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Algorithms for Finding the Inverses of Factor Block Circulant Matrices†

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Abstract. In this paper, algorithms for finding the inverse of a factor block circulant matrix, a factor block retrocirculant matrix and partitioned matrix with factor block circulant blocks over the complex field are presented respectively. In addition, two algorithms for the inverse of a factor block circulant matrix over the quaternion division algebra are proposed.

Key words: Inverses matrix; factor block circulant matrix; partitioned matrix; factor block retrocirculant matrix.

AMS subject classifications: 15A21, 65F15

1 Introduction

Factor block circulant matrices arise in diverse fields of applications [1–3], especially on the differential equations involving circulant matrices. So, computing the inverse of the factor block circulant matrix has become an important problem. In order to solve differential equations involving circulants, we consider in this work the inverses of factor block circulants over the complex field and the quaternion division algebra.

In Section 1, a computation formula for the inverse of a factor block circulant matrix over the complex field is presented by utilizing only the interpolation methods and basic properties of matrix. A remarkable character of the method needs neither the diagonalization method of a factor block circulant matrix nor the theory of the Jordan canonical form.

In Section 2, a computation formula for the inverse of partitioned matrix with factor block circulant blocks over the complex field is presented by using Schur complements.

In Section 3, we consider a new kind of matrices which are factor block circulant matrices over the quaternion division algebra and give a sufficient and necessary condition to determine

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whether a factor block circulant matrix is singular or not and propose two algorithms for the inverse of a factor block circulant matrix over the quaternion division algebra.

In Section 4, by utilizing only the relationship between a factor block retrocirculant matrix and a factor block circulant matrix, a computation formula for the inverse of a factor block retrocirculant matrix over the complex field is presented.

Definition 1.1. Let $C_1, C_2, \cdots, C_m, A$ be square matrices each of order n over the complex field $\mathbb{C}$. We assume that A is nonsingular and that it commutes with each of the C_k's. By an A-factor block circulant matrix of type (m, n) over the complex field $\mathbb{C}$ is meant an $mn \times mn$ matrix of the form

$$\Re = \mathrm{circ}_A(C_1, C_2, \cdots, C_m) = \begin{pmatrix} C_1 & C_2 & \cdots & C_{m-1} & C_m \\ AC_m & C_1 & \cdots & C_{m-2} & C_{m-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ AC_3 & AC_4 & \cdots & C_1 & C_2 \\ AC_2 & AC_3 & \cdots & AC_m & C_1 \end{pmatrix}.$$

We define π_A as the basic A- factor circulant over $\mathbb{C}$, that is,

$$\pi_A = \begin{pmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I \\ A & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

The following useful results are well known [1].

Lemma 1.1. *$\Re$ is an A- factor block circulant matrix over $\mathbb{C}$ if and only if $\Re = \mathcal{F}(\pi_A)$ for some matrix polynomial $\mathcal{F}(z)$. The polynomial $\mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1} z^k$ will be called the representer of the factor circulant over $\mathbb{C}$.*

Lemma 1.2. *Two A-factor circulants over $\mathbb{C}$ $\mathcal{B} = circ_A(B_1, \cdots, B_m)$, $\Re = circ_A(C_1, \cdots, C_m)$ commute if the B_j's commute with the C_j's.*

Lemma 1.3. *Let $\Re$ be an A- factor block circulant over $\mathbb{C}$. Then*

$$\Re = V_A \mathcal{F}(D_A) V_A^{-1},$$

where

$$V_A = V_n(K, \omega K, \ldots, \omega^{m-1} K), \quad \mathcal{F}(D_A) = \mathrm{diag}[\mathcal{F}(K), \mathcal{F}(\omega K), \ldots, \mathcal{F}(\omega^{m-1} K)],$$
$$\omega = \exp(2\pi i/m), \quad \mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1} z^k.$$

Lemma 1.4. *The inverse matrix $\Re^{-1}$ of a nonsingular factor block circulant matrix $\Re$ over $\mathbb{C}$ is also a factor block circulant matrix of the same type.*

Lemma 1.5. *Let K denote the principal mth root of the nonsingular matrix A over $\mathbb{C}$. Then $V_n(K, \omega K, \ldots, \omega^{m-1} K)$ is nonsingular, and its inverse equals*

$$F_{mn} X^{-1}/\sqrt{m} = \frac{1}{m}[V_n(K^{-1}, \varpi K^{-1}, \ldots, \varpi^{m-1} K^{-1})]^T,$$

where

$$X = \mathrm{diag}[I, K, K^2, \ldots, K^{m-1}].$$

2 Inverse of factor block circulant matrices over the complex field $\mathbb{C}$

Theorem 2.1. *Let $\Re = circ_A(C_1, C_2, \ldots, C_m)$ be a nonsingular A-factor block circulant matrix over $\mathbb{C}$, if $\Re^{-1} = circ_A(B_1, B_2, \ldots, B_m) = \sum_{j=0}^{m-1} B_{j+1}\pi_A^j$, then*

$$\mathcal{G}(\omega^i K) = \mathcal{F}(\omega^i K)^{-1}, i = 0, 1, \ldots, m-1.$$

where $\omega = \exp(2\pi i/m), \mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1}z^k$ is the representer of the A-factor block circulant matrix $\Re, \mathcal{G}(z) = \sum_{j=0}^{m-1} B_{j+1}z^j$ is the representer of the A-factor block circulant matrix $\Re^{-1}$, K denotes the principal mth root of the nonsingular matrix A.

