NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **15** No. 1, pp. 50-59
February 2006

Numerical Quadratures for Hadamard Hypersingular Integrals †

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Received October 13, 2003; Accepted (in revised version) August 23, 2004

Abstract. In this paper, we develop Gaussian quadrature formulas for the Hadamard finite part integrals. In our formulas, the classical orthogonal polynomials such as Legendre and Chebyshev polynomials are used to approximate the density function $f(x)$ so that the Gaussian quadrature formulas have degree $n-1$. The error estimates of the formulas are obtained. It is found from the numerical examples that the convergence rate and the accuracy of the approximation results are satisfactory. Moreover, the rate and the accuracy can be improved by choosing appropriate weight functions.

Key words: Gaussian quadrature; finite part; hypersingular integral; orthonormal polynomial.

AMS subject classifications: 65D30,65D32

1 Introduction

The numerical methods for the hypersingular integrals are most frequently encountered in many problems of mechanics. Particularly, they have been applied to solve the elasticity problems ([1-3]). In this paper,we consider hypersingular integrals of the form

$$I(t) = \text{f.p.} \int_a^b \frac{f(x)}{(x-t)^2} w(x) \mathrm{d}x, \quad t \in (a,b) \tag{1.1}$$

where f.p. denotes the finite part integral in the sense of Hadamard which is divergent in the classical sense, and $f(x)$ is a regular function on the interval $[a,b]$. The integral (1.1) is of second-order singularity. This type of integrals arises in the mechanics problems and the numerical methods of partial differential equations and integral equations, which have attracted considerable attention from researchers (see, for example, [1-8]). In numerical analysis of integrals arising from partial differential equations the chief difficulties in many cases are not only the loss

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†This research is supported in part by China NSF under grant 10071096, NSF of Guangdong under grant 990228, NSF of Hainan under grant 80525 and the One Hundred Distinguished Young Chinese Scientists Program of the Chinese Academy of Sciences from Yuesheng Xu.

of smoothness of the solution but also (and more crucially) the singularity of the solution. When smooth function are to be integrated , ordinary numerical methods are adequate, but when singular functions are to be integrated, the situation would be not reliable and satisfactory. Hence , it is desirable to have the simple and efficient quadrature formulas for the hypersingular integrals.

One of efficient methods for evaluating hypersingular integral (1.1) is generalizing the classical Gaussian quadrature rule. This was first done by Kutt[9] who has developed a set of Gaussian quadrature formulas. Ioakimidis, Pitta[10], Tsamasphyros and Dimou[11] developed the theorem of Kutt's. In the above works, the nodes and weights generally are complex numbers. This can reduce the precision in the numerical evaluation since $I(t)$ is real-valued. This is because the imaginary part will be required to cancel out exactly. Hui and Shia[12] have generalized Kutt's works to constant real nodes and weights that use classical orthogonal polynomials such as Legendre and Chebyshev, where the weight function is $w(x) = 1$ or $w(x) = \sqrt{1-x^2}$. Although Hui's method has more precision than that of Kutt's, it has some issues need to be further addressed: First the classical Gaussian quadrature formulas in general have degree $2n-1$, which integrate all polynomials up to its degree exactly. But Hui's formulas do not have any degree result. Second Hui's Gaussian quadrature formulas require parameter t not equal to the roots of orthogonal polynomials that will limit their application in the practical computation. The third Hui's method uses a deduction:

$$\text{if} \quad \int_{-1}^{1} \frac{f(x)}{x-t} w(x)\mathrm{d}x \approx g(t), \quad \text{then} \quad \frac{d}{dt}\int_{a}^{b} \frac{f(x)}{x-t} w(x)\mathrm{d}x \approx g'(t), \tag{$*$}$$

which is clearly coarse and would cause greater error and it is difficult to give an error estimate.

In this paper, we also use classical orthogonal polynomials such as Legendre and Chebyshev to establish Gaussian quadrature formulas for Cauchy principal value integrals and then use them to develop Gaussian quadrature formulas for finite part integrals . Nodes and weights in our approach are real-valued like Hui's. However, we avoid directly to use the formula $(*)$, but we apply the polynomials (the classical orthogonal polynomials or Taylor expansions) to approximate the considered functions. Consequently, we can prove that our Gaussian formulas have degree $n-1$ and the formulas do not require the parameter t not equal to the roots of orthogonal polynomials that is more convenient in practical computations. Moreover the inaccurate deduction $(*)$ is not used in our methods and we can establish error estimates for our Gaussian quadrature formulas and it seems possible that the accuracy of our formulas would be better than Hui's.

