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A Note on Generic Fiedler Matrices

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Abstract

In this paper, we first show that a generic $m \times n$ Fiedler matrix may have $2^{m-n-1}-1$ kinds of factorizations which are very complicated when m is much larger than n. In this work, two special cases are examined, one is an $m \times n$ Fiedler matrix being factored as a product of $(m-n)$ Fiedler matrices, the other is an $m \times (m-2)$ Fiedler matrix's factorization. Then we discuss the relation among the numbers of parameters of three generic $m \times n$, $n \times p$ and $m \times p$ Fiedler matrices, and obtain some useful results.

Keywords: Stochastic; column-rhomboidal; centrosymmetric; Fiedler matrix.
Mathematics subject classification: 15A27, 15A57

1. Introduction

Fiedler matrix was introduced by Start and Wearer in [1], which comes from a discrete time, Markov-like process where the number of states available may increase with time. Such a process may represent a long term investment model where new investment vehicles may be introduced as time evolves. It may also represent an epidemiological model where new treatments leading to new stages in the disease course may be developed. The size of the matrices reflects the number of available states. Since the matrices that we examine represent transition matrices, thus the entries are nonnegative. In [1], the authors discussed the properties of the Fiedler matrices, and investigated the factorization of Fiedler matrix into Fiedler matrices. They also presented some open questions such as when a Fiedler matrix is factorizable as a product of Fiedler matrices; and if factorizable, are the factors unique? and if not, are the dimensions of the factors unique?

In this paper, we show that a generic $m \times n$ Fiedler matrix may have $2^{m-n-1}-1$ kinds of factorizations, and as m is much larger than n the factorizations are very complicated.

2. Fiedler matrices

For the sake of brevity, all matrices in this paper are assumed to be real, and if the Fiedler matrix A is $m \times n$, it is understood that $m > n$.

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Definition 2.1. [3] *Let A be an $m \times n$ matrix with $m > n$. Then A is called column-rhomboidal if A satisfies both of the following conditions: the $n \times n$ submatrix of A consisting of the first n rows is a nonsingular, lower triangular matrix, and the $n \times n$ submatrix of A consisting of the last n rows is a nonsingular, upper triangular matrix.*

Definition 2.2. [1] *An $m \times n$ matrix with $m > n$ is called a Fiedler matrix if A is column-rhomboidal, centrosymmetric and if A has constant row sums.*

Definition 2.3. [1] *If A is an $m \times n$ column-rhomboidal matrix with exactly l diagonal bands of length n, then A is called l-banded.*

Lemma 2.1. [1] *If A is an $m \times n$ Fiedler matrix, then A is l-banded, where $l = m - n + 1$. Moreover, the row sums of A all equal to n/m, and each entry of A is bounded above by n/m.*

Lemma 2.2. [1] *If A is an $m \times n$, l-banded Fiedler matrix, then A has $\left\lceil (n-1)\left(\frac{1}{2}l - 1\right)\right\rceil$ independent parameters which was specified in Theorem* 2.4 *of* [1].

By the properties of Fiedler matrix, we can consider the following example:

Example 2.1. A generic 6×2 Fiedler matrix A is 5-banded, and has 2 independent parameters:

$$A = \frac{2}{6}\begin{pmatrix} 1 & \\ 1-y_1 & y_1 \\ 1-y_2 & y_2 \\ y_2 & 1-y_2 \\ y_1 & 1-y_1 \\ & 1 \end{pmatrix},$$

where $0 < y_1 \leqslant 1$ and $0 \leqslant y_2 \leqslant 1$.

Lemma 2.3. [3] *If A and B are Fiedler matrices for which the product is defined, then AB is also a Fiedler matrix.*

By induction, we can easily get the following corollary.

Corollary 2.1. *If $A_1, A_2, \cdots, A_n$ are all Fiedler matrices for which the product is defined, then $A_1 A_2 \cdots A_n$ is also a Fiedler matrix.*

3. On factorizations

Theorem 3.1. *Let A be a generic $m \times n$ Fiedler matrix. If A is factorizable, then the sum total of its factorizations is no more than $2^{m-n-1} - 1$, and the number of its factorizations with k Fiedler matrices is no more than $\binom{m-n-1}{k-1}$, with $2 \leqslant k \leqslant m - n$.*

Proof. If A can be factored to a product of k Fiedler matrices with dimensions $m \times m_1, m_1 \times m_2, \cdots, m_{k-1} \times n$, respectively, then $m_i \in S = \{m-1, m-2, \cdots, n+1\}$. Thus the number of its factorizations with k Fiedler matrices is no more than $\binom{m-n-1}{k-1}$, for k from 2 to $m-n$. So the sum total is

$$\binom{m-n-1}{1}+\binom{m-n-1}{2}+\cdots+\binom{m-n-1}{m-n-1}=2^{m-n-1}-1,$$

namely, the sum total of its factorizations is no more than $2^{m-n-1}-1$. ■

Theorem 3.2. *If A is a $m\times n$ Fiedler matrix, then the following are equivalent:*

1. *A can be factored as a product of $(m-n)$ Fiedler matrices;*
2. *the factors are all 2-banded;*
3. *A has no independent parameter;*
4. *$A=\left(a_{ij}\right)$, where*

$$a_{ij}=\frac{\binom{i-1}{j-1}\binom{m-i}{n-j}}{\binom{m}{n}},$$

for $i=1,\cdots,m$ and $j=1,\cdots,n$.

