such that, for every $v \in L^2(\Omega)$ with $v|_K \in H^1(K)$ for every $K \in \mathcal{T}_h$,*

$$\int_{\partial K} v^2 \mathrm{d}s \leq C(h_K^{-1}||v||_{0,K}^2 + h_K|v|_{1,K}^2), \quad \forall K \in \mathcal{T}_h.$$

Let $\tilde{u}_i^I$ be a continuous piecewise linear function in Ω_i equal to u in all vertices of $\mathcal{T}_h^i$. We have $\tilde{u}_i^I \in \widetilde{V}_h(\Omega_i)$ and (see [8])

$$||u - \tilde{u}_i^I||_{0,\Omega_i} + h_i|u - \tilde{u}_i^I|_{1,\Omega_i} \leq Ch_i^2|u|_{2,\Omega_i}. \tag{21}$$

The function $\tilde{u}^I = (\tilde{u}_1^I, \cdots, \tilde{u}_N^I) \in \widetilde{V}_h$ may not satisfy the mortar condition across the interfaces. From (3), we know that $u^I = \tilde{u}^I + \sum_{\delta_j \in \Gamma} \mathcal{E}_{\delta_j} \tilde{u}^I \in V_h$.

The following interpolation error estimates are proved in [3].

$$|u - u^I|_{1,h} \leq C\left(\sum_{i=1}^N h_i^2|u|_{2,\Omega_i}^2\right)^{\frac{1}{2}}; \qquad |u - u^I|_{0,\Omega} \leq C\left(\sum_{i=1}^N h_i^4|u|_{2,\Omega_i}^2\right)^{\frac{1}{2}}. \tag{22}$$

4 Error estimation

In this section, we establish the error estimate for the mortar upwind finite volume element method. For this purpose, we split the error into two parts: $u - u_h = \rho_h + e_h$, where $\rho_h = u - u^I$, $e_h = u^I - u_h$.

As an auxiliary tool, we introduce the following bilinear form:

$$\bar{a}^{(1)}(u, I_h^* v_h) = \sum_{i=1}^{N} \sum_{e \in E_{h,i}^{\text{in}}} v_h(m_e) \int_{\partial b_e} (b \cdot \mathbf{n}) u \mathrm{d}s$$
$$+ \sum_{\gamma_i = \delta_j \in \Gamma} \left(\sum_{e \subset \gamma_i} v_h(m_e) \int_{\partial b_e \setminus \gamma_i} b \cdot \mathbf{n} u \mathrm{d}s + \sum_{e \subset \delta_j} v_h(m_e) \int_{\partial b_e \setminus \delta_j} b \cdot \mathbf{n} u \mathrm{d}s \right).$$

From (6) and the definition of $\bar{a}^{(1)}(u, I_h^* v_h)$, we obtain

$$a^{(1)}(u, I_h^* v_h) = \bar{a}^{(1)}(u, I_h^* v_h) + \sum_{\gamma_i = \delta_j \in \Gamma} \left(\int_{\gamma_i} b \cdot \mathbf{n} u I_h^* v_h \mathrm{d}s + \int_{\delta_j} b \cdot \mathbf{n} u I_h^* v_h \mathrm{d}s \right). \tag{23}$$

Lemma 4.1. *For any $u \in H^1(\Omega)$ and $v_h \in V_h$, we have*

$$|\bar{a}^{(1)}(u, I_h^* v_h) - a_h^{(1)}(u, I_h^* v_h)| \le C \left(\sum_{i=1}^{N} h_i^2 ||u||_{2,\Omega_i}^2 \right)^{\frac{1}{2}} |v_h|_{1,h}. \tag{24}$$

Proof We introduce the function $H(x) = 0, x < 0$; $H(x) = 1, x \ge 0$, and rewrite the upwind value as follows:

$$\begin{aligned} \beta_{kl}^{+} u(m_{e_k}) - \beta_{kl}^{-} u(m_{e_l}) &= (H(\beta_{kl}) u(m_{e_k}) + (1 - H(\beta_{kl})) u(m_{e_l})) \beta_{kl} \\ &= \int_{\gamma_{kl}} b \cdot \mathbf{n} (H(\beta_{kl}) u(m_{e_k}) + (1 - H(\beta_{kl})) u(m_{e_l})) \mathrm{d}s. \end{aligned}$$