The paper is organized as follows. In Section 2, we develop Gaussian quadrature formulas for finite part integrals (1.1). In Section 3, the Gaussian quadrature formulas established in Section 2 are applied to the three concrete cases of weight function: $w(x) = 1, w(x) = \sqrt{1-x^2}$ and $w(x) = 1/\sqrt{1-x^2}$, and the error estimates of the formulas are given respectively. Finally, we present the results of the numerical experiments in Section 4.

2 Gaussian quadrature formulas

To simplify the exposition, we will take $a = -1, b = 1$ in (1.1), but our results are valid in the more general cases for finite a, b.

Definition 2.1. Suppose that $f(x)$ is a real function defined on $[-1, 1]$. If the limit

$$\lim_{\varepsilon\to 0}\left(\int_{-1}^{t-\varepsilon} \frac{f(x)}{(x-t)^2}\mathrm{d}x + \int_{t+\varepsilon}^{1} \frac{f(x)}{(x-t)^2}\mathrm{d}x - \frac{2f(x)}{\varepsilon}\right) \tag{2.1}$$

exists and is finite, then we denote the limit (2.1) by

$$\text{f.p.} \int_{-1}^{1} \frac{f(x)}{(x-t)^2} \mathrm{d}x$$

and refer to it as the finite part integral in Hadamard sense.

It has been shown ([13], chapter I, §4.2) that

$$\frac{d}{dt} \text{p.v.} \int_{-1}^{1} \frac{f(x)}{x-t} \mathrm{d}x = \text{f.p.} \int_{-1}^{1} \frac{f(x)}{(x-t)^2} \mathrm{d}x.$$

The classical Gaussian quadrature formula of degree $2n-1$ for the function $f(x)$ on interval $[-1,1]$ is expressed as

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x \approx \sum_{i=1}^{n} w_i f(x_i) \tag{2.2}$$

which will take equality when $f(x)$ is the polynomial of degree $2n-1$ or less, where the weights w_i and nodes x_i are determined by weight function $w(x)$. (cf. [14]).

Theorem 2.1. *Suppose polynomials φ_i of degree i ($i = 1, 2, 3, \cdots$) satisfies the orthonormal relation*

$$\int_{-1}^{1} \varphi_i(x)\varphi_j(x)w(x)\mathrm{d}x = \delta_{ij} \tag{2.3}$$

where δ_{ij} is the Kronecker delta. Let the Cauchy principal value

$$q_j(t) = \int_{-1}^{1} \frac{\varphi_j(x)}{x-t} w(x)\mathrm{d}x, \qquad t \in (-1,1) \tag{2.4}$$

and

$$\widetilde{w}_i(t) = w_i \sum_{j=0}^{n-1} \varphi_j(x_i) q_j(t) \tag{2.5}$$

where x_i and w_i are nodes and weights of the Gaussian quadrature formula (2.2) respectively. Then the Gaussian quadrature rule

$$\int_{-1}^{1} \frac{f(x)}{(x-t)^2} w(x)\mathrm{d}x \approx f(t) \int_{-1}^{1} \frac{w(x)}{(x-t)^2} \mathrm{d}x + \sum_{i=1}^{n} \widetilde{w}_i(t) \frac{f(x_i) - f(t)}{x_i - t} \tag{2.6}$$

has the degree $n-1$.

Proof. Suppose $f(x)$ is a polynomial of degree $n-1$ or less, then $f(x)$ can be expanded as

$$f(x) = \sum_{j=0}^{n-1} c_j \varphi_j(x) \tag{2.7}$$

where the coefficients c_j are given by

$$c_j = \int_{-1}^{1} f(x)\varphi_j(x)w(x)\mathrm{d}x. \tag{2.8}$$

From (2.2), the coefficients c_j are further expressed as

$$c_j = \sum_{i=1}^{n} w_i f(x_i)\varphi_j(x_i). \tag{2.9}$$

Combining this with (2.7) and (2.4) we get

$$\begin{aligned}
&\int_{-1}^{1} \frac{f(x)}{x-t} w(x)\mathrm{d}x \\
= \ & \sum_{j=0}^{n-1} c_j q_j(t) = \sum_{j=0}^{n-1}\sum_{i=1}^{n} w_i f(x_i)\varphi_j(x_i) q_j(t) \\
= \ & \sum_{i=1}^{n} w_i \left(\sum_{j=0}^{n-1} \varphi_j(x_i) q_j(t) \right) f(x_i) = \sum_{i=1}^{n} \widetilde{w}_i(t) f(x_i).
\end{aligned} \tag{2.10}$$