Proof. By Corollary 2.1, A can be factored as a product of $(m-n)$ Fiedler matrices if and only if the dimensions of the factors are $m\times(m-1),(m-1)\times(m-2),\cdots,(n+1)\times n$, respectively. By Lemma 2.1, we know $(m-i)-(m-i-1)+1=2$, for $i=0,\cdots,m-n-1$, and by Lemma 2.2, the factors are all 2-banded if and only if $\left\lceil (m-i)(\frac{1}{2}l-1)\right\rceil=0$, for $i=0,\cdots,m-n-1$, the factors are all unique. Thus, A must be unique and has no independent parameter. In the light of definition of Fiedler matrix, by computing we know (3) and (4) are equivalent. ■

Theorem 3.3. *Let k be a positive integer. Then the generic $m\times(m-2)$ Fiedler matrix A is $(m-3)$-banded, and depends on k parameters. The nonnegative matrix A is an $m\times(m-2)$ Fiedler matrix, for $m=2k+2$, $m=2k+3$, and the first $\lceil m/2\rceil$ rows of A are respectively shown:*

$$\frac{m-2}{m}\begin{pmatrix}
1 & & & & & \\
1-x_1 & x_1 & & & & \\
-\frac{k-1}{k}+x_1 & \frac{2k-1}{k}-x_1-x_2 & x_2 & & & \\
 & -\frac{k-2}{k}+x_2 & \frac{2k-2}{k}-x_2-x_3 & x_3 & & \\
 & \ddots & \ddots & \ddots & & \\
 & & -\frac{1}{k}+x_{k-1} & \frac{k+1}{k}-x_{k-1}-x_k & x_k &
\end{pmatrix},$$

$$\frac{m-2}{m}\begin{pmatrix} 1 & & & & \\ 1-x_1 & x_1 & & & \\ -\frac{2k-1}{2k+1}+x_1 & \frac{4k}{2k+1}-x_1-x_2 & x_2 & & \\ & -\frac{2k-3}{2k+1}+x_2 & \frac{4k-2}{2k+1}-x_2-x_3 & x_3 & \\ & \ddots & \ddots & \ddots & \\ & -\frac{3}{2k+1}+x_{k-1} & \frac{2k+4}{2k+1}-x_{k-1}-x_k & x_k & \\ & & -\frac{1}{2k+1}+x_k & \frac{2k+3}{2k+1}-2x_k & -\frac{1}{2k+1}+x_k \end{pmatrix},$$

where $x_i > 0$, for $1 \leqslant i \leqslant k$. The $m \times (m-2)$ Fiedler matrix A is factorizable if and only if $x_i = \binom{m-1-i}{2} \Big/ \binom{m-1}{2}$ for $1 \leqslant i \leqslant \lceil \frac{m-3}{2} \rceil$.

Proof. Using Lemmas 2.1 and 2.2, we can easily get the first result. Noting that each row sum of A must be n/m, and A is column stochastic and centrosymmetric, we can get the second conclusion. By Theorem 3.1, the sum total of A's factorizatons is no more than $2^{m-n-1} - 1 = 1$ if A is factorizable. Using Theorem 3.2, we have

$$\frac{m-2}{m}x_i = \frac{\binom{i}{i}\binom{m-(i+1)}{m-2-(i+1)}}{\binom{m}{m-2}} = \frac{\binom{m-1-i}{2}}{\binom{m}{2}}.$$

Consequently,

$$x_i = \frac{m}{m-2}\frac{\binom{m-1-i}{2}}{\binom{m}{2}} = \frac{\binom{m-1-i}{2}}{\binom{m-1}{2}} \quad \text{for } 1 \leq i \leq \left\lceil \frac{m-3}{2} \right\rceil.$$

This completes the proof of this theorem. ■

4. Results on parameters

Theorem 4.1. *Given positive integers $m > n > p > 1$, and r_1, r_2 and r_3 denote the number of parameters of generic $m \times n, n \times p$, and $m \times p$ Fiedler matrices, respectively. Then $r_3 \geqslant \max\{r_1, r_2\}$, for $n + p \geqslant m$, or, for $m = 4k+2, p = 2k$ and $n + p \geqslant m - 1$; $r_2 \leqslant r_3 \leqslant r_1$, for others.*