Since $\gamma_{kl} = \partial b_{e_k} \cap \partial b_{e_l}$, the integral along γ_{kl} in $a_h^{(1)}(u, I_h^* v_h)$ appears twice with opposite normal directions $\mathbf{n}$. Writing two such terms together, we obtain

$$\begin{aligned} a_h^{(1)}(u, I_h^* v_h) &= \frac{1}{2} \sum_{i=1}^{N} \sum_{e_k \in E_{h,i}} \sum_{l \in \Lambda_k} (v_h(m_{e_k}) - v_h(m_{e_l})) \\ &\quad \times \int_{\gamma_{kl}} b \cdot \mathbf{n} (H(\beta_{kl}) u(m_{e_k}) + (1 - H(\beta_{kl})) u(m_{e_l})) \mathrm{d}s. \end{aligned}$$

Similarly, we have

$$\bar{a}^{(1)}(u, I_h^* v_h) = \frac{1}{2} \sum_{i=1}^{N} \sum_{e_k \in E_{h,i}} \sum_{l \in \Lambda_k} (v_h(m_{e_k}) - v_h(m_{e_l})) \int_{\gamma_{kl}} b \cdot \mathbf{n} u \mathrm{d}s.$$

Set

$$E^{(1)}(u, I_h^* v_h) = \bar{a}^{(1)}(u, I_h^* v_h) - a_h^{(1)}(u, I_h^* v_h). \tag{25}$$

Thus,

$$\begin{aligned} E^{(1)}(u, I_h^* v_h) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{e_k\in E_{h,i}}\sum_{l\in\Lambda_k}(v_h(m_{e_k})-v_h(m_{e_l})) \\ &\quad\times\int_{\gamma_{kl}} b\cdot\mathbf{n}\{H(\beta_{kl})(u-u(m_{e_k}))+(1-H(\beta_{kl}))(u-u(m_{e_l}))\}\mathrm{d}s. \end{aligned}$$

It follows from the Cauchy-Schwarz inequality and Lemma 3.3 that

$$\begin{aligned} & |E^{(1)}(u, I_h^* v_h)| \\ \leq\ & \sum_{i=1}^{N}\sum_{e_k\in E_{h,i}}\sum_{l\in\Lambda_k}|v_h(m_{e_k})-v_h(m_{e_l})|\int_{\gamma_{kl}}|b\cdot\mathbf{n}|\left(|u-u(m_{e_k})|+|u-u(m_{e_l})|\right)\mathrm{d}s. \\ \leq\ & C|v_h|_{1,h}\left(\sum_{i=1}^{N}\sum_{e_k\in E_{h,i}}\sum_{l\in\Lambda_k}\left(\int_{\gamma_{kl}}|b\cdot\mathbf{n}|(|u-u(m_{e_k})|+|u-u(m_{e_l})|)\mathrm{d}s\right)^2\right)^{\frac{1}{2}} \\ \leq\ & C|v_h|_{1,h}\Bigg(\sum_{i=1}^{N}\sum_{e_k\in E_{h,i}}\sum_{l\in\Lambda_k}\left(\int_{\gamma_{kl}}|b\cdot\mathbf{n}||u-u(m_{e_k})|\mathrm{d}s\right)^2 \\ & \quad +\left(\int_{\gamma_{kl}}|b\cdot\mathbf{n}||u-u(m_{e_l})|\mathrm{d}s\right)^2\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{26}$$

From the Cauchy-Schwarz inequality and Lemma 3.4, we get

$$\begin{aligned} \left(\int_{\gamma_{kl}}|b\cdot\mathbf{n}||u-u(m_{e_k})|\mathrm{d}s\right)^2 &\leq \int_{\gamma_{kl}}|b\cdot\mathbf{n}|^2\mathrm{d}s\int_{\gamma_{kl}}|u-u(m_{e_k})|^2\mathrm{d}s \\ &\leq Ch_i\left(h_i^{-1}||u-u(m_{e_k})||_{0,K}^2+h_i|u|_{1,K}^2\right) \\ &\leq Ch_i^2||u||_{2,K}^2, \end{aligned} \tag{27}$$

where $\gamma_{kl}\subset K$ and $||u-u(m_{e_k})||_{0,K}\leq||u-u^I||_{0,K}+||u^I-I_h^*u^I||_{0,K}\leq Ch_K||u||_{2,K}$.

