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A New Homotopy Method for Nonlinear Complementarity Problems

Jundi Ding*   and Hongyou Yin
(*Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*
E-mail: dingjundi@nuaa.edu.cn, hyyin@nuaa.edu.cn)

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Abstract

In this paper, we present a new homotopy method for the nonlinear complementarity problems. Without the regularity or non-singulary assumptions for $\nabla F(x)$, we prove that our homotopy equations have a bounded solution curve. The numerical tests confirm the efficiency of our proposed method.

Keywords: Nonlinear complementarity problem (NCP); homotopy equations; bounded.
Mathematics subject classification: 90C33

1. Introduction

We are interested in finding a solution of the following nonlinear complementarity problem (NCP):

$$\begin{cases} \text{find} \quad x \in \mathbb{R}^n \\ \text{s. t.} \quad x_i \geq 0,\ F_i(x) \geq 0,\ x_i F_i(x) = 0,\ \text{for } i = 1, \cdots, n, \end{cases}$$

where $F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuously differentiable.

There have been extensive studies for complementarity problems, see for example [5] and a recent review paper [7]. The NCP was introduced by Cottle [4] in his Ph.D. thesis in 1964. At the beginning, it was recognized that the NCP is a special case of a variation inequality problem [11]. The nonlinear complementarity problem and the linear complementarity problem, where $F(x)$ is an affine map, have been studied extensively, see, e.g., [2, 8, 10].

There exist many methods for solving the NCP, such as semi-smooth equation method [15], smoothing or non-smoothing Newton method [17, 19], neural network method [13, 14] etc. Homotopy method, as a kind of fixed-point method, had been developed in 1970s: Scarf [18] introduced the notion of primitive sets and described the first algorithm to approximate a fixed point of a continuous mapping; some powerful theoretical tools were available [3]; Eaves [6] introduced piecewise-linear maps into the computational fixed-point literature. The classical reference by Todd [20] and the text by Garcia and Zangwill

*Corresponding author.

[9] provided discussions of these techniques in detail. As a historical note, homotopy-type methods were directly applied to the linear complementarity problems with some success in [21, 22], but [23] was the only one with homotopy method to solve NCP. In this paper, we can avoid the requirement that Jacobian matrix of $F(x)$ is regular as compared to [23].

The organization of the paper is as follows. In Section 2, we prove the existence and boundedness for the solution curve of the homotopy equation. We also give the conditions which ensure the end point of the solution curve to be the solution of NCP. In Section 3, some numerical examples are provided. In Section 4, some concluding remarks will be made.

2. Main results of a homotopy method for NCP

Suppose

$$K^{(M)} = \{x \in \mathbb{R}^n; 0 \le x_i \le M, i = 1, \cdots, n\},$$
$$\tilde{K}^{(M)} = \{x \in \mathbb{R}^n; 0 < x_i < M, i = 1, \cdots, n\},$$

and let $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x_i > 0, i = 1, \cdots, n\}$, $\mathbb{R}^n_{+} = \{x \in \mathbb{R}^n | x_i \ge 0, i = 1, \cdots, n\}$.

Lemma 2.1. (Parametric Form of Sard Theorem [1]) *Let $V \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$ be open sets, and let $F : V \times U \to \mathbb{R}^k$ be a C^r mapping, where $r > \max\{0, m-k\}$. If $0 \in \mathbb{R}^k$ is a regular value of F, then for almost all $a \in V$, 0 is a regular value of $F_a = F(a, \cdot)$.*

Lemma 2.2. (Inverse Image Theorem [16]) *If 0 is a regular value of the mapping F_a, then $F_a^{-1}(0)$ consists of some smooth manifolds.*

Lemma 2.3. (Classification Theorem of One-Dimensional Smooth Manifolds [16]) *A one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.*

In this paper, we consider the following homotopy equation:

$$H(w^{(0)}, w, t) = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{bmatrix} (1-t)(F(x) - y + z) + t(x - x^{(0)}) \\ -\begin{pmatrix} x_1 y_1 \\ \vdots \\ x_n y_n \end{pmatrix} + t \begin{pmatrix} x_1^{(0)} y_1^{(0)} \\ \vdots \\ x_n^{(0)} y_n^{(0)} \end{pmatrix} \\ -\begin{pmatrix} (M - x_1) z_1 \\ \vdots \\ (M - x_n) z_n \end{pmatrix} + t \begin{pmatrix} (M - x_1^{(0)}) z_1^{(0)} \\ \vdots \\ (M - x_n^{(0)}) z_n^{(0)} \end{pmatrix} \end{bmatrix} = 0, \quad (2.1)$$

where $x, y, z \in \mathbb{R}^n$, $w = (x, y, z)$, $w^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)})$ and $t \in (0, 1]$.

