NUMERICAL MATHEMATICS, A Journal of Chinese Universities (English Series)
Numer. Math. J. Chinese Univ. (English Ser.), issue 2, vol. **16**: 171-180, 2007

Computational Aspect For Function-Valued Padé-Type Approximation

Chuanqing Gu* **and Jindong Shen**
(*Department of Mathematics, Shanghai University, Shanghai 200444, China*
E-mail: cqgu@staff.shu.edu.cn, dong81206@163.com)

Received September 2, 2005; Accepted (in revised version) June 27, 2006

Abstract

The computational problems of two special determinants are investigated. Those determinants appear in the construction of the function-valued Padé-type approximation for computing Fredholm integral equation of the second kind. The main tool to be used in this paper is the well-known Schur complement theorem.

Keywords: Function-valued Padé-type approximation; Fredholm integral equation of the second kind; determinant; Schur complement.
Mathematics subject classification: O241.83

1. Introduction

Consider a Fredholm integral equation of the second kind

$$x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt, \quad a \le s, \quad t \le b, \tag{1.1}$$

where $K(s,t)$ and $y(s)$ are both continuous functions in the square area $[a,b] \times [a,b]$ and the interval $[a,b]$, respectively. The solution of equation (1.1) can be expressed as a power series with function-valued coefficient

$$x(s) = f(s,\lambda) = y_0(s) + y_1(s)\lambda + y_2(s)\lambda^2 + \cdots + y_n(s)\lambda^n + \cdots, \tag{1.2}$$

where

$$y_i(s) = \int_a^b K^i(s,t)y(t)dt, \quad i \ge 1, \quad y_0(s) = y(s), \tag{1.3}$$

and $K^i(s,t)$ in (1.3) is called the i-th iterative kernel.

Suppose that $x(s) = f(s,\lambda)$ as a function about λ is analytic when $\lambda = 0$, so the series (1.2) is convergent when $|\lambda|$ is sufficiently small. Meanwhile, $y_i(s)$ is a continuous function in the interval $[a,b]$. For $y_i(s),\ y_j(s) \in L^2[a,b]$, define the inner product by

$$(y_i, y_j) = (y_i(s), y_j(s)) = \int_a^b y_i(s)y_j(s)ds, \tag{1.4}$$

*Corresponding author.

and define the corresponding norm by

$$\| y_i(s) \| = \sqrt{(y_i(s), y_i(s))} = \left\{ \int_a^b y_i^2(s) ds \right\}^{\frac{1}{2}}.$$

A function-valued Padé-type approximation of type (m/n) for the power series (1.2) is denoted by $(m/n)_f(s,\lambda)$.

Theorem 1.1. ([1,2]) *If* $\mathbf{det}(A_n) \neq 0$*, then* $(m/n)_f(s,\lambda)$ *exists, and it holds that*

$$(m/n)_f(s,\lambda) = p_{m,n}(s,\lambda)/q_{m,n}(\lambda),$$

with

$$q_{m,n}(\lambda) = \mathbf{det} \begin{bmatrix} A_n & l^{(n)} \\ \lambda^{(n)^T} & 1 \end{bmatrix}, \tag{1.5}$$

$$p_{m,n}(s,\lambda) = \mathbf{det} \begin{bmatrix} A_n & l^{(n)} \\ \tilde{\lambda}^{(n)^T} & \eta \end{bmatrix}, \tag{1.6}$$

where

$$A_n = \begin{bmatrix} (y_{m-n+1}, y_{m-n+1}) & (y_{m-n+1}, y_{m-n+2}) & \cdots & (y_{m-n+1}, y_m) \\ (y_{m-n+1}, y_{m-n+2}) & (y_{m-n+1}, y_{m-n+3}) & \cdots & (y_{m-n+1}, y_{m+1}) \\ \cdots & \cdots & \cdots & \\ (y_{m-n+1}, y_m) & (y_{m-n+1}, y_{m+1}) & \cdots & (y_{m-n+1}, y_{m+n-1}) \end{bmatrix}, \tag{1.7}$$

