NUMERICAL MATHEMATICS, A Journal of Chinese Universities (English Series)
Numer. Math. J. Chinese Univ. (English Ser.), issue 2, vol. **16**: 97-111, 2007

Finite Difference Method for Reaction-Diffusion Equation with Nonlocal Boundary Conditions

Jianming Liu*
(*Department of Mathematics, Xuzhou Normal University, Xuzhou 221116, China*
E-mail: jmliu@xznu.edu.cn)
Zhizhong Sun
(*Department of Mathematics, Southeast University, Nanjing 210096, China*
E-mail: zzsun@seu.edu.cn)

Received May 14, 2005; Accepted (in revised version) November 5, 2006

Abstract

In this paper, we present a numerical approach to a class of nonlinear reaction-diffusion equations with nonlocal Robin type boundary conditions by finite difference methods. A second-order accurate difference scheme is derived by the method of reduction of order. Moreover, we prove that the scheme is uniquely solvable and convergent with the convergence rate of order two in a discrete L_2-norm. A simple numerical example is given to illustrate the efficiency of the proposed method.

Keywords: Reaction-diffusion; nonlocal Robin type boundary; finite difference; solvability; convergence.
Mathematics subject classification: 65M06, 65M12, 65M15

1. Introduction

Reaction-diffusion equations with nonlocal boundary conditions have been given considerable attention in recent years, and various methods have been developed for the treatment of these equations (see [1-9]). Most of the discussions in the current literatures are developed to the Dirichlet type nonlocal boundary conditions problem (see [10-17]), and much less is given to the problem with nonlocal Robin type boundary conditions (see [18]). The purpose of this article is to give a numerical treatment to a class of reaction-diffusion equations with nonlocal boundary conditions by finite difference method. The system of equations to be considered is as follows

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\Big(a(x,t)\frac{\partial u}{\partial x}\Big)+b(x,t)\frac{\partial u}{\partial x}+f(u,x,t), \qquad 0<x<1, 0<t\le T, \quad (1.1a)$$

$$a(0,t)\frac{\partial u}{\partial x}(0,t)-\sigma_1(t)u(0,t)=\int_0^1 \alpha(s)u(s,t)ds+g_1(t), \qquad 0\le t\le T, \quad (1.1b)$$

$$a(1,t)\frac{\partial u}{\partial x}(1,t)+\sigma_2(t)u(1,t)=\int_0^1 \beta(s)u(s,t)ds+g_2(t), \qquad 0\le t\le T, \quad (1.1c)$$

$$u(x,0)=\varphi(x), \qquad 0\le x\le 1, \quad (1.1d)$$

*Corresponding author.

where u is an unknown function, $a,\ b,\ f,\ \sigma_j,\ g_j(j=1,2),\ \alpha,\ \beta$ and φ are given functions.

By the influence of Robin type boundary, it is not easy to derive a higher accurate scheme. In [18], the author supposed a and b are independent of t and developed a non-linear monotone iterative difference scheme of (1.1) using the method of upper and lower solutions. However, the truncation error is only $\mathcal{O}(\tau+h)$, and the proof of convergence is not provided. It is also noticed that the proofs of numerical methods in all the recent articles including [18] use the conditions

$$\|\alpha\|_{L_1([0,1])}\equiv\int_0^1|\alpha(s)|ds<1,\ \ \|\beta\|_{L_1([0,1])}\equiv\int_0^1|\beta(s)|ds<1. \tag{1.2}$$

In this article, we develop a linear difference scheme by the method of reduction of order (see [19,20]), and prove the difference scheme is uniquely solvable and of second order rate of convergence in L_2-norm by the energy method. In our proof, it is found that the condition (1.2) is not necessary.

Throughout this article, we suppose that problem (1.1) has a unique smooth solution $u(x,t)\in C_{x,t}^{(4,3)}(\Omega_T)$, where $(x,t)\in\Omega_T\equiv\{0\le x\le 1,\ 0\le t\le T\}$. In addition, the following basic conditions are always assumed: when $|\,\varepsilon_i\,|\le\varepsilon_0,\ i=1,2$ and $(x,t)\in Q_T$, we have

$$\Big|f(u(x,t)+\varepsilon_1,x,t)-f(u(x,t)+\varepsilon_2,x,t)\Big|\le c_1\,|\,\varepsilon_1-\varepsilon_2\,|, \tag{1.3}$$

$$c_0\le a(x,t)\le c_1,\ \ |\,b(x,t)\,|\le c_1,\ \ |\,\sigma_1(t)\,|\le c_1,\ \ |\,\sigma_2(t)\,|\le c_1. \tag{1.4}$$

where $c_0,\ c_1$ and ε_0 are positive constants.

The outline of the article is as follows. In Section 2, a difference scheme for (1.1) is derived and the finite difference system is tri-diagonal at each time level, which can be solved by Thomas' algorithm. In Section 3, it is proved that the difference scheme is uniquely solvable and of second order rate of convergence in L_2-norm. Finally, a numerical example is given to verify the validity of the analytic results.

2. Difference scheme

In order to approximate the boundary condition, we give some lemmas at first.

Lemma 2.1. ([21]) *Let M be an integer, $h=1/M$, and $x_{i+\frac12}=(i+\frac12)h,\ 0\le i\le M-1.$ If $g(x)\in C^2[0,1]$, then*

$$\int_0^1 g(x)dx-h\sum_{i=0}^{M-1}g(x_{i+\frac12})=\frac{1}{24}h^2\frac{d^2g(x)}{dx^2}\bigg|_{x=\eta},\quad \eta\in(0,1).$$

Lemma 2.2. *Let M be an integer, $h=1/M,\ x_i=ih,\ 0\le i\le M;\ x_{i+\frac12}=(i+\frac12)h,\ 0\le i\le M-1.$ If $f(x),\ g(x)\in C^2[0,1]$, then*

$$\int_0^1 f(x)g(x)dx-\frac{h}{2}\sum_{i=0}^{M-1}f(x_{i+\frac12})\Big[g(x_i)+g(x_{i+1})\Big]=\mathcal{O}(h^2).$$

According to Lemma 2.1 and Taylor expansion, the proof is obvious, so we omit it.

Take two positive integers K and M. Let $\tau = T/K,\ h = 1/M$. Cover the domain Ω_T by Q_h^τ, where $Q_h^\tau = \{(x_i, t_k) \equiv (ih,\ k\tau) \mid 0 \le i \le M,\ \ 0 \le k \le K\}$. Let $u_h^\tau = \{u_i^k \mid 0 \le i \le M,\ 0 \le k \le K\}$ and $v_h^\tau = \{v_i^k \mid 0 \le i \le M,\ 0 \le k \le K\}$ be two net functions on Q_h^τ. We introduce the following notations

$$
\begin{aligned}
&u_i^{\widehat{k}} = \frac{1}{2}(u_i^{k+1} + u_i^{k-1}), \ \ \Delta_t u_i^k = (u_i^{k+1} - u_i^{k-1})/(2\tau),\\
&u_{i-\frac{1}{2}}^k = \frac{1}{2}(u_i^k + u_{i-1}^k), \ \ \delta_x u_{i-\frac{1}{2}}^k = \frac{1}{h}(u_i^k - u_{i-1}^k),\\
&\delta_x(a_i^k \delta_x u_i^{\widehat{k}}) = \frac{1}{h}\left(a(x_{i+\frac{1}{2}}, t_k)\delta_x u_{i+\frac{1}{2}}^{\widehat{k}} - a(x_{i-\frac{1}{2}}, t_k)\delta_x u_{i-\frac{1}{2}}^{\widehat{k}}\right),\\
&\delta_x(u^k v^k)_{i-\frac{1}{2}} = \frac{1}{h}(u_i^k v_i^k - u_{i-1}^k v_{i-1}^k),\\
&\langle u^k, v^k\rangle = h\sum_{i=1}^{M} u_{i-\frac{1}{2}}^k v_{i-\frac{1}{2}}^k, \quad \|u^k\| = \sqrt{\langle u^k, u^k\rangle},\\
&\langle \alpha, u^k\rangle = h\sum_{i=1}^{M} \alpha(x_{i-\frac{1}{2}}) u_{i-\frac{1}{2}}^k, \quad \langle \beta, u^k\rangle = h\sum_{i=1}^{M} \beta(x_{i-\frac{1}{2}}) u_{i-\frac{1}{2}}^k.
\end{aligned}
$$