Proposition 2.1. *For every $M > 0$ and almost all $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$, 0 is a regular value of the mapping $H_{w^{(0)}} = H(w^{(0)}, w, t) : \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0, 1] \to \mathbb{R}^{2n}$ and $H^{-1}_{w^{(0)}}(0) = \{(w, t) \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0, 1];\ H_{w(0)}(w, t) = 0\}$ is composed by some smooth and simple curves, with one of the curves has the initial point $(w^{(0)}, 1)$.*

Proof. For every $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$, $t \in (0,1]$, we have

$$\frac{\partial H(w^{(0)},w,t)}{\partial w^{(0)}} = \begin{bmatrix} \partial H_1/\partial x_0 & \partial H_1/\partial y_0 & \partial H_1/\partial z_0 \\ \partial H_2/\partial x_0 & \partial H_2/\partial y_0 & \partial H_2/\partial z_0 \\ \partial H_3/\partial x_0 & \partial H_3/\partial y_0 & \partial H_3/\partial z_0 \end{bmatrix} = \begin{bmatrix} -tI_{n\times n} & 0 & 0 \\ t\text{diag}\{y_i^{(0)}\}_i & t\text{diag}\{x_i^{(0)}\}_i & 0 \\ -t\text{diag}\{z_i^{(0)}\}_i & 0 & t\text{diag}\{M-x_i^{(0)}\}_i \end{bmatrix}.$$

Consequently,

$$\left|\frac{\partial H(w^{(0)},w,t)}{\partial w^{(0)}}\right| = (-1)^n t^{3n}\prod_{i=1}^{n} x_i^{(0)}(M-x_i^{(0)}).$$

Since $x_0 \in \tilde{K}^{(M)}$, we have $x_i^{(0)} > 0$ and $M - x_i^{(0)} > 0$, which imply that

$$\left|\frac{\partial H(w^{(0)},w,t)}{\partial w^{(0)}}\right| \neq 0.$$

Thus, as a mapping of the variables w, $w^{(0)}$ and t, the Jacobi matrix

$$\frac{\partial H(w^{(0)},w,t)}{\partial (w^{(0)},w,t)}$$

is of full row rank. Therefore, 0 is a regular value of $H(w^{(0)},w,t)$ for $t > 0$. By the parametric form of Sard theorem (Lemma 2.1), for almost all $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$, 0 is a regular value of the mapping $H_{w^{(0)}} : \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0,1] \to \mathbb{R}^{3n}$. Choose such a $w^{(0)}$, by the inverse image theorem (Lemma 2.2), $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. Since

$$H_{w^{(0)}}(w^{(0)},1) = 0,$$

there must be a smooth curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)},1)$. ■

Proposition 2.2. *For a given* $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$ *(*$M > 0$*), if 0 is a regular value of the mapping* $H_{w^{(0)}}$*, then the smooth curve* $\Gamma_{w^{(0)}}$ *starting from* $(w^{(0)},1)$ *is a bounded curve in* $\tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0,1]$.

Proof. If the curve $\Gamma_{w^{(0)}} \subset \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0,1]$ is unbounded, and note that $K^{(M)}$ and $(0,1]$ is bounded, then there exists a sequence of points $\{w^{(k)}, t_k\} \subset \Gamma_{w^{(0)}}$ such that

$$\lim_{k\to\infty} x^{(k)} = x^* \in K^{(M)}, \lim_{k\to\infty} t_k = t^* \in [0,1], \tag{2.2}$$

and

$$\lim_{k\to\infty} \|y^{(k)}\| = \infty \ \text{ or } \ \lim_{k\to\infty} \|z^{(k)}\| = \infty.$$