Proof. Since $\Re\Re^{-1} = I_{mn}$, by Lemmas 1.3 and 1.4, we obtain

$$[V_A\mathcal{F}(D_A)V_A^{-1}][V_A\mathcal{G}(D_A)V_A^{-1}] = I_{mn},$$

$$\mathrm{diag}[\mathcal{F}(K), \mathcal{F}(\omega K), \ldots, \mathcal{F}(\omega^{m-1}K)]\mathrm{diag}[\mathcal{G}(K), \mathcal{G}(\omega K), \ldots, \mathcal{G}(\omega^{m-1}K)] = I_{mn}.$$

Therefore,

$$\mathrm{diag}[\mathcal{F}(K)\mathcal{G}(K), \mathcal{F}(\omega K)\mathcal{G}(\omega K), \ldots, \mathcal{F}(\omega^{m-1}K)\mathcal{G}(\omega^{m-1}K)] = \mathrm{diag}[I_n, \ldots, I_n].$$

Then $\mathcal{F}(\omega^i K)\mathcal{G}(\omega^i K) = I_n, i = 0, \ldots, m-1$. This implies $\mathcal{G}(\omega^i K) = \mathcal{F}(\omega^i K)^{-1}, i = 0, \ldots, m-1$. ■

Theorem 2.2. *Let $\Re = circ_A(C_1, C_2, \ldots, C_m)$ be a nonsingular A-factor block circulant matrix over $\mathbb{C}$, if $\Re^{-1} = circ_A(B_1, B_2, \ldots, B_m) = \sum_{j=0}^{m-1} B_{j+1}\pi_A^j$, then*

$$B_{j+1} = \frac{1}{m}\sum_{k=0}^{m-1}(\omega^k K)^{-j}[\mathcal{F}(\omega^k K)]^{-1}, j = 0, 1, \ldots, m-1. \tag{1}$$

where $\omega = \exp(2\pi i/m), \mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1}z^k$ is the representer of the factor circulant $\Re$, and K denotes the principal mth root of the nonsingular matrix A.

Proof Let the representer of $\Re^{-1}$ be

$$\mathcal{G}(x) = B_1 + B_2x + \ldots + B_mx^{m-1}. \tag{2}$$

Replacing x in the equation (2) with $K, \omega K, \ldots, \omega^{m-1}K$, respectively, we obtain the following system of equations

$$\begin{cases} B_1 + B_2K + \ldots + B_mK^{m-1} = \mathcal{G}(K) \\ B_1 + B_2\omega K + \ldots + B_m(\omega K)^{m-1} = \mathcal{G}(\omega K) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B_1 + B_2\omega^{m-1}K + \ldots + B_m(\omega^{m-1}K)^{m-1} = \mathcal{G}(\omega^{m-1}K), \end{cases}$$

which is equivalent to

$$\begin{pmatrix} 1 & K & \ldots & K^{m-1} \\ 1 & \omega K & \ldots & (\omega K)^{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{m-1}K & \ldots & (\omega^{m-1}K)^{m-1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix} = \begin{pmatrix} \mathcal{G}(K) \\ \mathcal{G}(\omega K) \\ \vdots \\ \mathcal{G}(\omega^{m-1}K) \end{pmatrix} \tag{3}$$

The coefficient matrix of the system of equations (3) is precisely $[V_n(K, \omega K, \ldots, \omega^{m-1}K)]^T$,where $V_n(K, \omega K, \ldots, \omega^{m-1}K)$ denotes the block Vandermonde matrix of the $\omega^k K$'s. From Lemma 1.5, we know that $V_n(K, \omega K, \ldots, \omega^{m-1}K)$ is nonsingular, then $[V_n(K, \omega K, \ldots, \omega^{m-1}K)]^T$ is nonsingular. Thus, the system of equations (3) will have the unique solution $B_1, B_2, \ldots, B_m$. By the system of equations (3), we have

$$\begin{aligned}
\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix} &= \begin{pmatrix} 1 & K & \ldots & K^{m-1} \\ 1 & \omega K & \ldots & (\omega K)^{m-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{m-1}K & \ldots & (\omega^{m-1}K)^{m-1} \end{pmatrix} \begin{pmatrix} \mathcal{G}(K) \\ \mathcal{G}(\omega K) \\ \vdots \\ \mathcal{G}(\omega^{m-1}K) \end{pmatrix} \\
&= \{[V_n(K, \omega K, \ldots, \omega^{m-1}K)]^T\}^{-1} \begin{pmatrix} \mathcal{G}(K) \\ \mathcal{G}(\omega K) \\ \vdots \\ \mathcal{G}(\omega^{m-1}K) \end{pmatrix} \\
&= \{[V_n(K, \omega K, \ldots, \omega^{m-1}K)]^{-1}\}^T \begin{pmatrix} \mathcal{G}(K) \\ \mathcal{G}(\omega K) \\ \vdots \\ \mathcal{G}(\omega^{m-1}K) \end{pmatrix}.
\end{aligned}$$