In addition, if $g \in C[0,1]$, we denote

$$
\|g\|_{L_2} = \sqrt{\int_0^1 g(s)^2 ds}.
$$

The difference scheme we will consider for (1.1) is as follows

$$
\begin{aligned}
&\frac{1}{2}\left(\Delta_t u_{i+\frac{1}{2}}^k + \Delta_t u_{i-\frac{1}{2}}^k\right) = \delta_x(a_i^k \delta_x u_i^{\widehat{k}}) + \frac{1}{2}(b_{i-\frac{1}{2}}^k \delta_x u_{i-\frac{1}{2}}^{\widehat{k}} + b_{i+\frac{1}{2}}^k \delta_x u_{i+\frac{1}{2}}^{\widehat{k}})\\
&\qquad + \frac{1}{2}\left[f_{i-\frac{1}{2}}^k(u) + f_{i+\frac{1}{2}}^k(u)\right] - \frac{1}{4}(\sigma_1^k + \sigma_2^k)h^2\delta_x^2 u_i^{\widehat{k}}, \ 1 \le i \le M-1, \ 1 \le k \le K-1,\\
&a_{\frac{1}{2}}^k \delta_x u_{\frac{1}{2}}^{\widehat{k}} - \sigma_1^k u_0^{\widehat{k}} = \langle \alpha, u^k\rangle + g_1^k + \frac{h}{2}\left[\Delta_t u_{\frac{1}{2}}^k - b_{\frac{1}{2}}^k \delta_x u_{\frac{1}{2}}^{\widehat{k}} - f_{\frac{1}{2}}^k(u)\right]\\
&\qquad + \frac{h^2}{4}(\sigma_1^k + \sigma_2^k)\delta_x u_{\frac{1}{2}}^{\widehat{k}}, \ 1 \le k \le K-1, \qquad\qquad (2.1)\\
&a_{M-\frac{1}{2}}^k \delta_x u_{M-\frac{1}{2}}^{\widehat{k}} + \sigma_2^k u_M^{\widehat{k}} = \langle \beta, u^k\rangle + g_2^k - \frac{h}{2}\left[\Delta_t u_{M-\frac{1}{2}}^k - b_{M-\frac{1}{2}}^k \delta_x u_{M-\frac{1}{2}}^{\widehat{k}}\right.\\
&\qquad \left. - f_{M-\frac{1}{2}}^k(u)\right] + \frac{h^2}{4}(\sigma_1^k + \sigma_2^k)\delta_x u_{M-\frac{1}{2}}^{\widehat{k}}, \ 1 \le k \le K-1,\\
&u_i^0 = \varphi(x_i), \quad u_i^1 = \varphi(x_i) + \tau\psi(x_i), \quad 0 \le i \le M,
\end{aligned}
$$

where

$$a^k_{i-\frac{1}{2}}=a(x_{i-\frac{1}{2}},t_k),\ \ b^k_{i-\frac{1}{2}}=b(x_{i-\frac{1}{2}},t_k),\ \ f^k_{i-\frac{1}{2}}(u)=f(u^k_{i-\frac{1}{2}},x_{i-\frac{1}{2}},t_k),$$
$$\sigma^k_j=\sigma_j(t_k),\ \ g^k_j=g_j(t_k),\ \ j=1,2,$$
$$\psi(x)=\frac{d}{dx}\Big(a(x,0)\frac{d\varphi(x)}{dx}\Big)+b(x,0)\frac{d\varphi(x)}{dx}+f(\varphi(x),x,0).$$

At each time level, (2.1) is a tri-diagonal system of linear algebraic equations, which can be solved by Thomas' algorithm.

The remainder of the section is arranged to give the derivation of difference scheme (2.1). If we let

$$v=a(x,t)\frac{\partial u}{\partial x}+[(x-1)\sigma_1(t)+x\sigma_2(t)]u,$$

then (1.1) is equivalent to the following system of equations

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial x}+\frac{\tilde{b}(x,t)}{a(x,t)}v-\Big\{\sigma_1(t)+\sigma_2(t)+\frac{\tilde{b}(x,t)}{a(x,t)}\Big[(x-1)\sigma_1(t)+x\sigma_2(t)\Big]\Big\}u$$
$$+f(u,x,t),\ \ 0<x<1,\ \ 0<t\le T, \tag{2.2a}$$
$$\frac{1}{a(x,t)}v=\frac{\partial u}{\partial x}+\frac{1}{a(x,t)}\Big[(x-1)\sigma_1(t)+x\sigma_2(t)\Big]u,\quad 0<x<1,\ \ 0<t\le T, \tag{2.2b}$$
$$u(x,0)=\varphi(x),\ \ 0\le x\le 1, \tag{2.2c}$$
$$v(0,t)=\int_0^1\alpha(s)u(s,t)ds+g_1(t),\ \ v(1,t)=\int_0^1\beta(s)u(s,t)ds+g_2(t),\ \ 0\le t\le T, \tag{2.2d}$$

where $\tilde{b}(x,t)=b(x,t)-[(x-1)\sigma_1(t)+x\sigma_2(t)]$. Here no derivative boundary conditions occur explicitly.

Define the grid functions

$$U^k_i=u(x_i,t_k),\ \ V^k_i=v(x_i,t_k),\quad 0\le i\le M,\ \ 0\le k\le K.$$

Using Lemma 2.2 to approximate boundary conditions and Taylor expansion at $(x_{i-\frac{1}{2}},t_k)$, we have

$$\Delta_t U^k_{i-\frac{1}{2}}=\delta_x V^{\widehat{k}}_{i-\frac{1}{2}}+\tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1}V^{\widehat{k}}_{i-\frac{1}{2}}-\Big\{\sigma^k_1+\sigma^k_2+\tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1$$
$$+x\sigma_2\Big]^k_{i-\frac{1}{2}}\Big\}\cdot U^{\widehat{k}}_{i-\frac{1}{2}}+f^k_{i-\frac{1}{2}}(U)+P^k_{i-\frac{1}{2}},\ \ 1\le i\le M,\ \ 1\le k\le K-1, \tag{2.3a}$$
$$(a^k_{i-\frac{1}{2}})^{-1}V^{\widehat{k}}_{i-\frac{1}{2}}=\delta_x U^{\widehat{k}}_{i-\frac{1}{2}}+(a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1+x\sigma_2\Big]^k_{i-\frac{1}{2}}U^{\widehat{k}}_{i-\frac{1}{2}}+Q^k_{i-\frac{1}{2}},$$
$$1\le i\le M,\ \ 1\le k\le K-1, \tag{2.3b}$$
$$U^0_i=\varphi(x_i),\quad U^1_i=\varphi(x_i)+\tau\psi(x_i)+e_i,\quad 0\le i\le M, \tag{2.3c}$$
$$V^{\widehat{k}}_0=\langle\alpha,U^k\rangle+g^k_1+r^k_0,\ \ V^{\widehat{k}}_M=\langle\beta,U^k\rangle+g^k_2+r^k_M,\ \ 1\le k\le K-1, \tag{2.3d}$$