$$\begin{aligned} &\lambda^{(n)} = (\lambda^n, \lambda^{n-1}, \cdots, \lambda)^T, \\ &\tilde{\lambda}^{(n)} = \left(\sum_{i=n}^{m} y_{i-n}\lambda^i, \sum_{i=n-1}^{m} y_{i-n+1}\lambda^i, \cdots, \sum_{i=1}^{m} y_{i-1}\lambda^i \right)^T, \\ &l^{(n)} = \big((y_{m-n+1}, y_{m+1}), (y_{m-n+1}, y_{m+2}), \cdots, (y_{m-n+1}, y_{m+n}) \big)^T, \\ &\eta = \sum_{i=0}^{m} y_i \lambda^i. \end{aligned}$$

From Theorem 1.1, we observe that the central point to construct a $(m/n)_f(s,\lambda)$ is how to compute two determinants (1.5) and (1.6). Therefore, we need the following well-known result.

Lemma 1.1 (Schur complement)**.** *Let A be an $n \times n$ real matrix and partitioned into* 2×2 *block matrix* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. *If* $A_{11} \in C^{k \times k}$ *is nonsingular, then*

$$\mathbf{det}(A) = \mathbf{det}(A_{11})\mathbf{det}(A_{22} - A_{21}A_{11}^{-1}A_{12}).$$

2. Tridiagonal reduction for a real symmetric matrix

Assume $A_n = (a_{ij})$ in (1.7) be an $n \times n$ real symmetric matrix, i.e., $A_n^T = A_n$ (a tridiagonal reduction of a complex skew symmetric matrix has been given in [3]). Using the following Lanczos process

$$\begin{cases} q_1 = e_1, \quad \beta_0 = 0, \\ \alpha_i = (A_n q_i, q_i), \quad \beta_i q_{i+1} = A_n q_i - \alpha_i q_i - \beta_{i-1} q_{i-1}, \\ \beta_i = \|A_n q_i - \alpha_i q_i - \beta_{i-1} q_{i-1}\|, \\ q_{i+1} = (A_n q_i - \alpha_i q_i - \beta_{i-1} q_{i-1})/\beta_i, \quad i = 1, 2, \cdots, n-1, \\ \alpha_n = (A_n q_n, q_n), \end{cases} \tag{2.1}$$

we reduce A_n to a symmetric tridiagonal matrix

$$T_n = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \ddots & \vdots \\ 0 & \beta_2 & \alpha_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_{n-1} \\ 0 & \cdots & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}. \tag{2.2}$$

Thus, we obtain an orthogonal matrix $Q = [q_1, q_2, \ldots, q_n]$ such that $Q^T A_n Q = T_n$.

3. Inverse of a symmetric tridiagonal matrix

For the matrix T_n in (2.2), assume that $\alpha_i \neq 0$ for $i = 1, \cdots, n$. If for $i = 1, \cdots, n-1$, one of the β_i is equal to 0, say $\beta_l = 0$, for some $1 \leq l \leq n-1$, then it holds that $T_n = \begin{bmatrix} T_l & 0 \\ 0 & T_{n-l} \end{bmatrix}$, where

$$T_l = \begin{bmatrix} \alpha_1 & \beta_1 & & 0 \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{l-1} \\ 0 & & \beta_{l-1} & \alpha_l \end{bmatrix}, \quad T_{n-l} = \begin{bmatrix} \alpha_{l+1} & \beta_{l+1} & & 0 \\ \beta_{l+1} & \alpha_{l+2} & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ 0 & & \beta_{n-1} & \alpha_n \end{bmatrix}$$

are all nonsingular square matrices. It follows that

$$\mathbf{det}(T_n) = \mathbf{det}(T_l)\mathbf{det}(T_{n-l}), \qquad T_n^{-1} = \begin{bmatrix} T_l^{-1} & 0 \\ 0 & T_{n-l}^{-1} \end{bmatrix}.$$