Without loss of generality, we suppose that $\lim_{k\to\infty}\|y^{(k)}\|=\infty$. Then there is at least one $i\in\{1,\cdots,n\}$ such that

$$\lim_{k\to\infty} y_i^{(k)}=+\infty. \tag{2.3}$$

We let the set

$$I_y^*=\{i\in\{1,\cdots,n\};\lim_{k\to\infty} y_i^{(k)}=+\infty\}.$$

It obvious that $I_y^*\neq\emptyset$. Since $\Gamma_{w^{(0)}}\subset H_{w^{(0)}}^{-1}(0)$, we have

$$(1-t_k)(F(x^{(k)})-y^{(k)}+z^{(k)})+t_k(x^{(k)}-x^{(0)})=0, \tag{2.4}$$

$$-\begin{pmatrix} x_1^{(k)}y_1^{(k)} \\ \vdots \\ x_n^{(k)}y_n^{(k)} \end{pmatrix}+t_k\begin{pmatrix} x_1^{(0)}y_1^{(0)} \\ \vdots \\ x_n^{(0)}y_n^{(0)} \end{pmatrix}=0, \tag{2.5}$$

$$-\begin{pmatrix} (M-x^{(k)})_1z_1^{(k)} \\ \vdots \\ (M-x^{(k)})_nz_n^{(k)} \end{pmatrix}+t_k\begin{pmatrix} (M-x_1^{(0)})z_1^{(0)} \\ \vdots \\ (M-x_n^{(0)})z_n^{(0)} \end{pmatrix}=0, \tag{2.6}$$

for all positive integers k.

It is worthy pointing out that when $i\in I_y^*$, $\{z_i^{(k)}\}_k$ must be bounded. In fact, if $z_i^{(k)}\to+\infty$, from the ith element of Eqs. (2.5) and (2.6), we have

$$x_i^{(k)}=t_kx_i^{(0)}y_i^{(0)}/y_i^{(k)}, \tag{2.7}$$

$$M-x_i^{(k)}=t_k(M-x_i^{(0)})z_i^{(0)}/z_i^{(k)}. \tag{2.8}$$

Since $y_i^{(k)}\to+\infty$ and $z_i^{(k)}\to+\infty$, let $k\to\infty$ in (2.7) and (2.8), we have $x_i^*=0$ and $x_i^*=M$. This contradict with $M>0$.

If $t^*\in[0,1)$, we consider the ith element of Eq. (2.4) with $i\in I_y^*$. We have

$$y_i^{(k)}=F_i(x^{(k)})+z_i^{(k)}+\frac{t_k}{1-t_k}(x_i^{(k)}-x_i^{(0)}). \tag{2.9}$$

Since $x^{(k)}\in K^{(M)}$, $F(x)$ is continuous on the bounded closed set $K^{(M)}$ and $\{z_i^{(k)}\}_k$ is bounded, we know that the right hand side of (2.9) is bounded for all k. This leads to a contradiction since the left hand side of (2.9) goes to infinity when $k\to\infty$.

If $t^*=1$, Eq. (2.4) can be written as

$$t_kx^{(0)}=t_kx^{(k)}+(1-t_k)(F(x^{(k)})+z^{(k)})-(1-t_k)y^{(k)}.$$

Let $k\to\infty$, we have

$$x^{(0)}=x^*-\lim_{k\to\infty}(1-t_k)y^{(k)}. \tag{2.10}$$

Due to (2.7), we obtain $x_i^*=0$ for $i\in I_y^*$ by letting $k\to\infty$. Then the ith element of (2.10) gives

$$x_i^{(0)}=-\lim_{k\to\infty}(1-t_k)y_i^{(k)}\le 0. \tag{2.11}$$

This contradicts with $x^{(0)} \in \tilde{K}^{(M)}$. Thus, $\Gamma_{w^{(0)}}$ is bounded. ∎

By Propositions 2.1 and 2.2, for almost all $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$, 0 is a regular value of $H_{w^{(0)}}$ and $H^{-1}_{w^{(0)}}(0)$ contains a bounded smooth curve $\Gamma_{w^{(0)}} \subset \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0,1]$ starting from $(w^{(0)},1)$. According to Lemma 2.3, $\Gamma_{w^{(0)}}$ must be diffeomorphic to a unit circle or a unit interval $(0,1]$. Notice that the matrix