and there exists a positive constant c_1 such that

$$|P^k_{i-\frac{1}{2}}| \le c_1(\tau^2+h^2),\ \ 1\le i\le M,\ \ 1\le k\le K-1, \tag{2.4a}$$

$$|Q^k_{i-\frac{1}{2}}| \le c_1(\tau^2+h^2),\ \ 1\le i\le M,\ \ 1\le k\le K-1, \tag{2.4b}$$

$$|e_i| \le c_1\tau^2,\ \ 0\le i\le M, \tag{2.4c}$$

$$|r^k_0| \le c_1(\tau^2+h^2),\ \ |r^k_M| \le c_1(\tau^2+h^2),\ \ 1\le k\le K-1. \tag{2.4d}$$

Neglecting the small quantity terms in (2.3), we construct a difference scheme for (2.2) as follows

$$\begin{aligned}\Delta_t u^k_{i-\frac{1}{2}} = {} & \delta_x v^{\widehat{k}}_{i-\frac{1}{2}} + \tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1} v^{\widehat{k}}_{i-\frac{1}{2}} - \Big\{\sigma^k_1+\sigma^k_2+\tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1 \\ & +x\sigma_2\Big]^k_{i-\frac{1}{2}}\Big\} u^{\widehat{k}}_{i-\frac{1}{2}} + f^k_{i-\frac{1}{2}}(u),\ \ 1\le i\le M,\ \ 1\le k\le K-1,\end{aligned} \tag{2.5a}$$

$$\begin{aligned}(a^k_{i-\frac{1}{2}})^{-1} v^{\widehat{k}}_{i-\frac{1}{2}} = {} & \delta_x u^{\widehat{k}}_{i-\frac{1}{2}} + (a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1+x\sigma_2\Big]^k_{i-\frac{1}{2}} u^{\widehat{k}}_{i-\frac{1}{2}}, \\ & 1\le i\le M,\ 1\le k\le K-1,\end{aligned} \tag{2.5b}$$

$$u^0_i=\varphi(x_i),\quad u^1_i=\varphi(x_i)+\tau\psi(x_i),\quad 0\le i\le M, \tag{2.5c}$$

$$v^{\widehat{k}}_0=\langle\alpha,u^k\rangle+g^k_1,\ \ v^{\widehat{k}}_M=\langle\beta,u^k\rangle+g^k_2,\ \ 1\le k\le K-1\ . \tag{2.5d}$$

At the $(k+1)$-th time level, (2.5) is regarded as a system of linear algebraic equations with respect to $\Big\{u^{k+1}_i, v^{\hat{k}}_i\Big\}^M_{i=0}$. We have the following theorem.

Theorem 2.1. *The difference scheme* (2.5) *is equivalent to* (2.1) *and*

$$\begin{aligned} v^{\widehat{k}}_i = {} & a^k_{i+\frac{1}{2}}\delta_x u^{\widehat{k}}_{i+\frac{1}{2}} + \Big[(x-1)\sigma_1+x\sigma_2\Big]^k_{i+\frac{1}{2}} u^{\widehat{k}}_{i+\frac{1}{2}} \\ & -\frac{1}{2}h\left[\Delta_t u^k_{i+\frac{1}{2}} - \tilde{b}^k_{i+\frac{1}{2}}\delta_x u^{\widehat{k}}_{i+\frac{1}{2}} + (\sigma^k_1+\sigma^k_2)u^{\widehat{k}}_{i+\frac{1}{2}} - f^k_{i+\frac{1}{2}}(u)\right], \\ & \qquad 0\le i\le M-1,\quad 1\le k\le K-1, \\ v^{\widehat{k}}_M = {} & a^k_{M-\frac{1}{2}}\delta_x u^{\widehat{k}}_{M-\frac{1}{2}} + \Big[(x-1)\sigma_1+x\sigma_2\Big]^k_{M-\frac{1}{2}} u^{\widehat{k}}_{M-\frac{1}{2}} \\ & +\frac{1}{2}h\left[\Delta_t u^k_{M-\frac{1}{2}} - \tilde{b}^k_{M-\frac{1}{2}}\delta_x u^{\widehat{k}}_{M-\frac{1}{2}} + (\sigma^k_1+\sigma^k_2)u^{\widehat{k}}_{M-\frac{1}{2}} - f^k_{M-\frac{1}{2}}(u)\right],\ \ 1\le k\le K-1\ . \end{aligned}$$

The proof is similar to that of Theorem 1 in [19], so we omit it.

3. The unique solvability and convergence

In this section, we discuss the solvability of the system of difference equations, stability and convergence of the difference scheme.

Theorem 3.1. *The difference scheme* (2.1) *is uniquely solvable.*

Proof. According to Theorem 2.1, it suffices to prove that difference scheme (2.5) has a unique solution. Since at any time level it is a system of linear algebraic equations with respect to $\{u_i^{k+1}, 0 \le i \le M\} \cup \{v_i^{\hat{k}}, 0 \le i \le M\}$, we only need to prove that its homogeneous system

$$\frac{1}{2\tau} u_{i-\frac{1}{2}}^{k+1} = \delta_x v_{i-\frac{1}{2}}^{\widehat{k}} + \tilde{b}_{i-\frac{1}{2}}^{k} (a_{i-\frac{1}{2}}^{k})^{-1} v_{i-\frac{1}{2}}^{\widehat{k}} - \frac{1}{2} \Big\{ \sigma_1^k + \sigma_2^k + \tilde{b}_{i-\frac{1}{2}}^{k} (a_{i-\frac{1}{2}}^{k})^{-1} \Big[(x-1)\sigma_1 + x\sigma_2 \Big]_{i-\frac{1}{2}}^{k} \Big\} u_{i-\frac{1}{2}}^{k+1}, \quad 1 \le i \le M, \tag{3.1a}$$

$$0 = \frac{1}{2} \delta_x u_{i-\frac{1}{2}}^{k+1} - (a_{i-\frac{1}{2}}^{k})^{-1} v_{i-\frac{1}{2}}^{\widehat{k}} + \frac{1}{2} (a_{i-\frac{1}{2}}^{k})^{-1} \Big[(x-1)\sigma_1 + x\sigma_2 \Big]_{i-\frac{1}{2}}^{k} u_{i-\frac{1}{2}}^{k+1}, \quad 1 \le i \le M, \tag{3.1b}$$

$$v_0^{\widehat{k}} = 0, \quad v_M^{\widehat{k}} = 0\ , \tag{3.1c}$$

has only a trivial solution. Multiplying (3.1a) by $2u_{i-\frac{1}{2}}^{k+1}$, (3.1b) by $4v_{i-\frac{1}{2}}^{\widehat{k}}$, and adding the resulting equations yield

$$\begin{aligned} \frac{1}{\tau} (u_{i-\frac{1}{2}}^{k+1})^2 = {} & 2\delta_x (u^{k+1} v^{\widehat{k}})_{i-\frac{1}{2}} + 2\tilde{b}_{i-\frac{1}{2}}^{k} (a_{i-\frac{1}{2}}^{k})^{-1} u_{i-\frac{1}{2}}^{k+1} v_{i-\frac{1}{2}}^{\widehat{k}} \\ & - \Big\{ \sigma_1^k + \sigma_2^k + \tilde{b}_{i-\frac{1}{2}}^{k} (a_{i-\frac{1}{2}}^{k})^{-1} \Big[(x-1)\sigma_1 + x\sigma_2 \Big]_{i-\frac{1}{2}}^{k} \Big\} (u_{i-\frac{1}{2}}^{k+1})^2 \\ & - 4 (a_{i-\frac{1}{2}}^{k})^{-1} (v_{i-\frac{1}{2}}^{\widehat{k}})^2 + 2 (a_{i-\frac{1}{2}}^{k})^{-1} \Big[(x-1)\sigma_1 + x\sigma_2 \Big]_{i-\frac{1}{2}}^{k} u_{i-\frac{1}{2}}^{k+1} v_{i-\frac{1}{2}}^{\widehat{k}}. \end{aligned}$$