By Lemma 1.5, we have

$$\begin{aligned}
\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix} &= \{\frac{1}{m}[V_n(K^{-1}, \omega K^{-1}, \ldots, \omega^{m-1}K^{-1})]^T\}^T \begin{pmatrix} \mathcal{G}(K) \\ \mathcal{G}(\omega K) \\ \vdots \\ \mathcal{G}(\omega^{m-1}K) \end{pmatrix} \\
&= \frac{1}{m}[V_n(K^{-1}, \varpi K^{-1}, \ldots, \varpi^{m-1}K^{-1})] \begin{pmatrix} \mathcal{G}(K) \\ \mathcal{G}(\omega K) \\ \vdots \\ \mathcal{G}(\omega^{m-1}K) \end{pmatrix}.
\end{aligned}$$

By Theorem 2.1, we have

$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix} = \frac{1}{m}[V_n(K^{-1}, \varpi K^{-1}, \ldots, \varpi^{m-1}K^{-1})] \begin{pmatrix} \mathcal{F}(K)^{-1} \\ \mathcal{F}(\omega K)^{-1} \\ \vdots \\ \mathcal{F}(\omega^{m-1}K)^{-1} \end{pmatrix} \tag{4}$$

Multiplying the $(j+1)$th row of the $\frac{1}{m}[V_n(K^{-1}, \varpi K^{-1}, \ldots, \varpi^{m-1}K^{-1})]$ by

$$(\mathcal{F}(K)^{-1}, \mathcal{F}(\omega K)^{-1}, \ldots, \mathcal{F}(\omega^{m-1}K)^{-1})^T$$

respectively in the system of equations (4), we have

$$B_{j+1} = \frac{1}{m}\sum_{k=0}^{m-1}(\omega^k K)^{-j}[\mathcal{F}(\omega^k K)]^{-1}, \qquad j = 0, 1, \ldots, m-1.$$

This completes the proof of this theorem. ■

Let $\Re = \mathrm{circ}_A(C_1, C_2, \ldots, C_m)$ be a nonsingular A-factor block circulant matrix over $\mathbb{C}$, by Theorem 2.2, we have the following algorithm which can find the inverse of the matrix $\Re$:

- **Step 1.** Find out the principal mth root K of the nonsingular matrix A.
- **Step 2.** By $\mathcal{F}(z)=\sum_{k=0}^{m-1} C_{k+1}z^k$ for computing $\mathcal{F}(\omega^k K), k=0,1,\ldots,m-1$, respectively.
- **Step 3.** By Step 2 for computing $[\mathcal{F}(\omega^k K)]^{-1}, k=0,1,\ldots,m-1$, respectively.
- **Step 4.** By Equation(1) for computing $B_{j+1}, j=0,1,\ldots,m-1$, respectively, we have

$$\Re^{-1}=\mathrm{circ}_A(B_1,B_2,\ldots,B_m).$$

3 Inverse of partitioned matrix with factor block circulant blocks over $\mathbb{C}$

Let $\Re_1,\Re_2,\Re_3,\Re_4$ be A-factor block circulant matrices over $\mathbb{C}$. If $\Re_1$ is nonsingular, and if let

$$\Omega=\begin{pmatrix}\Re_1 & \Re_2\\ \Re_3 & \Re_4\end{pmatrix}, \mathcal{H}_1=\begin{pmatrix} I & 0\\ -\Re_3\Re_1^{-1} & I\end{pmatrix}, \mathcal{H}_2=\begin{pmatrix} I & -\Re_1^{-1}\Re_2\\ 0 & I\end{pmatrix},$$

then

$$\mathcal{H}_1\Omega\mathcal{H}_2=\begin{pmatrix}\Re_1 & 0\\ 0 & \Re_4-\Re_3\Re_1^{-1}\Re_2\end{pmatrix}. \tag{5}$$