$$
\begin{aligned}
\frac{\partial H_{w^{(0)}}(w^{(0)},1)}{\partial w} &= \left.\begin{bmatrix} \partial H_1/\partial x & \partial H_1/\partial y & \partial H_1/\partial z \\ \partial H_2/\partial x & \partial H_2/\partial y & \partial H_2/\partial z \\ \partial H_3/\partial x & \partial H_3/\partial y & \partial H_3/\partial z \end{bmatrix}\right|_{\substack{w=w^{(0)}\\ t=1}} \\
&= \begin{bmatrix} I_{n\times n} & 0 & 0 \\ -\mathrm{diag}\{y_i^{(0)}\}_i & -\mathrm{diag}\{x_i^{(0)}\}_i & 0 \\ \mathrm{diag}\{z_i^{(0)}\}_i & 0 & -\mathrm{diag}\{M-x_i^{(0)}\}_i \end{bmatrix}
\end{aligned}
$$

is non-singular. So, it is impossible that $\Gamma_{w^{(0)}}$ is diffeomorphic to the unit circle. Consequently, $\Gamma_{w^{(0)}}$ is diffeomorphic to the unit interval $(0,1]$. For bounded smooth curve which diffeomorphic to the unit interval $(0,1]$, there are two possible cases as demonstrated in Figs. 1 and 2.

Since the equation $H_{w^{(0)}}(w,1)=0$ has only one solution $(w^{(0)},1)$ in $\tilde{K}^{(M)} \times \mathbb{R}^{n}_{++} \times \{1\}$, the curve Γ^1 does not exist. So, $\Gamma_{w^{(0)}}$ has the form of Γ^2. Thus, we know that $w^* = (x^*,y^*,z^*) \in K^{(M)} \times \mathbb{R}^{2n}_{+}$ satisfies (let $t=0$ in the homotopy equation (2.1)):

$$F(x^*) - y^* + z^* = 0, \tag{2.12}$$
$$x_i^* y_i^* = 0, \ \ (M - x_i^*) z_i^* = 0, \ \ \text{for all} \ \ i = 1, \cdots, n. \tag{2.13}$$

Thus, if the solution x^* satisfies $x_i^* < M$ for every $i = 1, \cdots, n$, then from (2.13) we have $z^* = 0$. Consequently, it follows from (2.12) and (2.13) that

$$\begin{cases} x_i^* F_i(x^*) = 0, \\ x_i \geq 0, \quad F_i(x^*) \geq 0, \ \ \text{for all} \ \ i = 1, \cdots, n. \end{cases}$$

That is, x^* is a solution of the NCP.

In conclusion, we give the following main result of the paper.

Theorem 2.1. *For every $M > 0$, the homotopy equation* (2.1) *satisfies:*

1. *For almost all $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$, 0 is a regular value of the mapping $H_{w^{(0)}} = H(w^{(0)},w,t) : \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0,1] \to \mathbb{R}^{2n}$;*

2. *For a given $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$ ($M > 0$), if 0 is a regular value of the mapping $H_{w^{(0)}}$, then $H^{-1}_{w^{(0)}}(0) = \{(w,t) \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0,1];\ H_{w(0)}(w,t) = 0\}$ is composed by some smooth and simple curves, and one of the curves has the initial point $(w^{(0)},1)$, denoted by $\Gamma_{w^{(0)}}$;*

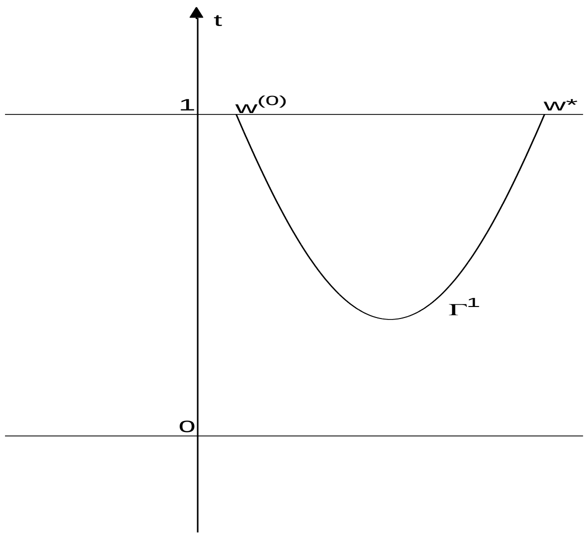

Fig. 2.1. Smooth curve which is diffeomorphic to (0,1]: possible case 1.