Using conditions (1.4) and the Cauchy inequality gives

$$\begin{aligned} \frac{1}{\tau} (u_{i-\frac{1}{2}}^{k+1})^2 \le {} & 2\delta_x (u^{k+1} v^{\widehat{k}})_{i-\frac{1}{2}} + \frac{6c_1}{c_0} \Big| u_{i-\frac{1}{2}}^{k+1} v_{i-\frac{1}{2}}^{\widehat{k}} \Big| + 2c_1 (1 + \frac{c_1}{c_0}) (u_{i-\frac{1}{2}}^{k+1})^2 - \frac{4}{c_1} (v_{i-\frac{1}{2}}^{\widehat{k}})^2 \\ \le {} & 2\delta_x (u^{k+1} v^{\widehat{k}})_{i-\frac{1}{2}} - \frac{2}{c_1} (v_{i-\frac{1}{2}}^{\widehat{k}})^2 + \Big[\frac{9c_1^3}{2c_0^2} + 2c_1 (1 + \frac{c_1}{c_0}) \Big] (u_{i-\frac{1}{2}}^{k+1})^2. \end{aligned}$$

Consequently, for $1 \le i \le M$,

$$\frac{1}{\tau} (u_{i-\frac{1}{2}}^{k+1})^2 + \frac{2}{c_1} (v_{i-\frac{1}{2}}^{\widehat{k}})^2 \le 2\delta_x (u^{k+1} v^{\widehat{k}})_{i-\frac{1}{2}} + \left[\frac{9c_1^3}{2c_0^2} + 2c_1 (1 + \frac{c_1}{c_0}) \right] (u_{i-\frac{1}{2}}^{k+1})^2.$$

Multiplying the above inequality by h, summing up for i from 1 to M, and noticing

$$h \sum_{i=1}^{M} \delta_x (u^{k+1} v^{\widehat{k}})_{i-\frac{1}{2}} = u_M^{k+1} v_M^{\widehat{k}} - u_0^{k+1} v_0^{\widehat{k}} = 0,$$

we can obtain

$$\frac{1}{\tau} \|u^{k+1}\|^2 + \frac{2}{c_1} \|v^{\widehat{k}}\|^2 \le c_2 \|u^{k+1}\|^2,$$

where $c_2 = \frac{9c_1^3}{2c_0^2} + 2c_1(1+\frac{c_1}{c_0})$. Thus, we have

$$(1-\tau c_2)\|u^{k+1}\|^2 + \frac{2}{c_1}\tau\|v^{\widehat{k}}\|^2 \le 0.$$

Therefore,

$$u^{k+1}_{i-\frac{1}{2}} = v^{\widehat{k}}_{i-\frac{1}{2}} = 0, \quad 1 \le i \le M,$$

for τ less than $1/c_2$. By (3.1b), we can obtain $\delta_x u^{k+1}_{i-\frac{1}{2}} = 0,\ 1 \le i \le M$. So we have $u_i^{k+1} = 0,\ 0 \le i \le M$. From (3.1c), we get $v_i^{\widehat{k}} = 0,\ 0 \le i \le M$. This completes the proof. ■

Theorem 3.2. *If there are two positive constants c_3 and ϵ such that*

$$\tau = c_3 h^{\frac{1}{4}+\epsilon}, \tag{3.2}$$

then for h sufficiently small, the solution $\{u_i^k\}$ of difference scheme (2.1) is convergent to the solution $u(x,t)$ of (1.1) with the convergence rate of $\mathcal{O}(\tau^2+h^2)$ in the L_2-norm.

Proof. According to Theorem 2.1, it suffices to prove that the solution of difference scheme (2.5) is convergent to the solution of (2.2), and the convergence order is $\mathcal{O}(\tau^2+h^2)$ in the L_2-norm.

Define the following net functions by

$$\widetilde{u}_i^k = U_i^k - u_i^k,\ \widetilde{v}_i^k = V_i^k - v_i^k,\ 0 \le i \le M,\ 0 \le k \le K.$$

Subtracting (2.5) from (2.3) respectively, we obtain the error equations as follows

$$\begin{aligned}
\Delta_t \widetilde{u}^k_{i-\frac{1}{2}} &= \delta_x \widetilde{v}^{\widehat{k}}_{i-\frac{1}{2}} + \tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1}\widetilde{v}^{\widehat{k}}_{i-\frac{1}{2}} \\
&\quad - \left\{\sigma_1^k + \sigma_2^k + \tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1 + x\sigma_2\Big]^k_{i-\frac{1}{2}}\right\}\widetilde{u}^{\widehat{k}}_{i-\frac{1}{2}} \\
&\quad + \Big[f^k_{i-\frac{1}{2}}(U) - f^k_{i-\frac{1}{2}}(u)\Big] + P^k_{i-\frac{1}{2}},\ \ 1 \le i \le M,\ \ 1 \le k \le K-1, \\
0 &= \ \delta_x \widetilde{u}^{\widehat{k}}_{i-\frac{1}{2}} - (a^k_{i-\frac{1}{2}})^{-1}\widetilde{v}^{\widehat{k}}_{i-\frac{1}{2}} + (a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1 + x\sigma_2\Big]^k_{i-\frac{1}{2}}\widetilde{u}^{\widehat{k}}_{i-\frac{1}{2}} + Q^k_{i-\frac{1}{2}}, \\
&\qquad 1 \le i \le M,\ \ 1 \le k \le K-1, \\
\widetilde{u}_i^0 &= \ 0, \quad \widetilde{u}_i^1 = \ e_i, \quad 0 \le i \le M, \\
\widetilde{v}_0^{\widehat{k}} &= \ \langle \alpha, \widetilde{u}^k\rangle + r_0^k,\ \ \widetilde{v}_M^{\widehat{k}} = \langle \beta, \widetilde{u}^k\rangle + r_M^k, \quad 1 \le k \le K-1.
\end{aligned} \tag{3.3}$$

We shall then show that

$$\|\widetilde{u}^k\| \le c_7(\tau^2+h^2),\ \ 0 \le k \le K \tag{3.4}$$

is valid for sufficiently small τ and h, where

$$c_4 = (1+2c_1c_0^{-1})(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)+2c_1c_0^{-1}+4c_1^2c_0^{-1}+5c_1+18c_1^3c_0^{-2}+2\,,$$
$$c_5 = (1+2c_1c_0^{-1})(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)+2c_1c_0^{-2}(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)^2\,,$$
$$c_6 = 4c_1^2(1+2c_1c_0^{-1})+8c_1^3c_0^{-2}+c_1^2+2c_1^3,$$
$$c_7 = \exp\Big(\frac{3}{4}\max\{c_1+c_5,c_4\}T\Big)\cdot\sqrt{c_1^2+\frac{c_6}{\max\{c_1+c_5,c_4\}}}.$$