So Ω is nonsingular if and only if $\Re_4-\Re_3\Re_1^{-1}\Re_2$ is nonsingular. Furthermore, if Ω is nonsingular, by equation (5), we have

$$\begin{aligned}\Omega^{-1} &= \mathcal{H}_2\begin{pmatrix}\Re_1^{-1} & 0\\ 0 & (\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\end{pmatrix}\mathcal{H}_1\\
&= \begin{pmatrix} I & -\Re_1^{-1}\Re_2\\ 0 & I\end{pmatrix}\begin{pmatrix}\Re_1^{-1} & 0\\ 0 & (\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\end{pmatrix}\begin{pmatrix} I & 0\\ -\Re_3\Re_1^{-1} & I\end{pmatrix}\\
&= \begin{pmatrix}\Re_1^{-1} & -\Re_1^{-1}\Re_2(\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\\ 0 & (\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\end{pmatrix}\begin{pmatrix} I & 0\\ -\Re_3\Re_1^{-1} & I\end{pmatrix}\\
&= \begin{pmatrix}\Re_1^{-1}+\Re_1^{-1}\Re_2(\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\Re_3\Re_1^{-1} & -\Re_1^{-1}\Re_2(\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\\ -(\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\Re_3\Re_1^{-1} & (\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\end{pmatrix}.\end{aligned}$$

Therefore, we have

Theorem 3.1. *Let*

$$\Omega=\begin{pmatrix}\Re_1 & \Re_2\\ \Re_3 & \Re_4\end{pmatrix},$$

where $\Re_1,\Re_2,\Re_3,\Re_4$ are all A-factor block circulant matrices over $\mathbb{C}$. If $\Re_1$ is nonsingular, then Ω is nonsingular if and only if $\Re_4-\Re_3\Re_1^{-1}\Re_2$ is nonsingular. Moreover, if Ω is nonsingular, then

$$\Omega^{-1}=\begin{pmatrix}\Re_1^{-1}+\Re_1^{-1}\Re_2(\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\Re_3\Re_1^{-1} & -\Re_1^{-1}\Re_2(\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\\ -(\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\Re_3\Re_1^{-1} & (\Re_4-\Re_3\Re_1^{-1}\Re_2)^{-1}\end{pmatrix}. \tag{6}$$

In particular, we have

Corollary 3.1. *Let $\Re_i = circ_A(C_{i1}, C_{i2}, \ldots, C_{im})$,with each C_{ik} and a-factor scalar circulant over $\mathbb{C}$, for $i = 1, 2, 3, 4$, and let us assume that A is an a-factor circulant over $\mathbb{C}$. Let*

$$\Omega = \begin{pmatrix} \Re_1 & \Re_2 \\ \Re_3 & \Re_4 \end{pmatrix}.$$

If $\Re_1$ is nonsingular, then Ω is nonsingular if and only if $\Re_1\Re_4 - \Re_2\Re_3$ is nonsingular. Moreover, if Ω is nonsingular, then

$$\Omega^{-1} = \begin{pmatrix} \Re_1^{-1} + (\Re_1\Re_4 - \Re_2\Re_3)^{-1}\Re_2\Re_3\Re_1^{-1} & -(\Re_1\Re_4 - \Re_2\Re_3)^{-1}\Re_2 \\ -(\Re_1\Re_4 - \Re_2\Re_3)^{-1}\Re_3 & (\Re_1\Re_4 - \Re_2\Re_3)^{-1}\Re_1 \end{pmatrix}. \tag{7}$$

Proof Now since each C_{ik} is a-factor scalar circulant, then the C_{ik}'s commute with the C_{jk}'s if $i \neq j$, for $i, j = 1, 2, 3, 4$ and $k = 1, 2, \ldots, m$. By Lemma 1.2, we obtain that the $\Re_i$ commutes with the $\Re_j$ if $i \neq j$ for $i, j = 1, 2, 3, 4$. Thus

$$\Re_1(\Re_4 - \Re_3\Re_1^{-1}\Re_2) = \Re_1\Re_4 - \Re_2\Re_3. \tag{8}$$

By Theorem 3.1 and equation (8), we conclude that Ω is nonsingular if and only if $\Re_1\Re_4 - \Re_2\Re_3$ is nonsingular and the validity of equation (7) is proved. ■

Using the proof similar to that of Theorem 3.1, we can obtain the following conclusion.

Theorem 3.2. *Let*

$$\Omega = \begin{pmatrix} \Re_1 & \Re_2 \\ \Re_3 & \Re_4 \end{pmatrix}, \tag{9}$$

where $\Re_1, \Re_2, \Re_3, \Re_4$ are all A-factor block circulant matrices over $\mathbb{C}$. If $\Re_4$ is nonsingular, then Ω is nonsingular if and only if $\Re_1 - \Re_2\Re_4^{-1}\Re_3$ is nonsingular. Moreover, if Ω is nonsingular, then

$$\Omega^{-1} = \begin{pmatrix} (\Re_1 - \Re_2\Re_4^{-1}\Re_3)^{-1} & -(\Re_1 - \Re_2\Re_4^{-1}\Re_3)^{-1}\Re_2\Re_4^{-1} \\ -\Re_4^{-1}\Re_3(\Re_1 - \Re_2\Re_4^{-1}\Re_3)^{-1} & \Re_4^{-1}\Re_3(\Re_1 - \Re_2\Re_4^{-1}\Re_3)^{-1}\Re_2\Re_4^{-1} + \Re_4^{-1} \end{pmatrix}. \tag{10}$$