Applying the above formula to function $(f(x) - f(t))/(x-t)$ which is a polynomial of degree less than $n-1$, we obtain

$$\begin{aligned}
\int_{-1}^{1} \frac{f(x)}{(x-t)^2} w(x)\mathrm{d}x &= \int_{-1}^{1} \frac{f(x)-f(t)}{(x-t)^2} w(x)\mathrm{d}x + f(t)\int_{-1}^{1} \frac{w(x)}{(x-t)^2}\mathrm{d}x \\
&= \sum_{i=1}^{n} \widetilde{w}_i(t) \frac{f(x_i)-f(t)}{x_i - t} + f(t)\int_{-1}^{1} \frac{w(x)}{(x-t)^2}\mathrm{d}x
\end{aligned}$$

which is the desired result (2.6). ■

For the general function $f(x)$, we define the error of Gaussian quadrature formula (2.6) as

$$r_n(t) = \int_{-1}^{1} \frac{f(x)w(x)}{(x-t)^2}\mathrm{d}x - \left(f(t)\int_{-1}^{1} \frac{w(x)}{(x-t)^2}\mathrm{d}x + \sum_{i=1}^{n} \widetilde{w}_i(t) \frac{f(x_i)-f(t)}{x_i-t} \right). \tag{2.11}$$

If $f \in C^n[-1,1]$, by writing

$$\psi_{n-1}(x,t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(x-t)^k \tag{2.12}$$

$$R_{n-1}(x,t) = f(x) - \psi_{n-1}(x,t) = \frac{f^{(n)}(\xi)}{n!}(x-t)^n, \qquad \xi \in (-1,1) \tag{2.13}$$

then the error of (2.6) can be expressed as

$$\begin{aligned}
r_n(t) \ = \ & \int_{-1}^{1} \frac{\psi_{n-1}(x,t) + R_{n-1}(x,t)}{(x-t)^2} w(x)\mathrm{d}x - (\psi_{n-1}(t,t) + R_{n-1}(t,t)) \int_{-1}^{1} \frac{w(x)}{(x-t)^2}\mathrm{d}x \\
& - \sum_{i=1}^{n} \widetilde{w}_i(t) \frac{(\psi_{n-1}(x_i,t) + R_{n-1}(x_i,t)) - (\psi_{n-1}(t,t) + R_{n-1}(t,t))}{x_i - t}.
\end{aligned}$$

Since the Gaussian quadrature rule (2.6) is of the degree $n-1$ and noting $\psi_{n-1}(\cdot,t)$ is a polynomial of degree $n-1$ and $R_{n-1}(t,t) = 0$, we reach at

$$r_n(t) = \int_{-1}^{1} \frac{R_{n-1}(x,t)}{(x-t)^2} w(x)\mathrm{d}x - \sum_{i=1}^{n} \widetilde{w}_i(t) \frac{R_{n-1}(x_i,t)}{x_i - t}. \tag{2.14}$$

3 Gauss-Legendre and Gauss-Chebyshev rules

In this section, we apply Theorem 2.1 to weight functions $w(x) = 1, w(x) = \sqrt{1-x^2}$ and $w(x) = 1/\sqrt{1-x^2}$ respectively.

Theorem 3.1. *Let*

$$\widetilde{w}_i(t) = -w_i \sum_{j=0}^{n-1} (2j+1) P_j(x_i) Q_j(t) \tag{3.1}$$

where $P_j(x)$ and $Q_j(x)$ are the Legendre polynomials of the first kind and Legendre function of second kind respectively, node x_i is the i-th root of $P_n(x)$ and weight w_i is given by

$$w_i = \frac{2}{(1-x_i^2)(P_n'(x_i))^2}. \tag{3.2}$$

Then the Gauss-Legendre quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{(x-t)^2} \mathrm{d}x \approx -2Q_0'(t) f(t) + \sum_{i=1}^{n} \widetilde{w}_i(t) \frac{f(x_i) - f(t)}{x_i - t} \tag{3.3}$$

has the degree $n-1$. Furthermore, if $f(x) \in C^n[-1,1]$, then there is an estimate of the error

$$|r_n(t)| \leq \frac{M}{n!} \left(\frac{1}{4} n^4 + n^2 |\ln \frac{1-t}{1+t}| + 1 \right) \Big[\max(1-t, 1+t) \Big]^{n-1}, \tag{3.4}$$

where $M = \max_{-1 \leq x \leq 1} |f^{(n)}(x)|$.