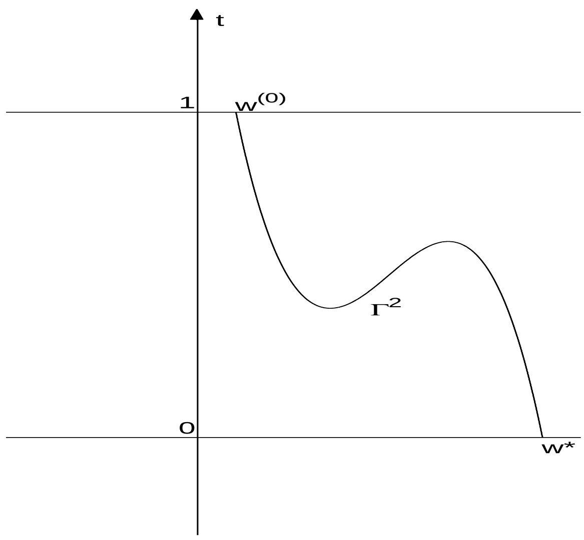

Fig. 2.2. Smooth curve which is diffeomorphic to (0,1]: possible case 2.

3. *The above smooth curve* $\Gamma_{w^{(0)}}$ *is bounded in* $\tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++} \times (0,1]$*, which has an end point* $(w^*, 1)$*, where* $w^* \in K^{(M)} \times \mathbb{R}^{2n}_{+}$*;*

4. *If the end point* $w^* = (x^*, y^*, z^*)$ *of the curve* $\Gamma_{w^{(0)}}$ *satisfies* $x^*_i < M$ *for all* $i = 1, \cdots, n$*, then* x^* *is a solution of NCP.*

From the above theorem, we obtain the following algorithm which searches the solution of NCP.

Algorithm

(i). For a given M_j, let $M = M_j$, we select $w^{(0)} \in \tilde{K}^{(M)} \times \mathbb{R}^{2n}_{++}$, such that 0 is a regular value of the mapping $H_{w^{(0)}}$;

(ii). Solve the following initial-value problem of ordinary differential equations

$$\dot{H}(w^{(0)}, w(s), t(s)) = \left[\frac{\partial H(w^{(0)}, w(s), t(s))}{\partial (w, t)} \right] \left[\begin{array}{c} \dot{w} \\ \dot{t} \end{array} \right] = 0,$$
$$w(0) = w^{(0)}, \quad t(0) = 1,$$

where s is the arc length of the curve $\Gamma_{w^{(0)}}$. Denote the solution by $w^* = (x^*, y^*, z^*)$.

(iii). If x^* satisfies $x_i^* < M$ for all $i = 1, \cdots, n$ or $\langle x^*, F(x^*) \rangle < \varepsilon$ (ε is a given tolerance), then x^* is a solution of NCP. Otherwise, we take $M_{j+1} > M_j$ and go to Step (i).

3. Numerical tests

In the following, we give some numerical examples.

Example 3.1. (Kojima-Shindo Nonlinear complementarity test problem [12])

(i) Degenerate example:

$$g(x) = x, \quad F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}.$$

(ii) Non-degenerate example:

$$g(x) = x, \quad F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 3x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 3x_4 - 1 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}.$$

Example 3.1(i) has two solutions, $x^* = (\sqrt{6}/2, 0, 0, 1/2)$ and $x^{**} = (1, 0, 3, 0)$. The first solution is degenerate (due to $F(x^*) = (0, 2 + \sqrt{6}/2, 0, 0)$) and the second one is non-degenerate (due to $F(x^{**}) = (0, 31, 0, 4)$). Example 3.1(ii) has only one solution $x^* = (\sqrt{6}/2, 0, 0, 1/2)$ which is non-degenerate ($F(x^*) = (0, 2 + \sqrt{6}/2, 5, 0)$).

We will solve the following initial-value problem of ordinary differential equations:

$$\begin{bmatrix} \dfrac{\partial H_1}{\partial x} & \dfrac{\partial H_1}{\partial y} & \dfrac{\partial H_1}{\partial z} & \dfrac{\partial H_1}{\partial t} \\ \dfrac{\partial H_2}{\partial x} & \dfrac{\partial H_2}{\partial y} & \dfrac{\partial H_2}{\partial z} & \dfrac{\partial H_2}{\partial t} \\ \dfrac{\partial H_3}{\partial x} & \dfrac{\partial H_3}{\partial y} & \dfrac{\partial H_3}{\partial z} & \dfrac{\partial H_3}{\partial t} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{t} \end{bmatrix} = 0,$$

$$x(0) = x^{(0)},\ y(0) = y^{(0)},\ z(0) = z^{(0)},\ t(0) = 1,$$

where H_1, H_2, H_3 are defined as (2.1) and we let $M = 10$.