4 Factor block circulant matrices over quaternion division algebra

Let $\mathbb{F}$ be a field, and $\mathbb{D} = \mathbb{F}[i, j, k] = \{a + bi + cj + dk | a, b, c, d \in \mathbb{F}\}$ a quaternion division algebra over the field $\mathbb{F}$, and suppose that $1, i, j, k$ is a basis of $\mathbb{D}$ as a vector space over the field $\mathbb{F}$, where i, j, k are elements in $\mathbb{D}$ such that $i^2 = j^2 = k^2 = -1, ij = k = -ji, ki = j = -ik$ and $jk = i = -kj$. It is easy to show that there exist matrices $\Re_0, \Re_1, \Re_2, \Re_3$ over the field $\mathbb{F}$ such that $\Re = \Re_0 + i\Re_1 + j\Re_2 + k\Re_3$ for a matrix $\Re$ over $\mathbb{D}$.

Definition 4.1. Let $C_1, C_2, \ldots, C_m, A$ be square matrices each of order n over $\mathbb{D}$. We assume that A is nonsingular and that it commutes with each of the C_k's. By an *A- factor block circulant matrix* of type (m, n) over the quaternion division algebra $\mathbb{D}$ is meant an $mn \times mn$ matrix of the form

$$\Re = \text{circ}_A(C_1, C_2, \ldots, C_m) = \begin{pmatrix} C_1 & C_2 & \ldots & C_{m-1} & C_m \\ AC_m & C_1 & \ldots & C_{m-2} & C_{m-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ AC_3 & AC_4 & \ldots & C_1 & C_2 \\ AC_2 & AC_3 & \ldots & AC_m & C_1 \end{pmatrix}.$$

We define π_A as the basic A-factor circulant over the quaternion division algebra $\mathbb{D}$, that is,

$$\pi_A = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I \\ A & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

This block matrix can be conveniently written as

$$\pi_A = (E_2, E_3, \dots, E_m, AE_1)^T = (Ae_m, e_1, e_2, \dots, e_{m-1}),$$

where $E_j = (0, 0, \dots, I, \dots, 0), e_j = \mathrm{col}(0, 0, \dots, I, \dots, 0)$ are the jth unit row and column block matrix respectively. The powers of π_A are easily seen to be

$$(E_{k+1}, \dots, E_m, AE_1 \dots, AE_k)^T = \begin{cases} (Ae_{m-k+1}, \dots, Ae_m, e_1, \dots, e_{m-k}), k = 1, \dots, m-1, \\ AI_{mn}, k = m, \\ A^q\pi_A^p, k = qm + p, p = 1, \dots, m-1, q = 1, 2, \dots, \end{cases}$$

where I_{mn} denotes the $mn \times mn$ identity matrix, and the product of a square matrix with a block matrix is to be understood as the block matrix obtained by multiplying the square matrix with each matrix component of the given block matrix. These powers can be visualized as follows. The matrix A which occupies the lower corner entry moves up, occupying each entry of the kth lower subdiagonal, while the nonzero upper subdiagonal shrinks into the next one. For $k \geq m$ there is a cyclic reproduction of the above, times a power of A.

In view of the structure of the powers of the basic factor circulant π_A over $\mathbb{D}$, it is clear that

$$\Re = \mathrm{circ}_A(C_1, C_2, \dots, C_m) = C_1 + C_2\pi_A + \dots + C_m\pi_A^{m-1}. \tag{11}$$

Thus, $\Re$ is an A-factor circulant matrix over $\mathbb{D}$ if and only if $\Re = \mathcal{F}(\pi_A)$ for some matrix polynomial $\mathcal{F}(z)$. The polynomial $\mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1}z^k$ will be called the representer of the factor circulant over $\mathbb{D}$.

Clearly, we have

Theorem 4.1. *$\Re = \Re_0 + i\Re_1 + j\Re_2 + k\Re_3$ is a factor block circulant matrix over $\mathbb{D}$ if and only if $\Re_0, \Re_1, \Re_2, \Re_3$ are all factor block circulant matrices over the field $\mathbb{F}$.*

Theorem 4.2. *The inverse matrix $\Re^{-1}$ of a nonsingular factor block circulant matrix $\Re$ over $\mathbb{D}$ is also a factor block circulant matrix of the same type.*

Proof From representation (9), we have $\Re = C_1 + C_2\pi_A + \dots + C_m\pi_A^{m-1}$. and the inverse matrix $\Re^{-1}$ of a nonsingular factor block circulant matrix $\Re$ over $\mathbb{D}$ is also a factor block circulant matrix of the same type if and only if there exists $B_1, B_2, \dots, B_m$ over $\mathbb{D}$ such that

$$\Re^{-1} = B_1 + B_2\pi_A + \dots + B_m\pi_A^{m-1}. \tag{12}$$

Since $\Re\Re^{-1} = I$ and $\pi_A^{m+k} = A\pi_A^k$, then

$$\begin{aligned} \Re\Re^{-1} &= (C_1 + C_2\pi_A + \dots + C_m\pi_A^{m-1})(B_1 + B_2\pi_A + \dots + B_m\pi_A^{m-1}) \\ &= D_1 + D_2\pi_A + \dots + D_m\pi_A^{m-1} = I. \end{aligned}$$