Proof. We first suppose $f(x)$ is a polynomial of degree $n-1$ or less, then from Theorem 2.1 and classical Gauss-Legendre formulas for weight function $w(x) = 1$, we know that x_i is the i-th root of Legendre polynomial $P_n(x)$ and w_i is given by (3.2)(cf. [16]). In the case of weight function $w(x) = 1$, $\varphi_j(x) = \sqrt{(2j+1)/2} P_j(x)$ (for the properties of orthogonal polynomials,we refer to [15]). then from (2.4) we have

$$q_j(t) = \sqrt{\frac{2j+1}{2}} \int_{-1}^{1} \frac{P_j(x)}{x-t} \mathrm{d}x = -\sqrt{2(2j+1)} Q_j(t). \tag{3.5}$$

By substituting $\varphi_j(x)$ and $q_j(t)$ into (2.5), the representation (3.1) follows immediately. By noting

$$\int_{-1}^{1} \frac{1}{(x-t)^2} \mathrm{d}x = \frac{d}{dt} \left(\int_{-1}^{1} \frac{1}{x-t} \mathrm{d}x \right) = -2Q_0'(t),$$

the Gaussian quadrature rule (2.6) implies (3.3).

In the following we estimate the error $r_n(t)$. From the properties of Legendre polynomial $P_j(x)$:

$$P_0(x) = 1,\ |P_j(x)| \leq 1,\ P_{j+1}'(x) = (j+1) P_j(x) + x P_j'(x),$$

we deduce that

$$|P_j'(x)| \leq 1 + 2 + \ldots + j = \frac{1}{2} j(j+1). \tag{3.6}$$

Consequently,

$$\begin{aligned}
|Q_j(t)| &= \frac{1}{2}|\int_{-1}^{1}\frac{P_j(x)}{x-t}\mathrm{d}x| \\
&\le \frac{1}{2}\left[|\int_{-1}^{1}\frac{P_j(x)-P_j(t)}{x-t}\mathrm{d}x| + |\int_{-1}^{1}\frac{P_j(t)}{x-t}\mathrm{d}x|\right] \\
&= \frac{1}{2}\left[\max|P_j'(\xi)| + |\int_{-1}^{1}\frac{P_j(t)}{x-t}\mathrm{d}x|\right] \qquad \text{for some } \xi\in(-1,1) \\
&\le \frac{1}{4}j(j+1)+\frac{1}{2}|\ln\frac{1-t}{1+t}|.
\end{aligned}$$

Further from (3.1) we have

$$\begin{aligned}
|\widetilde{w}_i(t)| &\le w_i\sum_{j=0}^{n-1}(2j+1)|P_j(x_i)||Q_j(t)| \\
&\le w_i\sum_{j=0}^{n-1}(2j+1)\left(\frac{1}{4}j(j+1)+\frac{1}{2}|\ln\frac{1-t}{1+t}|\right) \\
&= w_i\left(\frac{1}{8}n^2(n^2-1)+\frac{n^2}{2}|\ln\frac{1-t}{1+t}|\right) \\
&\le \frac{n^2}{8}w_i\left(n^2+4|\ln\frac{1-t}{1+t}|\right).
\end{aligned}$$

Combining this with (2.14) and (2.13), we find that

$$|r_n(t)|\le|\int_{-1}^{1}\frac{f^{(n)}(\xi)}{n!}(x-t)^{n-2}\mathrm{d}x|+\sum_{i=1}^{n}|\widetilde{w}_i(t)||\frac{f^{(n)}(\xi_i)}{n!}(x_i-t)^{n-1}| \quad \xi,\xi_i\in(-1,1)$$

$$\le \frac{M}{n!}[\max(1-t,1+t)]^{n-2}+\frac{n^2}{8}(n^2+4|\ln\frac{1-t}{1+t}|)\sum_{i=1}^{n}w_i\frac{M}{n!}\cdot[\max(1-t,1+t)]^{n-1}.$$

This implies (3.4) since $\sum\limits_{i=1}^{n}w_i=\int_{-1}^{1}\mathrm{d}x=2$ (set $f(x)=1$ in (2.2)). ■