Let

$$\widetilde{w}_i^{\widehat{k}} = \widetilde{v}_i^{\widehat{k}} - (1-x_i)\Big(\langle\alpha,\widetilde{u}^k\rangle + r_0^k\Big) - x_i\Big(\langle\beta,\widetilde{u}^k\rangle + r_M^k\Big).$$

Then, we have

$$\begin{aligned}
&\widetilde{v}_{i-\frac12}^{\widehat{k}} = \widetilde{w}_{i-\frac12}^{\widehat{k}} + (1-x_{i-\frac12})\Big(\langle\alpha,\widetilde{u}^k\rangle + r_0^k\Big) + x_i\Big(\langle\beta,\widetilde{u}^k\rangle + r_M^k\Big),\\
&\delta_x\widetilde{v}_{i-\frac12}^{\widehat{k}} = \delta_x\widetilde{w}_{i-\frac12}^{\widehat{k}} - \Big(\langle\alpha,\widetilde{u}^k\rangle + r_0^k\Big) + \Big(\langle\beta,\widetilde{u}^k\rangle + r_M^k\Big),\\
&\widetilde{w}_0^{\widehat{k}} = 0,\ \ \widetilde{w}_M^{\widehat{k}} = 0,\ \ 1\le k\le K-1.
\end{aligned}\tag{3.5}$$

Substituting (3.5) into (3.3) respectively, we obtain

$$\begin{aligned}
\Delta_t\widetilde{u}_{i-\frac12}^k =\ & \delta_x\widetilde{w}_{i-\frac12}^{\widehat{k}} - \Big(\langle\alpha,\widetilde{u}^k\rangle + r_0^k\Big) + \Big(\langle\beta,\widetilde{u}^k\rangle + r_M^k\Big)\\
&+\tilde{b}_{i-\frac12}^k(a_{i-\frac12}^k)^{-1}\Big[\widetilde{w}_{i-\frac12}^{\widehat{k}} + (1-x_{i-\frac12})\Big(\langle\alpha,\widetilde{u}^k\rangle + r_0^k\Big) + x_{i-\frac12}\Big(\langle\beta,\widetilde{u}^k\rangle + r_M^k\Big)\Big]\\
&-\Big\{\sigma_1^k + \sigma_2^k + \tilde{b}_{i-\frac12}^k(a_{i-\frac12}^k)^{-1}\Big[(x-1)\sigma_1 + x\sigma_2\Big]_{i-\frac12}^k\Big\}\widetilde{u}_{i-\frac12}^{\widehat{k}}\\
&+\Big[f_{i-\frac12}^k(U) - f_{i-\frac12}^k(u)\Big] + P_{i-\frac12}^k,\quad 1\le i\le M,\ \ 1\le k\le K-1,
\end{aligned}\tag{3.6a}$$

$$\begin{aligned}
0 =\ & \delta_x\widetilde{u}_{i-\frac12}^{\widehat{k}} - (a_{i-\frac12}^k)^{-1}\Big[\widetilde{w}_{i-\frac12}^{\widehat{k}} + (1-x_{i-\frac12})\Big(\langle\alpha,\widetilde{u}^k\rangle + r_0^k\Big) + x_{i-\frac12}\Big(\langle\beta,\widetilde{u}^k\rangle + r_M^k\Big)\Big]\\
&+(a_{i-\frac12}^k)^{-1}\Big[(x-1)\sigma_1 + x\sigma_2\Big]_{i-\frac12}^k\widetilde{u}_{i-\frac12}^{\widehat{k}} + Q_{i-\frac12}^k,\ 1\le i\le M,\ \ 1\le k\le K-1,
\end{aligned}\tag{3.6b}$$

$$\widetilde{u}_i^0 =\ 0,\quad \widetilde{u}_i^1 =\ e_i,\quad 0\le i\le M,\tag{3.6c}$$

$$\widetilde{w}_0^{\widehat{k}} =\ 0,\quad \widetilde{w}_M^{\widehat{k}} = 0,\quad 1\le k\le K-1.\tag{3.6d}$$

From (2.4c), and (3.6c)-(3.6d), we can find that (3.4) is valid for $k=0,1$. Suppose that

$$\|\widetilde{u}^k\| \le c_7(\tau^2+h^2),\quad 1\le k\le l$$

is true. Then, we shall prove (3.4) is also valid for $k=l+1$. From (3.2), we have

$$|\widetilde{u}_{i-\frac12}^k| \le \sqrt{2}c_7(\tau^2+h^2)/h^{\frac12} \le \varepsilon_0\,,\quad 1\le i\le M,\ \ 1\le k\le l\,.$$

Then, by (1.3), we now get

$$\left|f^k_{i-\frac{1}{2}}(U)-f^k_{i-\frac{1}{2}}(u)\right| \le c_1|\widetilde{u}^k_{i-\frac{1}{2}}|\,, \quad 1\le i\le M,\ 1\le k\le l\,. \tag{3.7}$$

Multiplying (3.6a) by $2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}$ and multiplying (3.6b) by $2\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}$, and adding the resulting equations yield

$$\begin{aligned}
2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Delta_t\widetilde{u}^k_{i-\frac{1}{2}} =\;& 2\delta_x(\widehat{\widetilde{u}^k}\widehat{\widetilde{w}^k})_{i-\frac{1}{2}} - 2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Big(\langle\alpha,\widetilde{u}^k\rangle + r^k_0\Big) + 2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Big(\langle\beta,\widetilde{u}^k\rangle + r^k_M\Big)\\
&+2\tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1}\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Big[\widehat{\widetilde{w}^k}_{i-\frac{1}{2}} + (1-x_{i-\frac{1}{2}})\Big(\langle\alpha,\widetilde{u}^k\rangle + r^k_0\Big) + x_{i-\frac{1}{2}}\Big(\langle\beta,\widetilde{u}^k\rangle + r^k_M\Big)\Big]\\
&-2\Big\{\sigma^k_1+\sigma^k_2+\tilde{b}^k_{i-\frac{1}{2}}(a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1+x\sigma_2\Big]^k_{i-\frac{1}{2}}\Big\}(\widehat{\widetilde{u}^k}_{i-\frac{1}{2}})^2\\
&+2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Big[f^k_{i-\frac{1}{2}}(U)-f^k_{i-\frac{1}{2}}(u)\Big] + 2(a^k_{i-\frac{1}{2}})^{-1}\Big[(x-1)\sigma_1+x\sigma_2\Big]^k_{i-\frac{1}{2}}\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}\\
&-2(a^k_{i-\frac{1}{2}})^{-1}\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}\Big[\widehat{\widetilde{w}^k}_{i-\frac{1}{2}} + (1-x_{i-\frac{1}{2}})\Big(\langle\alpha,\widetilde{u}^k\rangle + r^k_0\Big) + x_{i-\frac{1}{2}}\Big(\langle\beta,\widetilde{u}^k\rangle + r^k_M\Big)\Big]\\
&+2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}P^k_{i-\frac{1}{2}} + 2\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}Q^k_{i-\frac{1}{2}}\,, \qquad 1\le i\le M,\ 1\le k\le l\,.
\end{aligned}$$