if and only if

$$\begin{cases} D_m = C_1B_m + C_2B_{m-1} + \dots + C_{m-1}B_2 + C_mB_1 = 0, \\ D_{m-1} = AC_mB_m + C_1B_{m-1} + \dots + C_{m-2}B_2 + C_{m-1}B_1 = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ D_2 = AC_3B_m + AC_4B_{m-1} + \dots + C_1B_2 + C_2B_1 = 0, \\ D_1 = AC_2B_m + AC_3B_{m-1} + \dots + AC_mB_2 + C_1B_1 = I_n, \end{cases}$$

if and only if $\Re(B_m, \ldots, B_2, B_1)^T = (0, \ldots, 0, I_n)^T$. Since $\Re$ is nonsingular, so

$$(B_m, \ldots, B_2, B_1)^T = \Re^{-1}(0, \ldots, 0, I_n)^T. \tag{13}$$

By the above system of equations (11), the existence of $B_1, B_2, \ldots, B_m$ in the system of equations (10) has been proved. ■

Theorem 4.3. *Let $\Re$ be a factor block circulant matrix over $\mathbb{D}$, then $\Re$ is nonsingular if and only if $\overline{\Re} = \Re_0 - i\Re_1 - j\Re_2 - k\Re_3$ is nonsingular, where $\Re_0, \Re_1, \Re_2, \Re_3$ are all factor block circulant matrices over the field $\mathbb{F}$.*

Proof By computing, it is easy to prove that $\aleph_0 + i\aleph_1 + j\aleph_2 + k\aleph_3$ is the inverse of $\Re$ if and only if $\aleph_0 - i\aleph_1 - j\aleph_2 - k\aleph_3$ is the inverse of $\overline{\Re}$, where $\aleph_0, \aleph_1, \aleph_2, \aleph_3$ are all factor block circulant matrices over the field $\mathbb{F}$. ■

Theorem 4.4. *If matrices $\Re_0, \Re_1, \Re_2, \Re_3$ are all factor block circulant matrices over the field $\mathbb{F}$, then $\Re = \Re_0 + i\Re_1 + j\Re_2 + k\Re_3$ over $\mathbb{D}$ is nonsingular if and only if $\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2$ is nonsingular. Furthermore, if $\Re$ is nonsingular, then*

$$\Re^{-1} = (\Re_0 - i\Re_1 - j\Re_2 - k\Re_3)(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2)^{-1}.$$

Proof If $\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2$ is nonsingular, then

$$\Re[(\Re_0 - i\Re_1 - j\Re_2 - k\Re_3)(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2)^{-1}] = I.$$

Hence $\Re$ is nonsingular, and

$$\Re^{-1} = (\Re_0 - i\Re_1 - j\Re_2 - k\Re_3)(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2)^{-1}.$$

If $\Re$ is nonsingular,and suppose that $\aleph = \aleph_0 + i\aleph_1 + j\aleph_2 + k\aleph_3$ is the inverse of $\Re$. So $\aleph$ is also a factor block circulant matrix over $\mathbb{D}$ such that $\Re\aleph = \aleph\Re = I$. Hence we have the following system of equations:

$$\begin{pmatrix} \Re_0 & -\Re_1 & -\Re_2 & -\Re_3 \\ \Re_1 & \Re_0 & -\Re_3 & \Re_2 \\ \Re_2 & \Re_3 & \Re_0 & -\Re_1 \\ \Re_3 & -\Re_2 & \Re_1 & \Re_0 \\ \Re_1 & \Re_0 & \Re_3 & -\Re_2 \\ \Re_2 & -\Re_3 & \Re_0 & \Re_1 \\ \Re_3 & \Re_2 & -\Re_1 & \Re_0 \end{pmatrix} \begin{pmatrix} \aleph_0 \\ \aleph_1 \\ \aleph_2 \\ \aleph_3 \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

By solving the system of equations (12), we have the following system of equations:

$$\begin{cases} \aleph_0(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2) & = & \Re_0 \\ \aleph_1(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2) & = & -\Re_1 \\ \aleph_2(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2) & = & -\Re_2 \\ \aleph_3(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2) & = & -\Re_3 \end{cases}$$

Therefore

$$\aleph(\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2) = \Re_0 - i\Re_1 - j\Re_2 - k\Re_3 \tag{15}$$

Since $\Re$ is nonsingular, we know that $\Re_0 - i\Re_1 - j\Re_2 - k\Re_3$ is nonsingular by Theorem 4.3. So $\Re_0^2 + \Re_1^2 + \Re_2^2 + \Re_3^2$ is nonsingular by equation (13).