From (26) and (27), we get the desired result (24). This completes the proof. ■

The following Lemma 4.2 will be used in our later convergence analysis.

Lemma 4.2. *For $u\in H^2(\Omega)\cap H_0^1(\Omega), v_h\in V_h$, then there exists a constant independent of the mesh parameter and the number of the subdomains such that*

$$\begin{aligned} |I| &= \left|\sum_{\gamma_i=\delta_j\in\Gamma}\left(\int_{\gamma_i}(A\nabla u)\cdot\mathbf{n}I_h^*v_h\mathrm{d}s+\int_{\delta_j}(A\nabla u)\cdot\mathbf{n}I_h^*v_h\mathrm{d}s\right)\right| \\ &\leq C\left(\sum_{i=1}^{N}h_i^2||u||_{2,\Omega_i}^2\right)^{\frac{1}{2}}|v_h|_{1,h}. \end{aligned} \tag{28}$$

Proof From the triangle inequality we obtain

$$\begin{aligned}|I| &\leq \sum_{\gamma_i\in\Gamma}|\int_{\gamma_i}(A\nabla u)\cdot\mathbf{n}(I_h^*v_h|_{\gamma_i}-v_h|_{\gamma_i})\mathrm{d}s|+\sum_{\delta_j\in\Gamma}|\int_{\delta_j}(A\nabla u)\cdot\mathbf{n}(I_h^*v_h|_{\delta_j}-v_h|_{\delta_j})\mathrm{d}s| \\ &\quad+\sum_{\delta_j\in\Gamma}|\int_{\delta_j}(A\nabla u)\cdot\mathbf{n}(v_h|_{\gamma_i}-v_h|_{\delta_j})\mathrm{d}s| \\ &= R_1+R_2+R_3. \end{aligned} \tag{29}$$

The estimation for R_3 is given in the proof of Lemma 3.7 in [18]:

$$R_3 \leq C\left(\sum_{i=1}^{N}h_i^2||u||_{2,\Omega_i}^2\right)^{\frac{1}{2}}|v_h|_{1,h}. \tag{30}$$

Next, we estimate R_1. From the definitions of the function $I_h^*v_h|_{\gamma_i}$, we have

$$\int_{\gamma_i}(I_h^*v_h|_{\gamma_i}-v_h|_{\gamma_i})\mathrm{d}s=0,\quad I_h^*v_h|_{\gamma_i}=Q^{\gamma_i}v_h.$$

Based on this fact, by the Cauchy-Schwarz inequality and Lemma 2.1 we have

$$\begin{aligned}&|\int_{\gamma_i}(A\nabla u)\cdot\mathbf{n}(I_h^*v_h|_{\gamma_i}-v_h|_{\gamma_i})\mathrm{d}s| \\ =\ &|\int_{\gamma_i}(A\nabla u\cdot\mathbf{n}-Q^{\gamma_i}(A\nabla u\cdot\mathbf{n}))(Q^{\gamma_i}v_h|_{\gamma_i}-v_h|_{\gamma_i})\mathrm{d}s| \\ \leq\ &Ch_i||u||_{2,\Omega_i}|v_h|_{1,h,\Omega_i}. \end{aligned} \tag{31}$$

From the Cauchy-Schwarz inequality, we get

$$R_1 \leq C\left(\sum_{i=1}^{N}h_i^2||u||_{2,\Omega_i}^2\right)^{\frac{1}{2}}|v_h|_{1,h}. \tag{32}$$

A similar estimation also holds for R_2. Then, the desired result (28) follows from (29), (30) and (32). ■

From the proof of Lemma 4.2 we obtain the following result:

$$\left|\sum_{\gamma_i=\delta_j\in\Gamma}(\int_{\gamma_i}b\cdot\mathbf{n}uI_h^*v_h\mathrm{d}s+\int_{\delta_j}b\cdot\mathbf{n}uI_h^*v_h\mathrm{d}s)\right| \leq C(\sum_{i=1}^{N}h_i^2||u||_{2,\Omega_i}^2)^{\frac{1}{2}}|v_h|_{1,h}. \tag{33}$$