From (1.4) and (3.7), we obtain

$$\begin{aligned}
2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Delta_t\widetilde{u}^k_{i-\frac{1}{2}} \le\;& 2\delta_x(\widehat{\widetilde{u}^k}\widehat{\widetilde{w}^k})_{i-\frac{1}{2}} + 2\Big|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Big(\langle\alpha,\widetilde{u}^k\rangle + r^k_0\Big)\Big| + 2\Big|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Big(\langle\beta,\widetilde{u}^k\rangle + r^k_M\Big)\Big|\\
&+\frac{4c_1}{c_0}\Big|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\Big[\widehat{\widetilde{w}^k}_{i-\frac{1}{2}} + (1-x_{i-\frac{1}{2}})\Big(\langle\alpha,\widetilde{u}^k\rangle + r^k_0\Big) + x_{i-\frac{1}{2}}\Big(\langle\beta,\widetilde{u}^k\rangle + r^k_M\Big)\Big]\Big|\\
&+2(2c_1+\frac{2c_1^2}{c_0})(\widehat{\widetilde{u}^k}_{i-\frac{1}{2}})^2 + 2c_1\Big|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\widetilde{u}^k_{i-\frac{1}{2}}\Big| + \frac{2c_1}{c_0}\Big|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}\Big| - \frac{2}{c_1}(\widehat{\widetilde{w}^k}_{i-\frac{1}{2}})^2\\
&+\frac{2}{c_0}\Big|\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}\Big[(1-x_{i-\frac{1}{2}})\Big(\langle\alpha,\widetilde{u}^k\rangle + r^k_0\Big) + x_{i-\frac{1}{2}}\Big(\langle\beta,\widetilde{u}^k\rangle + r^k_M\Big)\Big]\Big|\\
&+2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}P^k_{i-\frac{1}{2}} + 2\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}Q^k_{i-\frac{1}{2}}\\
\le\;& 2\delta_x(\widehat{\widetilde{u}^k}\widehat{\widetilde{w}^k})_{i-\frac{1}{2}} + 2(1+\frac{2c_1}{c_0})(\|\alpha\|_h+\|\beta\|_h)|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}|\cdot\|\widetilde{u}^k\| + 2(1+\frac{2c_1}{c_0})(|r^k_0|+|r^k_M|)|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}|\\
&+2(2c_1+\frac{2c_1^2}{c_0})(\widehat{\widetilde{u}^k}_{i-\frac{1}{2}})^2 + 2c_1|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}|\cdot|\widetilde{u}^k_{i-\frac{1}{2}}| + \frac{6c_1}{c_0}|\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}| - \frac{2}{c_1}(\widehat{\widetilde{w}^k}_{i-\frac{1}{2}})^2\\
&+\frac{2}{c_0}(\|\alpha\|_h+\|\beta\|_h)|\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}|\cdot\|\widetilde{u}^k\| + \frac{2}{c_0}(|r^k_0|+|r^k_M|)|\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}| + 2\widehat{\widetilde{u}^k}_{i-\frac{1}{2}}P^k_{i-\frac{1}{2}} + 2\widehat{\widetilde{w}^k}_{i-\frac{1}{2}}Q^k_{i-\frac{1}{2}},
\end{aligned}$$

where $\|\alpha\|_h=\sqrt{\langle\alpha,\alpha\rangle}$, $\|\beta\|_h=\sqrt{\langle\beta,\beta\rangle}$. Noticing that

$$\|\alpha\|_h \le \|\alpha\|_{L^2}+\frac{1}{2}, \qquad \|\beta\|_h \le \|\beta\|_{L^2}+\frac{1}{2} \tag{3.8}$$

for sufficiently small h, (2.4a)-(2.4b), (2.4d) and (3.8), we obtain

$$
\begin{aligned}
&2\widehat{\widetilde{u}}^k_{i-\frac{1}{2}}\Delta_t\widetilde{u}^k_{i-\frac{1}{2}}\\
\le\,& 2\delta_x(\widehat{\widetilde{u}}^k\widehat{\widetilde{w}}^k)_{i-\frac{1}{2}}+(1+\frac{2c_1}{c_0})(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)\Big[(\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2+\|\widetilde{u}^k\|^2\Big]\\
&+(1+\frac{2c_1}{c_0})\Big[4c_1^2(\tau^2+h^2)^2+(\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2\Big]+2(2c_1+\frac{2c_1^2}{c_0})(\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2+c_1(\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2\\
&+c_1(\widetilde{u}^k_{i-\frac{1}{2}})^2+\frac{1}{2c_1}(\widehat{\widetilde{w}}^k_{i-\frac{1}{2}})^2+\frac{18c_1^3}{c_0^2}(\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2-\frac{2}{c_1}(\widehat{\widetilde{w}}^k_{i-\frac{1}{2}})^2+\frac{1}{2c_1}(\widehat{\widetilde{w}}^k_{i-\frac{1}{2}})^2\\
&+\frac{2c_1}{c_0^2}(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)^2\|u^k\|^2+\frac{1}{2c_1}(\widehat{\widetilde{w}}^k_{i-\frac{1}{2}})^2+\frac{8c_1^3}{c_0^2}(\tau^2+h^2)^2\\
&+(\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2+c_1^2(\tau^2+h^2)^2+\frac{1}{2c_1}(\widehat{\widetilde{w}}^k_{i-\frac{1}{2}})^2+2c_1^3(\tau^2+h^2)^2\\
\le\,& 2\delta_x(\widehat{\widetilde{u}}^k\widehat{\widetilde{w}}^k)_{i-\frac{1}{2}}+\Big[(1+\frac{2c_1}{c_0})(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)+(1+\frac{2c_1}{c_0})\\
&+2(2c_1+\frac{2c_1^2}{c_0})+c_1+\frac{18c_1^3}{c_0^2}+1\Big](\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2+\Big[(1+\frac{2c_1}{c_0})(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)\\
&+\frac{2c_1}{c_0^2}(\|\alpha\|_{L^2}+\|\beta\|_{L^2}+1)^2\Big]\|\widetilde{u}^k\|^2+c_1(\widetilde{u}^k_{i-\frac{1}{2}})^2\\
&+\Big[4c_1^2(1+\frac{2c_1}{c_0})+\frac{8c_1^3}{c_0^2}+c_1^2+2c_1^3\Big](\tau^2+h^2)^2,\quad 1\le i\le M,\ 1\le k\le l\,.
\end{aligned}
$$

So we have

$$
\begin{aligned}
2\widehat{\widetilde{u}}^k_{i-\frac{1}{2}}\Delta_t\widetilde{u}^k_{i-\frac{1}{2}}=\,&2\delta_x(\widehat{\widetilde{u}}^k\widehat{\widetilde{w}}^k)_{i-\frac{1}{2}}+c_1(\widetilde{u}^k_{i-\frac{1}{2}})^2+c_4(\widehat{\widetilde{u}}^k_{i-\frac{1}{2}})^2\\
&+c_5\|\widetilde{u}^k\|^2+c_6(\tau^2+h^2)^2,\ \ 1\le i\le M.
\end{aligned}\tag{3.9}
$$

Multiplying (3.9) by h and summing up for i from 1 to M, we can get

$$
\begin{aligned}
\frac{1}{\tau}\Big(\|\widetilde{u}^{k+1}\|^2-\|\widetilde{u}^{k-1})\|^2\Big)\ \le\ &2h\sum_{i=1}^{M}\delta_x(\widehat{\widetilde{u}}^k\widehat{\widetilde{w}}^k)_{i-\frac{1}{2}}+(c_1+c_5)\|\widetilde{u}^k\|^2\\
&+\frac{1}{2}c_4(\|\widetilde{u}^{k+1}\|^2+\|\widetilde{u}^{k-1}\|^2)+c_6(\tau^2+h^2)^2.
\end{aligned}
$$