In the following, let $\mathbb{D} = \mathbb{F}[i, j, k] = \{a + bi + cj + dk | a, b, c, d \in \mathbb{C}\}$ be a quaternion division algebra over the complex field $\mathbb{C}$, we give another algorithm for the inverse of $\Re = \Re_0 + i\Re_1 + j\Re_2 + k\Re_3$.

Suppose that $\aleph = \aleph_0 + i\aleph_1 + j\aleph_2 + k\aleph_3$ is the inverse of $\Re$. Then we have the following system of equations which is equivalent to the system of equations (12)

$$\begin{pmatrix} \Re_0 & -\Re_1 & -\Re_2 & -\Re_3 \\ \Re_1 & \Re_0 & -\Re_3 & \Re_2 \\ \Re_2 & \Re_3 & \Re_0 & -\Re_1 \\ \Re_3 & -\Re_2 & \Re_1 & \Re_0 \end{pmatrix} \begin{pmatrix} \aleph_0 \\ \aleph_1 \\ \aleph_2 \\ \aleph_3 \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{16}$$

Let

$$\mathcal{A} = \begin{pmatrix} \Omega & -W \\ W & \Omega \end{pmatrix}$$

where

$$\Omega = \begin{pmatrix} \Re_0 & -\Re_1 \\ \Re_1 & \Re_0 \end{pmatrix}, \quad W = \begin{pmatrix} \Re_2 & \Re_3 \\ \Re_3 & -\Re_2 \end{pmatrix}.$$

Then $\Re$ is nonsingular if and only if $\mathcal{A}$ is nonsingular. If Ω is nonsingular, then

$$\begin{pmatrix} I & 0 \\ -W\Omega^{-1} & I \end{pmatrix} \begin{pmatrix} \Omega & -W \\ W & \Omega \end{pmatrix} \begin{pmatrix} I & \Omega^{-1}W \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega + W\Omega^{-1}W \end{pmatrix}, \tag{17}$$

where I is an 2×2 identity matrix. Hence $\mathcal{A}$ is nonsingular if and only if $\Omega + W\Omega^{-1}W$ and Ω are both nonsingular. ■

We now introduce the following algorithm for the inverse of $\Re = \Re_0 + i\Re_1 + j\Re_2 + k\Re_3$.

- **Step 1.** If Ω is singular, stop. Otherwise, go to step 2.
- **Step 2.** By equation (6), we have

$$\Omega^{-1} = \begin{pmatrix} \wp_0 & \wp_1 \\ -\wp_1 & \wp_0 \end{pmatrix} = \wp. \tag{18}$$

- **Step 3.** Calculate

$$\Omega + W\Omega^{-1}W = \begin{pmatrix} (\Re_2^2 + \Re_3^2)\wp_0 & -(\Re_2^2 + \Re_3^2)\wp_1 \\ (\Re_2^2 + \Re_3^2)\wp_1 & (\Re_2^2 + \Re_3^2)\wp_0 \end{pmatrix}.$$

  If $\Omega + W\Omega^{-1}W$ is singular, stop. Otherwise, go to step 4.

- **Step 4.** By equation (6), we have

$$(\Omega + W\Omega^{-1}W)^{-1} = \Im \tag{19}$$

- **Step 5.** By equations (15), (16) and (17), we have

$$\mathcal{A}^{-1} = \begin{pmatrix} I & -\wp W \\ 0 & I \end{pmatrix} \begin{pmatrix} \wp & 0 \\ 0 & \Im \end{pmatrix} \begin{pmatrix} I & 0 \\ W\wp & I \end{pmatrix}. \tag{20}$$

- **Step 6.** By the system of equations (14) and equation (18), we have

$$\begin{pmatrix} \aleph_0 \\ \aleph_1 \\ \aleph_2 \\ \aleph_3 \end{pmatrix} = \begin{pmatrix} I & -\wp W \\ 0 & I \end{pmatrix} \begin{pmatrix} \wp & 0 \\ 0 & \Im \end{pmatrix} \begin{pmatrix} I & 0 \\ W\wp & I \end{pmatrix} \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The $\Re^{-1} = \aleph = \aleph_0 + i\aleph_1 + j\aleph_2 + k\aleph_3$ is then obtained.

5 Factor block retrocirculant matrices over $\mathbb{C}$

Definition 5.1. Let $C_1, C_2, \ldots, C_m, A$ be square matrices each of order n over $\mathbb{C}$. We assume that A is nonsingular and that it commutes with each of the C_k's. By an A- factor block retrocirculant matrix of type (m,n) is meant an $mn \times mn$ matrix of the form

$$\Im = \mathrm{retrocirc}_A(C_1, C_2, \ldots, C_m) = \begin{pmatrix} C_1 & C_2 & \ldots & C_{m-1} & C_m \\ C_2 & C_3 & \ldots & C_m & AC_1 \\ C_3 & C_4 & \ldots & AC_1 & AC_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ C_m & AC_1 & \ldots & AC_{m-2} & AC_{m-1} \end{pmatrix}.$$

The factor retrocirculants of type$(m,1)$ will be referred to as scalar factor retrocirculants. In this case the matrix A reduces to a scalar that we shall denote by a. When A is the identity matrix I, we drop the word "factor" in the above definition. This kind of matrices are just block retrocirculants. Further, when $C_1, C_2, \ldots, C_m$ are scalar $c_1, c_2, \ldots, c_m$, this kind of matrices are as in [10–12].