Now we introduce the linear functionals $\eta_{kl}(u)$:

$$\eta_{kl}(u)=-\int_{\gamma_{kl}}A\nabla(u^I-u)\cdot\mathbf{n}\mathrm{d}s,\quad \gamma_{kl}=b_{e_k}\cap b_{e_l},\quad e_k,e_l\in E_{h,i}. \tag{34}$$

For $e_k,e_l\in E_{h,i}$, since the restriction of u^I to any edge $\gamma_{kl}=b_{e_k}\cap b_{e_l}$ is the standard interpolation function, from Lemma 3.4 and the interpolation error estimates, we get the following lemma:

Lemma 4.3. *If $u\in H^2(\Omega)$, then there is a constant C independent of h such that for $\gamma_{kl}\subset K, K\in\mathcal{T}_h$,*

$$|\eta_{kl}(u)|\leq Ch||A||_{0,\infty}|u|_{2,K}.$$

Lemma 4.4. *If $u \in H^2(\Omega)$, then there exists a constant C independent of u and h such that*

$$|a_h^{(2)}(u-u^I, I_h^* v_h)| \leq C\left(\sum_{i=1}^N h_i^2 |u|_{2,\Omega_i}^2\right)^{1/2} |v_h|_{1,h}, \quad \forall v_h \in V_h.$$

Proof From the definitions of the bilinear form $a_h(\cdot,\cdot)$ and the linear functionals $\eta_{kl}(u)$ we obtain

$$\begin{aligned} a_h^{(2)}(u-u^I, I_h^* v_h) &= -\sum_{K\in\mathcal{T}_h}\sum_{e_k,e_l\in E(K)} v_h(m_{e_k})\int_{\partial b_{e_k}\cap\partial b_{e_l}} A\nabla(u-u^I)\cdot\mathbf{n}\mathrm{d}s \\ &= \frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{e_k,e_l\in E(K)}(\eta_{kl}(u)v_h(m_{e_k})+\eta_{lk}(u)v_h(m_{e_l})) \\ &= \frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{e_k,e_l\in E(K)}\eta_{kl}(u)(v_h(m_{e_k})-v_h(m_{e_l})). \end{aligned} \tag{35}$$

It then follows from the Cauchy-Schwarz inequality and from Lemmas 3.3 and 4.3 that

$$\begin{aligned} |a_h^{(2)}(u-u^I, I_h^* v_h)| &\leq C\left(\sum_{K\in\mathcal{T}_h}\sum_{e_k,e_l\in E(K)}\eta_{kl}^2\right)^{\frac{1}{2}}\cdot\left(\sum_{K\in\mathcal{T}_h}\sum_{e_k,e_l\in E(K)}(v_h(m_{e_k})-v_h(m_{e_l}))^2\right)^{\frac{1}{2}} \\ &\leq C\left(\sum_{i=1}^N h_i^2|u|_{2,\Omega_i}^2\right)^{1/2}|v_h|_{1,h}. \end{aligned}$$

This completes the proof. ■

Theorem 4.1. *Assume that u and u_h are the solutions of (4) and (7), respectively. Then, there exists a constant C independent of h and the number of the subdomains such that*

$$\begin{aligned} &||u-u_h||_0^2 + \int_0^t ||u-u_h||_{1,h}^2 \mathrm{d}s \\ &\leq C\sum_{i=1}^N h_i^2\left(||u||_{2,\Omega_i}^2 + \int_0^t ||u(s)||_{2,\Omega_i}^2\mathrm{d}s + \int_0^t ||u_t(s)||_{2,\Omega_i}^2\mathrm{d}s\right). \end{aligned} \tag{36}$$