Noticing that

$$
h\sum_{i=1}^{M}\delta_x(\widehat{\widetilde{u}}^k\widehat{\widetilde{w}}^k)_{i-\frac{1}{2}}=\widehat{\widetilde{u}}^k_M\widehat{\widetilde{w}}^k_M-\widehat{\widetilde{u}}^k_0\widehat{\widetilde{w}}^k_0=0\,,\tag{3.10}
$$

we have

$$\begin{aligned}&\frac{1}{\tau}(\|\widetilde{u}^{k+1}\|^2-\|\widetilde{u}^{k-1}\|^2)\\ \le\ &(c_1+c_5)\|\widetilde{u}^k\|^2+\frac{1}{2}c_4(\|\widetilde{u}^{k+1}\|^2+\|\widetilde{u}^{k-1}\|^2)+c_6(\tau^2+h^2)^2,\ \ 1\le k\le l.\end{aligned}$$

Denote $E^k=\|\widetilde{u}^k\|^2+\|\widetilde{u}^{k-1}\|^2$. Then we have

$$\frac{1}{\tau}(E^{k+1}-E^k)\le\frac{1}{2}\max\{c_1+c_5,c_4\}(E^{k+1}+E^k)+c_6(\tau^2+h^2)^2\ ,\ \ 1\le k\le l,$$

or

$$\begin{aligned}&\Big[1-\frac{1}{2}\max\{c_1+c_5,c_4\}\tau\Big]E^{k+1}\\ \le\ &\Big[1+\frac{1}{2}\max\{c_1+c_5,c_4\}\tau\Big]E^k+c_6\tau(\tau^2+h^2)^2,\ \ 1\le k\le l.\end{aligned}$$

Now we can easily see that

$$E^{k+1}\le\Big[1+\frac{3}{2}\max\{c_1+c_5,c_4\}\tau\Big]E^k+\frac{3}{2}c_6\tau(\tau^2+h^2)^2,\ 1\le k\le l\ ,$$

provided that $\max\{c_1+c_5,c_4\}\tau\le\frac{2}{3}$. With the help of Gronwall inequality, we have

$$E^{k+1}\le\exp\Big(\frac{3}{2}\max\{c_1+c_5,c_4\}k\tau\Big)\cdot\Big[E^1+\frac{c_6}{\max\{c_1+c_5,c_4\}}(\tau^2+h^2)^2\Big],\ \ 1\le k\le l\ .$$

By (2.4c),(3.6c) and (3.6d), we can get

$$\begin{aligned}E^{k+1}&\le\exp\Big(\frac{3}{2}\max\{c_1+c_5,c_4\}T\Big)\cdot\Big(c_1^2+\frac{c_6}{\max\{c_1+c_5,c_4\}}\Big)(\tau^2+h^2)^2\\&=c_7^2(\tau^2+h^2)^2,\ 1\le k\le l\ .\end{aligned}$$

That is

$$\|\widetilde{u}^{k+1}\|^2+\|\widetilde{u}^k\|^2\le c_7^2(\tau^2+h^2)^2,\ \ 1\le k\le l\ ,$$

and we have $\|\widetilde{u}^{l+1}\|\le c_7(\tau^2+h^2)$. By the inductive procedure, the theorem is proved. ■

4. Numerical example

Consider the following problem [18]

$$\begin{aligned}&u_t-0.1u_{xx}+5u=u(1-u)+q(x,t), \qquad 0<x<1,\quad t>0,\\&u_x(0,t)-u(0,t)=\frac{1}{2}\int_0^1 u(s,t)ds+g_0(t),\quad t>0,\\&u_x(1,t)+u(1,t)=\int_0^1\exp(-s)u(s,t)ds+g_1(t),\quad t>0,\\&u(x,0)=\psi(x),\quad 0<x<1.\end{aligned}\tag{4.1}$$

Table 4.1: Some numerical results at t=0.5.

$M \setminus (x,t)$	(0.0, 0.5)	(0.2, 0.5)	(0.4, 0.5)	(0.6, 0.5)	(0.8, 0.5)	(1.0, 0.5)
10	0.011153	0.375917	0.663794	0.871892	0.996069	1.032996
20	0.002601	0.365478	0.648974	0.852229	0.974096	1.013638
40	0.000619	0.363012	0.645394	0.847406	0.968646	1.008806
80	0.000150	0.362429	0.644540	0.846245	0.967328	1.007635
160	0.000037	0.362288	0.644332	0.845962	0.967006	1.007350
320	0.000009	0.362254	0.644282	0.845893	0.966927	1.007279
640	0.000002	0.362245	0.644269	0.845876	0.966907	1.007262
$u(x,0.5)$	0.000000	0.362242	0.644265	0.845870	0.966901	1.007256

Table 4.2: The absolute errors of some numerical solutions at t=0.5.

$M \setminus (x,t)$	(0.0, 0.5)	(0.2, 0.5)	(0.4, 0.5)	(0.6, 0.5)	(0.8, 0.5)	(1.0, 0.5)
10	1.115e-2	1.367e-2	1.953e-2	2.602e-2	2.917e-2	2.574e-2
20	2.601e-3	3.236e-3	4.709e-3	6.359e-3	7.195e-3	6.382e-3
40	6.189e-4	7.701e-4	1.130e-3	1.535e-3	1.745e-3	1.550e-3
80	1.499e-4	1.866e-4	2.748e-4	3.748e-4	4.269e-4	3.795e-4
160	3.683e-5	4.583e-5	6.764e-5	9.242e-5	1.054e-4	9.375e-5
320	9.121e-6	1.135e-5	1.677e-5	2.294e-5	2.617e-5	2.329e-5
640	2.270e-6	2.825e-6	4.175e-6	5.713e-6	6.520e-6	5.802e-6

If we let

$$
\begin{aligned}
&u^*(x,t) = x(2-x) + 0.1e^{-\alpha t}\sin\frac{\pi x}{2},\ \alpha = 5+\pi^2/40,\\
&q(x,t) = 0.2 + 5x(2-x) - u^*(x,t)(1-u^*(x,t)),\\
&g^{(0)}(t) = u_x^*(0,t) - u^*(0,t) - \frac{1}{2}\int_0^1 u^*(x,t)dx,\\
&g^{(1)}(t) = u_x^*(1,t) + u^*(1,t) - \int_0^1 \exp(-x)(x)u^*(x,t)dx,\\
&\psi(x) = x(2-x) + 0.1\sin\frac{\pi x}{2},
\end{aligned}
$$

we can easily find that $u^*(x,t)$ is the exact solution of (4.1). We compute the numerical solutions of this problem by using difference scheme (2.1) and let $\tau = \frac{1}{2}h$ or $K = 2M$. Then at each time level, the difference scheme can be written as systems of $M+1$ tri-diagonal linear algebraic equations, which is solved by Thomas' algorithm.