Proof. By Lemmas 2.1 and 2.2, we have

$$r_1 = \left\lceil \frac{1}{2}(n-1)(m-n-1) \right\rceil, \qquad r_2 = \left\lceil \frac{1}{2}(p-1)(n-p-1) \right\rceil,$$
$$r_3 = \left\lceil \frac{1}{2}(p-1)(m-p-1) \right\rceil,$$

which is specified by the eight cases as below:

1. m, n and p are all even:

$$
\begin{aligned}
r_3 - r_1 &= \left[\frac{1}{2}(p-1)(m-p-1)+\frac{1}{2}\right] - \left[\frac{1}{2}(n-1)(m-n-1)+\frac{1}{2}\right] \\
&= \frac{1}{2}(n-p)(n+p-m) \begin{cases} \geqslant 0 & \text{for } n+p \geqslant m, \\ < 0 & \text{for } n+p < m. \end{cases}
\end{aligned}
$$

2. m and p are even, n is odd:

$$
\begin{aligned}
r_3 - r_1 &= \left[\frac{1}{2}(p-1)(m-p-1)+\frac{1}{2}\right] - \frac{1}{2}(n-1)(m-n-1) \\
&= \frac{1}{2}\left[(n-p)(n+p-m)+1\right] \begin{cases} > 0 & \text{for } n+p \geqslant m, \\ \geqslant 0 & \text{for } m=4k+2, p=2k \\ & \ \text{and } n+p \geqslant m-1, \\ < 0 & \text{for others.} \end{cases}
\end{aligned}
$$

3. m and n are even, p is odd:

$$
\begin{aligned}
r_3 - r_1 &= \frac{1}{2}(p-1)(m-p-1) - \left[\frac{1}{2}(n-1)(m-n-1)+\frac{1}{2}\right] \\
&= \frac{1}{2}\left[(n-p)(n+p-m)-1\right] \begin{cases} \geqslant 0 & \text{for } n+p > m, \\ < 0 & \text{for } n+p < m. \end{cases}
\end{aligned}
$$

4. m is even, n and p are odd:

$$
\begin{aligned}
r_3 - r_1 &= \frac{1}{2}(p-1)(m-p-1) - \frac{1}{2}(n-1)(m-n-1) \\
&= \frac{1}{2}(n-p)(n+p-m) \begin{cases} \geqslant 0 & \text{for } n+p \geqslant m, \\ < 0 & \text{for } n+p < m. \end{cases}
\end{aligned}
$$

5. m is odd, n and p are even:

$$
\begin{aligned}
r_3 - r_1 &= \frac{1}{2}(p-1)(m-p-1) - \frac{1}{2}(n-1)(m-n-1) \\
&= \frac{1}{2}(n-p)(n+p-m) \begin{cases} > 0 & \text{for } n+p > m, \\ < 0 & \text{for } n+p < m. \end{cases}
\end{aligned}
$$

6. m and p are odd, n is even:

$$
\begin{aligned}
r_3 - r_1 &= \frac{1}{2}(p-1)(m-p-1) - \frac{1}{2}(n-1)(m-n-1) \\
&= \frac{1}{2}(n-p)(n+p-m) \begin{cases} \geqslant 0 & \text{for } n+p \geqslant m, \\ < 0 & \text{for } n+p < m. \end{cases}
\end{aligned}
$$

7. m and n are odd, p is even:

$$\begin{aligned} r_3 - r_1 =& \frac{1}{2}\left(p-1\right)\left(m-p-1\right) - \frac{1}{2}(n-1)(m-n-1) \\ =& \frac{1}{2}\left[\left(n-p\right)\left(n+p-m\right)+1\right] \begin{cases} >0 & \text{for } n+p \geqslant m, \\ <0 & \text{for } n+p < m. \end{cases} \end{aligned}$$

8. m, n and p are all odd:

$$\begin{aligned} r_3 - r_1 =& \frac{1}{2}\left(p-1\right)\left(m-p-1\right) - \frac{1}{2}(n-1)(m-n-1) \\ =& \frac{1}{2}\left(n-p\right)\left(n+p-m\right) \begin{cases} >0 & \text{for } n+p > m, \\ <0 & \text{for } n+p < m. \end{cases} \end{aligned}$$

Moreover, r_3 is obviously larger than r_2. Thus we can easily get the desired results. ■

Theorem 4.2. *Given positive integers m, n, k, t, with $m-k > n > 1$. If a generic $m \times n$ Fiedler matrix C can be factored as a product of an $m \times (m-k)$ Fiedler matrix A and an $(m-k) \times n$ Fiedler matrix B, each with a smaller number of parameters, then*

$$\begin{aligned} &1 \leqslant k \leqslant \min\{n+1, m-n-1\}, \textit{ for } m = 4t+2 \textit{ and } n = 2t; \\ &1 \leqslant k \leqslant \min\{n, m-n-1\}, \textit{ for others.} \end{aligned}$$