Now we find that the underlying problem is reduced to two subproblems with smaller dimensions. For the matrix T_n in (2.2), we define the n quantities $c_i, i = 1, \cdots, n$ (the case of the general tridiagonal matrix can be seen in [5]):

$$c_i = \begin{cases} \alpha_1, & i=1, \\ \alpha_i - \dfrac{\beta_{i-1}^2}{c_{i-1}}, & i=2,\cdots,n. \end{cases} \tag{3.1}$$

If all $c_i \neq 0$ for $i = 1,\cdots,n$, there exists a lower triangular matrix L and an upper triangular matrix U satisfying $T_n = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \dfrac{\beta_1}{c_1} & 1 & 0 & \ddots & \vdots \\ 0 & \dfrac{\beta_2}{c_2} & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \dfrac{\beta_{n-1}}{c_{n-1}} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} c_1 & \beta_1 & 0 & \cdots & 0 \\ 0 & c_2 & \beta_2 & \ddots & \vdots \\ 0 & 0 & c_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & c_n \end{bmatrix}.$$

It can be easily verified that

$$\mathbf{det}(T_n) = \prod_{i=1}^{n} c_i, \tag{3.2}$$

and

$$\begin{aligned} \mathbf{det}(T_k) &= \prod_{i=1}^{k} c_i = \left(\prod_{i=1}^{k-2} c_i\right) c_{k-1}c_n \\ &= \mathbf{det}(T_{k-2})c_{k-1}\left(\alpha_k - \frac{\beta_{k-1}^2}{c_{k-1}}\right) = \mathbf{det}(T_{k-2})(\alpha_k c_{k-1} - \beta_{k-1}^2). \end{aligned}$$

Theorem 3.1. *For the symmetric tridiagonal matrix in the form of* (2.2)*, it holds that*

$$\mathbf{det}(T_k) = \mathbf{det}(T_{k-2})(\alpha_n c_{k-1} - \beta_{k-1}^2), \quad k = 3,4,\cdots,n,$$

where c_k*,* $k = 3,4,\cdots,n$*, are given by* (3.1).

Since $T_n^{-1} = U^{-1}L^{-1}$, we may first compute the inverse of U and L. Let $R = (r_{ij}) = U^{-1}$. Then the elements of the i-th row of R $(i = 1,\cdots,n)$ can be expressed as

$$r_{it} = 0 \;\; (t < i), \qquad r_{it} = (-1)^{t-i}\left(\prod_{k=i}^{t-1} \frac{\beta_k}{c_k}\right)\frac{1}{c_t} \quad (t \geq i).$$

The analogical case of the general block tridiagonal matrix can be found in [6]. So we can give the following formulas to compute the elements of R:

$$\begin{cases} r_{ij} = 0, & i > j, \\ r_{ii} = \dfrac{1}{c_i}, & i = 1,2,\cdots,n, \\ r_{i,j} = -r_{i+1,j}\dfrac{\beta_i}{c_i}, & j = n,\cdots,n-1, \;\; i = j-1,\cdots,2,1. \end{cases}$$

Let $W = (w_{ij}) = L^{-1}$ and denote the j-th column of W by $w_j = (w_{1j}, \cdots, w_{jj}, \cdots, w_{nj})^T$. Then the elements of W_j are given by

$$w_{tj} = 0 \ \ (t < j), \qquad w_{tj} = (-1)^{t-j} \prod_{k=j}^{t-1} \frac{\beta_k}{c_k} \ \ (t \geq j).$$

Similarly to U, the elements of W can be given by

$$\begin{cases} w_{ij} = 0, & i < j. \\ w_{ii} = 1, & i = 1, 2, \cdots, n. \\ w_{i,j} = -w_{i,j+1}\dfrac{\beta_j}{c_j}, & i = n, \cdots, n-1, \ \ j = i-1, \cdots, 2, 1. \end{cases} \tag{3.3}$$

Lemma 3.1. *Let T_n in* (2.2) *and $X = (x_{ij}) = T_n^{-1}$. Then x_{ij} can be expressed as*

$$x_{ij} = (-1)^{i+j} \sum_{t=max(i,j)}^{n} \left[\left(\prod_{k=i}^{t-1} \frac{\beta_k}{c_k} \right) \frac{1}{c_t} \prod_{k=j}^{t-1} \frac{\beta_k}{c_k} \right], \quad i, j = 1, 2, \cdots, n.$$

Using Lemma 3.1, we may obtain the following theorem.