Table 4.1 gives some numerical results and exact values at some points at the time $t = 0.5$. Table 4.2 gives the absolute errors of the numerical solutions at some points at $t = 0.5$, and this is also shown in Fig. 4.1. Table 4.3 gives the maximum errors of the numerical solutions. The maximum error is defined as follows

$$
\|u - u_{h\tau}\|_\infty = \max_{1\le k\le K}\left\{\max_{0\le i\le M}|u(x_i,t_k) - u_i^k|\right\}.
$$

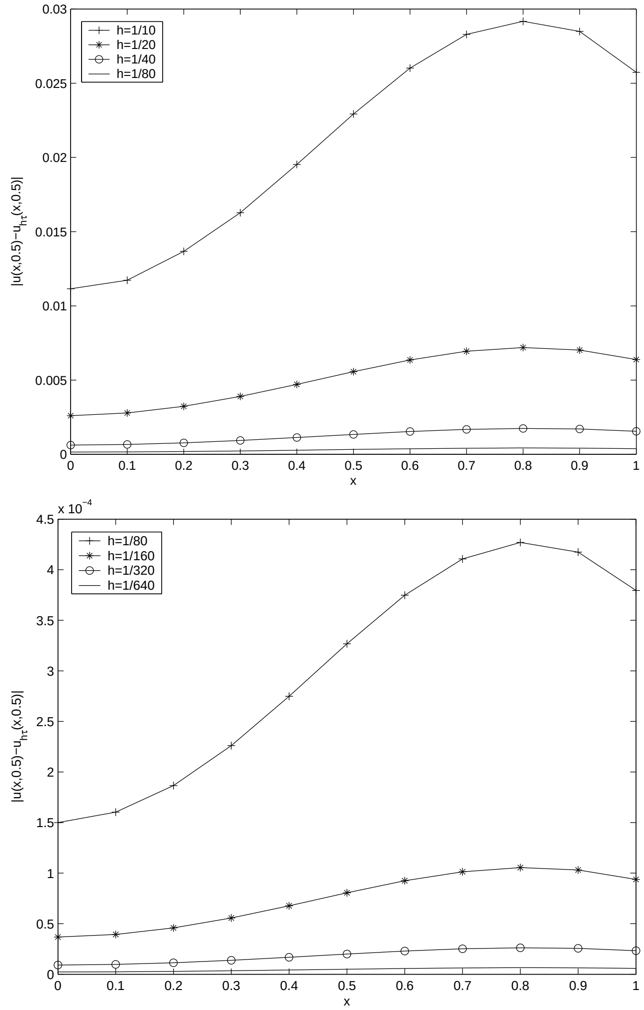

Fig. 4.1. The errors of the numerical solutions at t=0.5.

Table 4.3: The maximum errors of the numerical solutions.

M	10	20	40	80	160	320	640
$\|u-u_{h\tau}\|_\infty$	3.61e-1	9.24e-2	2.25e-2	5.51e-3	1.36e-3	3.37e-4	8.40e-5

From these tables, we may see the errors of difference scheme (2.1) decrease about by a factor of 4 as the mesh size is reduced by a factor of 2.

Now, suppose

$$\|u-u_{h\tau}\|_\infty \approx ch^p.$$

Then it can be verified that

$$-\log\|u-u_{h\tau}\|_\infty \approx -\log c + p(-\log h).$$

Using the data in Table 4.3 and with the help of MATLAB, we obtain linear fitting functions

$$-\log\|u-u_{h\tau}\|_\infty \approx -3.6377+2.0160(-\log h).$$

5. Conclusion

In this article, we present a difference scheme for the reaction-diffusion equation with nonlocal boundary conditions. The difference scheme is derived by the method of reduction of order. For nonlocal boundary conditions, we build a three-level scheme which make the integral term achieve value at intermediate level. Consequently, we can get a tri-diagonal system of linear algebraic equations at each time level, which can be solved easily by Thomas' algorithm. The solvability and convergence are proved by the energy method. The convergence order is $\mathcal{O}(\tau^2+h^2)$. In the proof, we also find the condition (1.2) is not necessary. A numerical example demonstrates the theoretical results. The method presented in this paper also can be used to solve the reaction-diffusion equation with nonlinear and nonlocal boundary conditions (see [22]).

Acknowledgment The work of Liu was supported by National Natural Science Foundation of China (Tian-yuan Foundation) under grant 10626044, and Foundation of Research Startup of Xuzhou Normal University (KY2004111). The work of Sun was supported by National Natural Science Foundation of China under grant 10471023.

The authors thank the referees for their many valuable suggestions.

References

[1] Day W A. Existence of a property of the heat equation to linear thermoelasticity and other theories. Quart. Appl. Math., 1982, 40: 319-330.
[2] Day W A. A decreasing property of solutions of parabolic equations with applications to thermoelasticity. Quart. Appl. Math., 1983, 41: 468-475.
[3] Boley P A, Weiner J H. Theory of thermal stresses, Wiley, New York, 1960.
[4] Carlson D E. Linear thermoelasticity. Encyclopedia of Physics, VI a/2, Springer, Berlin, 1972.
[5] Friedman A. Monotone decay of solutions of parabolic equations with nonlocal boundary conditions. Quart. Appl. Math., 1986, 44: 401-407.
[6] Kawohl B. Remarks on a paper by W.A.Day on a maximum principle under nonlocal boundary conditions. Quart. Appl. Math., 1987, 44: 751-752.
[7] Deng K. Comparison principle for some nonlocal problems. Quart. Appl. Math., 1992, 50: 517-522.
[8] Pao C V. Dynamics of reaction diffusion equations with nonlocal boundary conditions. Quart. Appl. Math., 1995, 53: 173-186.
[9] Pao C V. Asymptotic behavior of solutions of solutions of reaction-diffusion equations with nonlocal boundary conditions. J. Comput. Appl. Math., 1998, 88: 225-238.
[10] Ekolin G. Finite difference methods for a nonlocal boundary value problem for the heat equation. BIT, 1991, 31: 245-261.
[11] Araújo A L, Oliveira F A. Semi-discretization method for the heat equation with non-local boundary conditions. Commun. Numer. Meth. Engrg., 1994, 10: 751-758.

[12] Liu Y K. Numerical solution of the heat equation with nonlocal boundary conditions. J. Comput. Appl. Math., 1999, 110: 115-127.

[13] Borovykh N. Stability in the numerical solution of the heat equation with nonlocal boundary conditions. Appl. Numer. Math., 2002, 42: 17-27.

[14] Tullie T A. Numerical solutions of a reaction-diffusion equation with a nonlocal boundary conditions. Master's project, North Carolina State University, 1997.

[15] Lin Y, Xu S, Yin H M. Finite difference approximations for a class of nonlocal parabolic equations. Int. J. Math. Math. Sci., 1997, 20: 147-163.

[16] Fairweather G, López-Marcos J C. Galerkin methods for a semilinear parabolic problem with nonlocal boundary conditions. Adv. Comput. Math., 1996, 6: 243-262.

[17] Sun Z Z. A high-order difference scheme for a nonlocal boundary value probleme for the heat equation. Comput. Method. Appl. Math., 2001, 1: 398-414.

[18] Pao C V. Numerical solutions of reaction-diffusion equations with nonlocal boundary conditions. J. Comput. Appl. Math., 2001, 136: 227-243.

[19] Sun Z Z. A class of second-order accurate difference schemes for quasi-linear parabolic differencial equations. Math. Numer. Sinica (in Chinese), 1994, 16(4): 347-361.

[20] Sun Z Z, Zhu Y L. A second order accurate difference scheme for the heat equation with concentrated capacity. Numer. Math., 2004, 97(2): 379-395.

[21] Stoer J, Bulirsch R. Introduction to Numerical Analysis. Springer-Verlag, New York, 1993.

[22] Liu J. Finite difference method for reaction-diffusion equation with nonlinear and nonlocal boundary conditions. Numer. Math. J. Chinese Univ., accepted (in Chinese).