We use the Euler-Newton method to trace the solution curves. For Example 3.1(i), given the initial point $x(0) = (2, 1, 0.5, 2)$, we get the solution $(1.225, 0, 0, 0.5)$; given the initial point $x(0) = (2, 1, 4, 2)$, we get the solution $(1, 0, 3, 0)$. For Example 3.1(ii), given the initial point $x(0) = (2, 1, 0.5, 2)$ or $x(0) = (0, 0, 0, 0)$, we always get the solution $(1.225, 0, 0, 0.5)$.

4. Conclusion

In this paper, we present a new homotopy method for solving the nonlinear complementarity problems. Without the regularity or non-singulary assumptions on $\nabla F(x)$, which is the Jacobian matrix of nonlinear mapping $F(x)$, we prove that our homotopy equations have a bounded solution curve emanating from $(w^{(0)}, 1)$. The numerical tests confirm the efficiency of our homotopy method.

References

[1] Allgower E L, Georg K. Numerical Continuation Method: An Intruduction, Springer Verlag, Berlin, 1990.

[2] Billups S C, Murty K G. Complementarity problem. J. Comput. Appl. Math., 2000, 124: 303-318.

[3] Chow S, Mallet-Paret J, Yorke J A. Finding zeros of maps: Homotopy methods, that are constructive with probability one. Math. Comput., 1978, 32(143): 887-899.

[4] Cottle R W. Nonlinear programs with positively bounded Jacobians, Ph. D. dissertation, Department of Mathematics, University of California, Berkeley, CA, 1964.

[5] Cottle R, Pang J, Stone R. The Linear Complementarity Problem, Academic Press, Boston, 1992.

[6] Eaves B C. Homotopies for computation of fixed points. Math. Program., 1972, 3: 1-22.

[7] Ferris M, Pang J. Engineering and economic applications of complementarity problems. SIAM Rev., 1997, 39: 669-713.

[8] Fischer A. New constrained optimization reformulation of complementarity problems. J. Optimiz. Theory Appl., 1998, 97(1): 105-117.

[9] Garcia C B, Zangwill W I. Pathways to Solutions, Fixed Points and Equilibria, Prentice-Hall, Englewood Cliffs, NJ, 1981.

[10] Harker P T, Pang J S. Finite-dimensional variation inquality and nonlinear complementarity problems: a survey of theory, algorithms and applications. Math. Program., 1990, 48: 161-220.
[11] Karaamardian S. Generalized complementarity problem. J. Optimiz. Theory Appl., 1971, 8: 161-167.
[12] Kojima M, Shindo S. Extensions of Newton and quasi-Newton methods to systems of PC_1 equations. J. Oper. Res. Soc. Jpn., 1986, 29: 352-374.
[13] Liao L, Qi H. A neural network for the linear complementarity problem. Math. Comput. Model., 1999, 29: 9-18.
[14] Liao L, Qi H, Qi L. Solving nonlinear complementarity problems with neural networks: a reformulation method approach. J. Comput. Appl. Math., 2001, 131: 343-359.
[15] Luca T D, Facchinei F, Kanzow C. A semismooth equation approach to the solution of nonlinear complementarity problems. Math. Program., 1996, 75: 407-439.
[16] Naber G L. Topological Method in Euclidean Space, Cambridge University Press, London, 1980.
[17] Qi L, Sun J. A nonsmooth version of Newton's method. Math. Program., 1993, 58: 353-368.
[18] Scarf H E. The approximation of fixed points of a continuous mapping. SIAM J. Appl. Math., 1967, 15: 1328-1342.
[19] Taji K, Miyamoto M. A globally convergent smoothing Newton method for nonsmooth equations and its application to complementarity problems. Comput. Optimiz. Appl., 2002, 22: 81-101.
[20] Todd M J. The Computation of Fixed Points and Applications, Springer, Berlin, 1976.
[21] Watson L T. A variational approach to the linear complementarity problem, Doctoral dissertation, Department of Mathematics, University of Michigan, Ann Arbor, 1974.
[22] Watson L T. Some perturbation theorems for Q-matrics. SIAM J. Appl. Math., 1976, 31: 379-384.
[23] Watson L T. Solving the nonlinear complementarity problem by a homotopy method. SIAM J. Control Optimiz., 1979, 17(1): 36-46.