Theorem 3.2. *Let T_n be of the form of* (2.2) *and $X = (x_{ij}) = T_n^{-1}$. Then x_{ij} can be computed by (note $x_{ij} = x_{ji}$)*

$$\begin{aligned} &(i) \ \ x_{nn} = \frac{1}{c_n}, \quad x_{ii} = \frac{1}{c_i} + \frac{\beta_i^2}{c_i^2} x_{i+1,1+1}; \\ &(ii) \ \ x_{ij} = -x_{i,j+1}\frac{\beta_j}{c_j}, \quad i = n, \cdots, 2; \ \ j = i-1, \cdots, 2, 1. \end{aligned} \tag{3.4}$$

Proof. It is obvious that $x_{nn} = c_n^{-1}$. By Theorem 3.1, we have

$$\begin{aligned} x_{ii} &= \sum_{t=i}^{n} \left[\left(\prod_{k=i}^{t-1} \frac{\beta_k}{c_k} \right) \frac{1}{c_t} \prod_{k=i}^{t-1} \frac{\beta_k}{c_k} \right] \\ &= \frac{1}{c_i} + \sum_{t=i+i}^{n} \left[\left(\prod_{k=i}^{t-1} \frac{\beta_k}{c_k} \right) \frac{1}{c_t} \prod_{k=i}^{t-1} \frac{\beta_k}{c_k} \right] \\ &= \frac{1}{c_i} + \left(\frac{\beta_i}{c_i} \right)^2 \sum_{t=i+1}^{n} \left[\left(\prod_{k=i+1}^{t-1} \frac{\beta_k}{c_k} \right) \frac{1}{c_t} \prod_{k=i+1}^{t-1} \frac{\beta_k}{c_k} \right] \\ &= \frac{1}{c_i} + \left(\frac{\beta_i}{c_i} \right)^2 x_{i+1,i+1}. \end{aligned}$$

Similarly, we have the following recursive relationship for the i-th row elements x_{ij} $(i > j)$ for the matrix X, namely, the elements left to the diagonal element x_{ii},

$$\begin{aligned} x_{ij} &= (-1)^{i+j} \sum_{t=i}^{n} \left[\left(\prod_{k=i}^{t-1} \frac{\beta_k}{c_k} \right) \frac{1}{c_t} \prod_{k=j}^{t-1} \frac{\beta_k}{c_k} \right] \\ &= (-1)^{-1}(-1)^{i+j+1} \sum_{t=i}^{n} \left[\left(\prod_{k=i}^{t-1} \frac{\beta_k}{c_k} \right) \frac{1}{c_t} \prod_{k=j+1}^{t-1} \frac{\beta_k}{c_k} \right] \frac{\beta_j}{c_j} \\ &= x_{i,j+1} \frac{\beta_j}{c_j}. \end{aligned}$$

This completes the proof of this theorem. ■

As a direct consequence of Theorem 3.2, we may now establish the following algorithm:

Algorithm 3.1

Step 1 Compute x_{ii} $(i = 1, \cdots, n)$ by using (i) in (3.4). The process can be expressed as

$$\begin{bmatrix} x_{11} & & & \\ & \nwarrow & & \\ & & x_{n-1,n-1} & \\ & & \nwarrow & \\ & & & x_{nn} \end{bmatrix}.$$

Step 2 Use (ii) in (3.4) to compute the elements x_{ij} $(i > j)$ that are the left side of the diagonal element x_{ii}. The process can be expressed as

$$\begin{bmatrix} \cdots & \cdots & \cdots & & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ x_{n-1,1} & \leftarrow & \cdots & \leftarrow & x_{n-1,n-1} & & \\ x_{n1} & \leftarrow & \cdots & \leftarrow & x_{n,n-1} & \leftarrow & x_{nn} \end{bmatrix}.$$

Step 3 Compute c_i $(i = 1, \cdots, n)$ by using (3.1) and substitute them in the X that have been obtained by the above step.