Proof From (4), (7) and (23), we get

$$\begin{aligned} &\left(\frac{\partial e_h}{\partial t}, I_h^* v_h\right) + a_h^{(2)}(e_h, I_h^* v_h) + a_h^{(1)}(e_h, I_h^* v_h) \\ = \;& ((u^I)_t - u_t, I_h^* v_h) - a_h^{(2)}(\rho_h, I_h^* v_h) + a_h^{(1)}(u^I - u, I_h^* v_h) + \left(a_h^{(1)}(u, I_h^* v_h) - \bar{a}^{(1)}(u, I_h^* v_h)\right) \\ &+ \sum_{\gamma_i=\delta_j\in\Gamma}\left(\int_{\gamma_i}(A\nabla u)\cdot\mathbf{n}I_h^* v_h\mathrm{d}s + \int_{\delta_j}(A\nabla u)\cdot\mathbf{n}I_h^* v_h\mathrm{d}s\right) \\ &- \sum_{\gamma_i=\delta_j\in\Gamma}\left(\int_{\gamma_i}(b\cdot\mathbf{n})uI_h^* v_h\mathrm{d}s + \int_{\delta_j}(b\cdot\mathbf{n})uI_h^* v_h\mathrm{d}s\right) \\ = \;& J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned} \tag{37}$$

Using the Cauchy-Schwarz inequality and the interpolation error estimate (22), we obtain

$$|J_1| \leq C\left(\sum_{i=1}^{N} h_i^4 ||u||_{2,\Omega}^2\right)^{\frac{1}{2}} ||v_h||_0 \leq C\left(\sum_{i=1}^{N} h_i^4 ||u||_{2,\Omega}^2\right)^{\frac{1}{2}} |v_h|_{1,h}, \tag{38}$$

where the equivalence of $||I_h^* v_h||_0$ and $||v_h||_0$ and the Poincare inequality are used.

By means of Lemma 4.4, we get the estimation for J_2 :

$$|J_2| \leq C\left(\sum_{i=1}^{N} h_i^4 ||u||_{2,\Omega}^2\right)^{\frac{1}{2}} |v_h|_{1,h}. \tag{39}$$

Next, we estimate J_3. If $e_k, e_l \in E_{h,i}^{\text{in}} \cup \gamma_i$, $(u-u^I)(m_{e_k}) = 0$, $(u-u^I)(m_{e_l}) = 0$, then $\beta_{kl}^{+}(u-u^I)(m_{e_k}) - \beta_{kl}^{-}(u-u^I)(m_{e_l}) = 0$. Therefore,

$$-J_3 = \sum_{\delta_j \in \Gamma} \sum_{e_k \subset \delta_j} v_h(m_{e_k}) \sum_{l \in \Lambda_k} \{\beta_{kl}^{+}(u-u^I)(m_{e_k}) - \beta_{kl}^{-}(u-u^I)(m_{e_l})\}. \tag{40}$$

We know from the definition of the interpolation operator that, if $e_k \subset \delta_j$, $(u-u^I)(m_{e_k}) = \mathcal{E}_{\delta_j}\tilde{u}^I(m_{e_k})$; $l \in \Lambda_k$, $e_l \in E_{h,j}^{\text{in}}$, $(u-u^I)(m_{e_l}) = 0$. Since $\gamma_{kl} = \partial b_{e_k} \cap \partial b_{e_l}$, the integral along γ_{kl} in (40) appears twice with opposite normal directions $\mathbf{n}$. Write such two terms together to obtain

$$\begin{aligned} -J_3 &= \frac{1}{2}\sum_{\delta_j \in \Gamma} \sum_{e_k \subset \delta_j} \sum_{l \in \Lambda_k, e_l \in E_{h,j}^{\text{in}}} (v_h(m_{e_k}) - v_h(m_{e_l}))\beta_{kl}^{+}(u-u^I)(m_{e_k}) \\ &= \frac{1}{2}\sum_{\delta_j \in \Gamma} \sum_{e_k \subset \delta_j} \sum_{l \in \Lambda_k, e_l \in E_{h,j}^{\text{in}}} (v_h(m_{e_k}) - v_h(m_{e_l}))\int_{\gamma_{kl}} b\cdot\mathbf{n}H(\beta_{kl})(\mathcal{E}_{\delta_j}\tilde{u}^I)(m_{e_k})\mathrm{d}s. \end{aligned}$$