Lemma 5.1. *We have* $\Gamma_A^{-1} = \Gamma_{A^{-1}}$, *where*

$$\Gamma_A = \mathit{retrocic}_A(I_n, 0, \ldots, 0), \quad \Gamma_{A^{-1}} = \mathit{retrocic}_{A^{-1}}(I_n, 0, \ldots, 0).$$

Lemma 5.2. *Let* $\Re = \mathit{circ}_{A^{-1}}(C_1, C_2, \ldots, C_m)$ *be an* A^{-1}- *factor block circulant and let* $\Im = \mathit{retrocirc}_A(C_1, C_2, \ldots, C_m)$ *be an* A- *factor block retrocirculant. Then* $\Gamma_A \Re = \Im$.

Theorem 5.1. *Let* $\Im = \mathit{retrocirc}_A(C_1, C_2, \ldots, C_m)$ *be a nonsingular* A*-factor block retrocirculant matrix over the complex field* $\mathbb{C}$. *Then*

$$\Im^{-1} = \mathit{retrocirc}_{A^{-1}}(B_1, A^{-1}B_m, \ldots, A^{-1}B_3, A^{-1}B_2),$$

where

$$B_{j+1} = \frac{1}{m}\sum_{k=0}^{m-1} (\omega^k K)^{-j} [\mathcal{F}(\omega^k K)]^{-1}, j = 0, 1, \ldots, m-1 \tag{21}$$

and $\omega = \exp(2\pi i/m), \mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1} z^k$ *and* K *denotes the principal* m*th root of the nonsingular matrix* A^{-1}.

Proof By Lemma 5.2, Lemma 5.1 and Theorem 2.2, we have

$$\begin{aligned} \Im^{-1} &= [\mathrm{circ}_{A^{-1}}(C_1, C_2, \ldots, C_m)]^{-1}\Gamma_A^{-1} = \mathrm{circ}_{A^{-1}}(B_1, B_2, \ldots, B_m)\Gamma_{A^{-1}} \\ &= \begin{pmatrix} B_1 & B_2 & \ldots & B_{m-1} & B_m \\ A^{-1}B_m & B_1 & \ldots & B_{m-2} & B_{m-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ A^{-1}B_3 & A^{-1}B_4 & \ldots & B_1 & B_2 \\ A^{-1}B_2 & A^{-1}B_3 & \ldots & A^{-1}B_m & B_1 \end{pmatrix} \begin{pmatrix} I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & A^{-1} \\ 0 & 0 & \ldots & A^{-1} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & A^{-1} & \ldots & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} B_1 & A^{-1}B_m & \ldots & A^{-1}B_3 & A^{-1}B_2 \\ A^{-1}B_m & A^{-1}B_{m-1} & \ldots & A^{-1}B_2 & A^{-1}B_1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ A^{-1}B_3 & A^{-1}B_2 & \ldots & A^{-1}(A^{-1}B_5) & A^{-1}(A^{-1}B_4) \\ A^{-1}B_2 & A^{-1}B_1 & \ldots & A^{-1}(A^{-1}B_4) & A^{-1}(A^{-1}B_3). \end{pmatrix} \\ &= \mathrm{retrocirc}_{A^{-1}}(B_1, A^{-1}B_m, \ldots, A^{-1}B_3, A^{-1}B_2), \end{aligned}$$

where

$$B_{j+1} = \frac{1}{m}\sum_{k=0}^{m-1}(\omega^k K)^{-j}[\mathcal{F}(\omega^k K)]^{-1}, j = 0, 1, \ldots, m-1.$$

This implies the validity of Theorem 5.1. ■

Let $\Im = \text{retrocirc}_A(C_1, C_2, \ldots, C_m)$ be a nonsingular A-factor block retrocirculant matrix over the complex field $\mathbb{C}$, by Theorem 5.1, we have the following algorithm which can find the inverse of the matrix $\Im$:

- **Step 1.** Find out the inverse matrix A^{-1} of the nonsingular matrix A.
- **Step 2.** Find out the principal mth root K of A^{-1} .
- **Step 3.** By $\mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1}z^k$ for computing $\mathcal{F}(\omega^k K), k = 0, 1, \ldots, m-1$, respectively.
- **Step 4.** By Step 2 for computing $[\mathcal{F}(\omega^k K)]^{-1}, k = 0, 1, \ldots, m-1$, respectively.
- **Step 5.** By Equation(19) for computing $B_{j+1}, j = 0, 1, \ldots, m-1$, respectively, we have

$$\Im^{-1} = \text{retrocirc}_{A^{-1}}(B_1, A^{-1}B_m, \ldots, A^{-1}B_3, A^{-1}B_2).$$

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