Example 3.1. A simple symbolic example will be given to show the above algorithm. Let

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \beta_2 & \alpha_2 & \beta_2 \\ 0 & \beta_3 & \alpha_3 \end{bmatrix}. \tag{3.5}$$

Using (i) in (3.4), we first compute x_{ii} :

$$\begin{aligned} x_{33} &= \frac{1}{c_3}, \\ x_{22} &= \frac{1}{c_2} + (\frac{\beta_2}{c_2})^2 x_{33} = \frac{1}{c_2} + \frac{\beta_2^2}{c_2^2 c_3}, \\ x_{11} &= \frac{1}{c_1} + (\frac{\beta_1}{c_1})^2 x_{22} = \frac{1}{c_1} + (\frac{\beta_1}{c_1})^2 (\frac{1}{c_2} + \frac{\beta_2^2}{c_2^2 c_3}) = \frac{1}{c_1} + \frac{\beta_1^2}{c_1^2 c_2} + \frac{\beta_1^2 \beta_2^2}{c_1^2 c_2^2 c_3}. \end{aligned}$$

Then using (ii) in (3.4), we have the lower triangular part of X:

$$\begin{bmatrix} \frac{1}{c_1} + \frac{\beta_1^2}{c_1^2 c_2} + \frac{\beta_1^2 \beta_2^2}{c_1^2 c_2^2 c_3} & & \\ -\frac{\beta_1}{c_1}(\frac{1}{c_2} + \frac{\beta_2^2}{c_2^2 c_3}) & \frac{1}{c_2} + \frac{\beta_2^2}{c_2^2 c_3} & \\ \frac{\beta_1 \beta_2}{c_1 c_2 c_3} & -\frac{\beta_2}{c_2 c_3} & \frac{1}{c_3} \end{bmatrix}.$$

Finally we obtain the inverse of T in (3.5) as follows

$$T^{-1} = X = \begin{bmatrix} \frac{1}{c_1} + \frac{\beta_1^2}{c_1^2 c_2} + \frac{\beta_1^2 \beta_2^2}{c_1^2 c_2^2 c_3} & -\frac{\beta_1}{c_1 c_2} - \frac{\beta_1 \beta_2^2}{c_1 c_2^2 c_3} & \frac{\beta_1 \beta_2}{c_1 c_2 c_3} \\ -\frac{\beta_1}{c_1 c_2} - \frac{\beta_1 \beta_2^2}{c_1 c_2^2 c_3} & \frac{1}{c_2} + \frac{\beta_2^2}{c_2^2 c_3} & -\frac{\beta_2}{c_2 c_3} \\ \frac{\beta_1 \beta_2}{c_1 c_2 c_3} & -\frac{\beta_2}{c_2 c_3} & \frac{1}{c_3} \end{bmatrix},$$

where $c_1 = \alpha_1, c_2 = \alpha_2 - \beta_1^2 \alpha_1^{-1}, c_3 = \alpha_3 - \alpha_1 \beta_2^2 (\alpha_1 \alpha_2 - \beta_1^2)^{-1}$, which can be easily computed by (3.1).

A special case. The matrix T_n given in (2.2) is positive definite if and only if $c_i > 0$ for each $i = 1, \cdots, n$, which can be referred to in [7]. Then an additional LU factorization of the form LL^T is always possible for this matrix, i.e., $T_n = LL^T$, where the matrix $L = (l_{ij})_{n \times n}$ is a lower triangular matrix given by:

$$l_{ij} = \begin{cases} \sqrt{c_i}, & j = i, \\ \frac{\beta_{i-1}}{\sqrt{c_{i-1}}}, & j = i-1, \\ 0, & \text{otherwise}. \end{cases}$$

where c_i $(i = 1, \cdots, n)$ are given by (3.1).