Proof. As m, n, k are positive integers and $m-k > n > 1$, we have $1 < k < m-n$, which yields $1 \leqslant k \leqslant m-n-1$. Using Theorem 4.1, for $m = 4t+2$ and $n = 2t$, they must subject to $(m-k)+n \geqslant m-1$. Then $k \leqslant n+1$. For others, they must subject to $(m-k)+n \geqslant m$, then $k \leqslant n$. Thus we come to the desired conclusion. ■

Theorem 4.3. *If a generic $m \times n$ Fiedler matrix A can be factored as a product of k $(2 \leqslant k \leqslant m-n)$ Fiedler matrices $A_1, \cdots, A_k$ with dimensions $m \times m_1$, $m_1 \times m_2, \cdots, m_{k-1} \times n$ respectively, then when $n \geqslant \lfloor m/2 \rfloor$, the number of A's parameters is no less than that of A_i's $(1 \leqslant i \leqslant k)$; when $m = 4t+2$, $n = 2t$ and $n \geqslant \lfloor m/2 \rfloor - 1$, the number of A's parameters is no less than that of A_i's $(1 \leqslant i \leqslant k)$.*

Proof. Let $m_0 = m$ and $m_k = n$. Since $n \geqslant \lfloor m/2 \rfloor$ and $m_i > m_{i+1} \geqslant n$ $(0 \leqslant i \leqslant k-1)$, we have

$$m_i + m_{i+1} \geqslant n+1+n \geqslant \left\lfloor \frac{m}{2} \right\rfloor + 1 + \left\lfloor \frac{m}{2} \right\rfloor \geqslant m \quad (1 \leqslant i \leqslant k-1).$$

Using Theorem 4.1 repeatedly, we know that for any k $(2 \leqslant k \leqslant m-n)$ the number of A's parameters is no less than that of A_i's $(1 \leqslant i \leqslant k)$. When $m = 4t+2$, $n = 2t$ and $n \geqslant \lfloor m/2 \rfloor - 1$, we have

$$m_i + m_{i+1} \geqslant n+2+n+1 \geqslant \left\lfloor \frac{m}{2} \right\rfloor - 1 + 2 + \left\lfloor \frac{m}{2} \right\rfloor - 1 + 1 > m \quad (1 \leqslant i \leqslant k-2)$$

and

$$m_{k-1}+n \geqslant n+1+n \geqslant \left\lfloor \frac{m}{2} \right\rfloor -1+1+\left\lfloor \frac{m}{2} \right\rfloor -1 = m-1.$$

Using Theorem 4.1 repeatedly, we can get that for any k $(2 \leqslant k \leqslant m-n)$ the number of A's parameters is no less than that of A_i's $(1 \leqslant i \leqslant k)$. ■

Theorem 4.4. *If a generic $m \times n$ Fiedler matrix A can be factored as a product of k Fiedler matrices $A_1, \cdots, A_k$, with dimensions $m \times m_1, m_1 \times m_2, \cdots, m_{k-1} \times n$, respectively, then when $m-2n \leqslant k \leqslant m-n$, A has the greatest number of parameters; when $m = 4t+2$, $n = 2t$ and $m-2n-1 \leqslant k \leqslant m-n$, A has the greatest number of parameters.*

Proof. When $m-2n \leqslant k \leqslant m-n$, we have

$$\max\{m-m_1, m_1-m_2, \cdots, m_{k-1}-n\} = n.$$

Let $m_0 = m$ and $m_k = n$. Then

$$\max\left\{\left\lceil \frac{1}{2}(m_i-1)(m_{i-1}-m_i-1)\right\rceil, 1 \leqslant i \leqslant k\right\} = \left\lceil \frac{1}{2}(m_1-1)(n-1)\right\rceil.$$

If $m-m_1 = n$, then $m_1 = m-n, m_1+n \geqslant m$. By Theorem 4.1, we know that A has the greatest number of parameters. Similarly, when $m = 4t+2, n = 2t$ and $m-2n-1 \leqslant k \leqslant m-n$, we have

$$\max\{m-m_1, m_1-m_2, \cdots, m_{k-1}-n\} = n+1.$$

Let $m_0 = m$ and $m_k = n$. Then

$$\max\left\{\left\lceil \frac{1}{2}(m_i-1)(m_{i-1}-m_i-1)\right\rceil, 1 \leqslant i \leqslant k\right\} = \left\lceil \frac{1}{2}(m_1-1)n\right\rceil.$$

If $m_1+n \geqslant m-1$, then $m_1 = m-n-1, m_1+n \geqslant m-1$. By Theorem 4.1, we know that A has the greatest number of parameters. ■

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