From the Cauchy-Schwarz inequality and Lemma 3.3 we have

$$\begin{aligned} |J_3| &\leq C\sum_{\delta_j \in \Gamma} \sum_{e_k \subset \delta_j} \sum_{l \in \Lambda_k, e_l \in E_{h,j}^{\text{in}}} |v_h(m_{e_k}) - v_h(m_{e_l})||\mathcal{E}_{\delta_j}\tilde{u}^I(m_{e_k})|h_j \\ &\leq C|v_h|_{1,h}\left(\sum_{\delta_j \in \Gamma} \sum_{e_k \subset \delta_j} h_j^2(\mathcal{E}_{\delta_j}\tilde{u}^I(m_{e_k}))^2\right)^{\frac{1}{2}}. \end{aligned} \tag{41}$$

Let $Q_e^{\delta_j}$ be the $L^2(e)$-orthogonal projection operator onto the one-dimensional space of constant functions on e. By the definition of $\mathcal{E}_{\delta_j}$ and $Q_e^{\delta_j}$, we easily get

$$\begin{aligned} &(\mathcal{E}_{\delta_j}\tilde{u}^I(m_e))^2 = (Q_e^{\delta_j}((\tilde{u}^I)_{\gamma_i}^M - (\tilde{u}^I)_{\delta_j}^{NM})(m_e))^2 \\ = \; &(\frac{1}{|e|}\int_e((\tilde{u}^I)_{\gamma_i}^M - (\tilde{u}^I)_{\delta_j}^{NM})\mathrm{d}s)^2 \leq \frac{1}{|e|}\int_e((\tilde{u}^I)_{\gamma_i}^M - (\tilde{u}^I)_{\delta_j}^{NM})^2\mathrm{d}s. \end{aligned} \tag{42}$$

It then follows from the triangle inequality, Lemma 3.4 and (22) that

$$\begin{aligned} &\sum_{\delta_j \in \Gamma} \sum_{e_k \subset \delta_j} h_j^2(\mathcal{E}_{\delta_j}\tilde{u}^I(m_{e_k}))^2 \leq C\sum_{\delta_j \in \Gamma} h_j \int_{\delta_j}((\tilde{u}^I)_{\gamma_i}^M - (\tilde{u}^I)_{\delta_j}^{NM})^2 ds \\ \leq \; &C\sum_{\delta_j \in \Gamma} h_j\left(||(\tilde{u}^I)_{\gamma_i}^M - u||_{0,\delta_j}^2 + ||u - (\tilde{u}^I)_{\delta_j}^{NM}||_{0,\delta_j}^2\right) \leq C\sum_{i=1}^{N} h_i^4|u|_{2,\Omega_i}^2. \end{aligned} \tag{43}$$

Thus, by (40), (41) and (43), we obtain

$$|J_3| \le C \left(\sum_{i=1}^{N} h_i^2 |u|_{2,\Omega_i}^2 \right)^{\frac{1}{2}} |v_h|_{1,h}. \tag{44}$$

By means of (38), (39), (44), employing Lemma 4.1 to estimate J_4, Lemma 4.2 to estimate J_5 and (33) to estimate J_6, taking $v_h = e_h$ in (37), from (16), (12), Lemma 3.1 and the ε-inequality, we obtain

$$\frac{\partial}{\partial t}|||e_h|||_0^2 + \alpha_0 ||e_h||_{1,h}^2 \le C \sum_{i=1}^{N} h_i^2 \left(||u||_{2,\Omega_i}^2 + ||u_t||_{2,\Omega_i}^2 \right). \tag{45}$$

Integrating the above inequality, and noting that $e_h^0 = 0$ leads to

$$||e_h||_0^2 + \int_0^t ||e_h(t)||_{1,h}^2 \mathrm{d}t \le C \sum_{i=1}^{N} h_i^2 (\int_0^t ||u(s)||_{2,\Omega_i}^2 \mathrm{d}s + \int_0^t ||u_t(s)||_{2,\Omega_i}^2 \mathrm{d}s), \tag{46}$$

where the equivalence $|||e_h|||_0$ and $||e_h||_0$ has been used.

Combining (46) with the interpolation error estimate (22) yields the desired result (36). ■

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