The inverse matrix $L^{-1} = M = (m_{ij})_{n \times n}$ is also a lower triangular matrix and can be given by the formulas similar to (3.3):

$$\begin{cases} m_{ij} = 0, \quad i < j, \\ m_{ii} = \frac{1}{l_{ii}}, \quad i = 1, \cdots, n, \\ m_{i,j} = -m_{i,j+1} \frac{\beta_i}{c_i}, \quad i = n, \cdots, n-1, \ \ j = i-1, \cdots, 2, 1. \end{cases} \tag{3.6}$$

Therefore, the inverse of T_n is obtained naturally, i.e., $X = (x_{ij})_{n\times n} = T_n^{-1} = (L^T)^{-1}L^{-1}$ where x_{ij} is given by

$$x_{ij} = \sum_{k=\max(i,j)}^{n} m_{ki}m_{kj}, \quad i,j = 1,\cdots,n. \tag{3.7}$$

Example 3.2. For the symmetric tridiagonal matrix T in (3.5), suppose that $c_i > 0$ for $1 \le i \le n$. Then there exists a lower tridiagonal matrix L such that $T = LL^T$, where

$$L = \begin{bmatrix} \sqrt{c_1} & 0 & 0 \\ \dfrac{\beta_1}{\sqrt{c_1}} & \sqrt{c_2} & 0 \\ 0 & \dfrac{\beta_2}{\sqrt{c_2}} & \sqrt{c_3} \end{bmatrix}.$$

Using the recurrence formula in (3.6), we can obtain the inverse of L:

$$L^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{c_1}} & 0 & 0 \\ -\dfrac{\beta_1}{c_1\sqrt{c_2}} & \dfrac{1}{\sqrt{c_2}} & 0 \\ \dfrac{\beta_1\beta_2}{c_1c_2\sqrt{c_3}} & -\dfrac{\beta_2}{c_2\sqrt{c_3}} & \dfrac{1}{\sqrt{c_3}} \end{bmatrix}.$$

So the inverse matrix T^{-1} of the matrix T in (3.5) is obtained by using (3.7):

$$T^{-1} = (L^{-1})^T L^{-1} = \begin{bmatrix} \dfrac{1}{c_1} + \dfrac{\beta_1^2}{c_1^2c_2} + \dfrac{\beta_1^2\beta_2^2}{c_1^2c_2^2c_3} & -\dfrac{\beta_1}{c_1c_2} - \dfrac{\beta_1\beta_2^2}{c_1c_2^2c_3} & \dfrac{\beta_1\beta_2}{c_1c_2c_3} \\ -\dfrac{\beta_1}{c_1c_2} - \dfrac{\beta_1\beta_2^2}{c_1c_2^2c_3} & \dfrac{1}{c_2} + \dfrac{\beta_2^2}{c_2^2c_3} & -\dfrac{\beta_2}{c_2c_3} \\ \dfrac{\beta_1\beta_2}{c_1c_2c_3} & -\dfrac{\beta_2}{c_2c_3} & \dfrac{1}{c_3} \end{bmatrix},$$

where $c_1 = \alpha_1$, $c_2 = \alpha_2 - \beta_1^2\alpha_1{}^{-1}$, $c_3 = \alpha_3 - \alpha_1\beta_2^2(\alpha_1\alpha_2 - \beta_1^2)^{-1}$.

4. Main results

In this paper we consider the computation of the two determinants $q_{mn}(\lambda)$ in (1.5) and $p_{mn}(s,\lambda)$ in (1.6). Since $\lambda^{(n)}$ and $\tilde{\lambda}^{(n)}$ are two vectors with unknown variable in the determinants, we can not compute these determinants by some conventional methods. In order to compute $q_{mn}(\lambda)$ and $p_{mn}(s,\lambda)$, we must utilize the special block structure of the matrix of the right sides of (1.5) and (1.6). At the same time we note that the matrix A_n in (1.6) is a nonsingular real symmetric matrix. According to the above observations, we have the following main result.