Theorem 3.2. *Let*

$$\widetilde{w}_i(t)=-2w_i\sum_{j=0}^{n-1}U_j(x_i)T_{j+1}(t) \tag{3.7}$$

where $T_j(x)$ and $U_j(x)$ are the Chebyshev polynomials of the first kind and second kind respectively, and

$$x_i=\cos\frac{i\pi}{n+1}, \tag{3.8}$$

$$w_i=\frac{\pi}{n+1}\sin^2\frac{i\pi}{n+1} \tag{3.9}$$

then the Gauss-Chebyshev quadrature formula

$$\int_{-1}^{1}\frac{f(x)}{(x-t)^2}\sqrt{1-x^2}\mathrm{d}x\approx-\pi f(t)+\sum_{i=1}^{n}\widetilde{w}_i(t)\frac{f(x_i)-f(t)}{x_i-t} \tag{3.10}$$

has the degree $n-1$. *Furthermore, if* $f(x) \in C^n[-1,1]$ *then there is an estimate of the error*

$$|r_n(t)| \leq \frac{(n^2+n+1)M\pi}{2n!}\Big[\max(1-t,1+t)\Big]^{n-1}, \tag{3.11}$$

where $M = \max_{-1\leq x\leq 1} |f^{(n)}(x)|$.

Proof. From Theorem 2.1 and the classical Gauss-Chebyshev quadrature rule for weight function $w(x) = \sqrt{1-x^2}$, the node x_i and the weight w_i are given by (3.8) and (3.9) respectively (cf. [16]). In addition, the orthonormal polynomial $\varphi_j(x) = \sqrt{2/\pi}U_j(x)$ and hence by (2.4) and the properties of Chebyshev polynomials we have

$$q_j(t) = \sqrt{\frac{2}{\pi}}\int_{-1}^{1} \frac{U_j(x)}{x-t}\sqrt{1-x^2}\mathrm{d}x = -\sqrt{2\pi}T_{j+1}(t) \tag{3.12}$$

so that

$$\widetilde{w}_i(t) = w_i \sum_{j=0}^{n-1} \varphi_j(x_i)q_j(t) = -2w_i \sum_{j=0}^{n-1} U_j(x_i)T_{j+1}(t). \tag{3.13}$$

This with

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{(x-t)^2}\mathrm{d}x = \frac{d}{dt}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{x-t}\mathrm{d}x = -\pi$$

implies (3.10). In the following we discuss the estimate of the error term $r_n(t)$. From the properties of $T_j(x)$ and $U_j(x)$, we have

$$|T_j(x)| \leq 1, \tag{3.14}$$

$$|U_j(\cos\theta)| = \left|\frac{\sin[(j+1)\theta]}{\sin\theta}\right| \leq (j+1), \quad \theta \in [0,\pi] \tag{3.15}$$

and by (2.5) it follows that

$$|\widetilde{w}_i(t)| \leq 2w_i \sum_{j=0}^{n-1}(j+1) = w_i n(n+1).$$

Hence, according to (2.13) and (2.14), we get

$$\begin{aligned} |r_n(t)| &\leq |\int_{-1}^{1} \frac{f^{(n)}(\xi)}{n!}(x-t)^{n-2}\sqrt{1-x^2}\mathrm{d}x| \\ &\quad + \sum_{i=1}^{n} |\widetilde{w}_i(t)||\frac{f^{(n)}(\xi_i)}{n!}(x_i-t)^{n-1}| \quad \xi, \xi_i \in (-1,1) \\ &\leq \frac{M}{n!}(\frac{\pi}{2} + n(n+1)\sum_{i=1}^{n} w_i)[\max(1-t,1+t)]^{n-1}. \end{aligned}$$

Observe

$$\sum_{i=1}^{n} w_i = \int_{-1}^{1} \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{2}.$$

The above two results yield (3.11). ■

Theorem 3.3. *Let*

$$\widetilde{w}_i(t) = 2w_i \sum_{j=1}^{n-1} T_j(x_i) U_{j-1}(t) \tag{3.16}$$

where

$$x_i = \cos\frac{(2i-1)\pi}{2n}, \tag{3.17}$$

$$w_i = \frac{\pi}{n}. \tag{3.18}$$

Then the Gauss-Chebyshev quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{(x-t)^2} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x \approx \sum_{i=1}^{n} \widetilde{w}_i(t) \frac{f(x_i) - f(t)}{x_i - t} \tag{3.19}$$

has the degree $n-1$. Furthermore if $f(x) \in C^n[-1,1]$ then the error

$$|r_n(t)| \le \frac{(n^2-n+1)M\pi}{n!} \Big[\max(1-t, 1+t)\Big]^{n-1}, \tag{3.20}$$

where $M = \max_{-1\le x\le 1} |f^{(n)}(x)|$.