Theorem 4.1. *Let $q_{mn}(\lambda)$ and $p_{mn}(s,\lambda)$ be defined by* (1.5) *and* (1.6)*, respectively. Then*

$$q_{mn}(\lambda) = det(T) \times (1 - \lambda^{(n)^T} Q T^{-1} Q^T l^{(n)}),$$
$$p_{mn}(s,\lambda) = det(T) \times (\eta - \tilde{\lambda}^{(n)^T} Q T^{-1} Q^T l^{(n)}),$$

where

$$\lambda^{(n)} = (\lambda^n, \lambda^{n-1}, \cdots, \lambda)^T,$$
$$l^{(n)} = ((y_{m-n+1}, y_{m+1}), (y_{m-n+1}, y_{m+2}), \cdots, (y_{m-n+1}, y_{m+n}))^T,$$
$$\tilde{\lambda}^{(n)} = \left(\sum_{i=n}^{m} y_{i-n}\lambda^i, \sum_{i=n-1}^{m} y_{i-n+1}\lambda^i), \cdots, \sum_{i=1}^{m} y_{i-1}\lambda^i \right)^T,$$
$$\eta = \sum_{i=0}^{m} y_i \lambda^i.$$

Proof. From the hypothesis, we know that there exists a complex orthogonal matrix Q such that $Q^T A_n Q = T$. Taking $\tilde{Q} = \begin{bmatrix} Q & \\ & 1 \end{bmatrix}$ gives

$$\tilde{Q}^T \begin{bmatrix} A_n & l^{(n)} \\ \lambda^{(n)^T} & 1 \end{bmatrix} \tilde{Q} = \begin{bmatrix} T & Q^T l^{(n)} \\ \lambda^{(n)^T} Q & 1 \end{bmatrix},$$
$$\tilde{Q}^T \begin{bmatrix} A_n & l^{(n)} \\ \tilde{\lambda}^{(n)^T} & \eta \end{bmatrix} \tilde{Q} = \begin{bmatrix} T & Q^T l^{(n)} \\ \tilde{\lambda}^{(n)^T} Q & \eta \end{bmatrix}.$$

It follows from

$$q_{mn}(\lambda) = \mathbf{det} \begin{bmatrix} T & Q^T l^{(n)} \\ \lambda^{(n)^T} Q & 1 \end{bmatrix},$$

and by Lemma 1.1, we have

$$q_{mn}(\lambda) = det(T) \times (1 - \lambda^{(n)^T} Q T^{-1} Q^T l^{(n)}).$$

Similarly, from

$$p_{mn}(s,\lambda) = \mathbf{det} \begin{bmatrix} T & Q^T l^{(n)} \\ \tilde{\lambda}^{(n)^T} Q & \eta \end{bmatrix},$$

we obtain

$$p_{mn}(s,\lambda) = det(T) \times (\eta - \tilde{\lambda}^{(n)^T} Q T^{-1} Q^T l^{(n)}). \quad \blacksquare$$

Finally, to construct the determinants for $(m/n)_f(s,\lambda)$, we give the following algorithm:

Algorithm 4.1

Step 1 Compute A_n by (1.3) and (1.4),

Step 2 Reduce A_n to a symmetric tridiagonal matrix T_n in (2.2) by process (2.1), i.e., find a real orthogonal matrix Q such that $Q^T A_n Q = T_n$,

Step 3 Compute $det(T_n)$ and T_n^{-1} by using Algorithm 3.1 or Special case in Section 3,

Step 4 Form $\lambda^{(n)}, \tilde{\lambda}^{(n)}, l^{(n)}$, and η,

Step 5 Compute $q_{mn}(\lambda)$ in (1.5) and $p_{mn}(s,\lambda)$ in (1.6).

Acknowledgment The work is supported by the National Natural Science Foundation of China (10271074) and by the Special Funds for Major Specialities of Shanghai Education Committee.

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