Proof From Theorem 2.1 and the classical Gauss-Chebyshev quadrature rule for the weight function $w(x) = 1/\sqrt{1-x^2}$, the node x_i and the weight w_i are given by (3.17) and (3.18) respectively, and the orthonormal polynomial $\varphi_j(x) = a_j T_j(x)$, where a_j is just a normalization constant, precisely

$$a_j = \begin{cases} \sqrt{1/\pi}, & j = 0 \\ \sqrt{2/\pi}, & j \neq 0. \end{cases}$$

In virtue of (2.4) it is clear that

$$\begin{aligned} q_j(t) &= a_j \int_{-1}^{1} \frac{T_j(x)}{(x-t)\sqrt{1-x^2}} \mathrm{d}x \\ &= \begin{cases} 0, & j = 0 \\ \sqrt{2\pi} U_{j-1}(t), & j \ge 1 \end{cases} \end{aligned}$$

The remaining of the proof is similar to the proof of Theorem 3.2. ■

4 Numerical experiments

In this section,we present the numerical tests of formula (3.3) and (3.10) for $f(x) = e^x \cos x$ and $f(x) = (1-x^2)^{1/2}\cos x$. Exact values of integrals in the following tables are computed analytically by using MATHEMATICA.

Table 1 exhibits the exact and the computed values using (3.3) for the integral

$$\int_{-1}^{1} \frac{e^x \cos x}{x^2} \mathrm{d}x \tag{4.1}$$

and Table 2 exhibits the exact and the computed values using (3.3) and (3.10) for the integral

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}\cos x}{x^2} \mathrm{d}x. \tag{4.2}$$

Table 1: Numerical results for integral (4.1)

n	Exact	formula (3.3)
4	-2.1109977567176	-2.1109977532733
6	-2.1109977567176	-2.1109977567177
8	-2.1109977567176	-2.1109977567176
10	-2.1109977567176	-2.1109977567176
12	-2.1109977567176	-2.1109977567176

Table 2: Numerical results for integral (4.2)

n	Exact	formula (3.3)	formula (3.10)
4	-3.9108980428714	-3.9069888566016	-3.9108980412059
6	-3.9108980428714	-3.9094450576900	-3.9108980428714
8	-3.9108980428714	-3.9102186697286	-3.9108980428714
10	-3.9108980428714	-3.9105296248566	-3.9108980428714
12	-3.9108980428714	-3.9106769531136	-3.9108980428714

where we have selected $t = 0$ for the convenience of contrast with Hui's method. From Tables 1 and 2, we find that our numerical results agree fairly well with Hui's one. In Table 2, results for quadrature formula (3.3) convergent slowly and results for quadrature formula (3.10) convergent quickly. This shows that Chebyshev polynomials is more appropriate to the weight function of $\sqrt{1-x^2}$ to which Legendre polynomials are not satisfactory.

In order to evaluate the quadrature formula (3.3) in a more efficient way, we choose $t_i = -1 + 0.2i, i = 1, 2, \cdots, 9$. and compute the relative errors via the formula

$$E^{\text{rel}} = \frac{\sqrt{\sum_i E^{\text{abs}}(t_i)^2}}{\sqrt{\sum_i I(t_i)^2}} \tag{4.3}$$

where $E^{\text{abs}}(t_i)$ and $I(t_i)$ denote the absolute error $|r_n(t_i)|$ and the exact value for integral

$$I(t) = \int_{-1}^{1} \frac{e^x \cos x}{(x-t)^2} \mathrm{d}x \tag{4.4}$$

at point $t = t_i$ respectively. The numerical results are shown in Table 3 for $n = 4, 6, 8, \cdots, 12$.

The computational results show that by choosing appropriate weight function our algorithm converges quickly providing high accuracy.

Acknowledgments

The author would also like to thank Professor Wei Lin for his careful guidance.

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Table 3: The relative errors of quadrature formula (3.3) for integral (4.4)

n	Relative errors
4	0.0014219800
6	0.0000175004
8	3.33566×10^{-7}
10	5.30531×10^{-10}
12	1.99660×10^{-12}

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