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A Solution of Inverse Eigenvalue Problems for Unitary Hessenberg Matrices

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Abstract

Let $H \in \mathbb{C}^{n\times n}$ be an $n \times n$ unitary upper Hessenberg matrix whose subdiagonal elements are all positive. Partition H as

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \tag{0.1}$$

where H_{11} is its $k \times k$ leading principal submatrix; H_{22} is the complementary matrix of H_{11}. In this paper, H is constructed uniquely when its eigenvalues and the eigenvalues of $\widehat{H}_{11}$ and $\widehat{H}_{22}$ are known. Here $\widehat{H}_{11}$ and $\widehat{H}_{22}$ are rank-one modifications of H_{11} and H_{22} respectively.

Keywords: Unitary upper Hessenberg matrix; Schur parameters; inverse eigenvalue problem.
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1. Introduction

Let $\mathscr{H}_n$ denote the set of unitary upper Hessenberg matrices of order n with positive subdiagonal elements. It is known that any $H \in \mathscr{H}_n$ can be written uniquely as the products

$$H \doteq H(\gamma_1, \gamma_2, \cdots, \gamma_n) = G_1(\gamma_1)\cdots G_{n-1}(\gamma_{n-1})\widetilde{G}_n(\gamma_n) \tag{1.1}$$

where

$$G_k(\gamma_k) = diag\left[I_{k-1}, \begin{pmatrix} -\gamma_k & \sigma_k \\ \sigma_k & \overline{\gamma}_k \end{pmatrix}, I_{n-k-1}\right], \qquad k = 1, 2, \cdots, n-1, \tag{1.2}$$

and

$$\widetilde{G}_n(\gamma_n) = diag[I_{n-1}, -\gamma_n].$$

The parameters $\gamma_k \in \mathbb{C}$, $1 \le k \le n$, are called *reflection coefficients* or *Schur parameters* in signal processing and satisfy $|\gamma_k|^2 + \sigma_k^2 = 1$, $\sigma_k > 0$, $k = 1, \cdots, n-1$, and $|\gamma_n| = 1$. We also refer to (1.1) as *Schur parametric form* of H [7] and to (1.2) as the complex Givens

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matrices. In this paper, I_j denotes the $j \times j$ identity matrix, e_j denotes the j-th column of the identity matrix, and $\lambda(T)$ denotes the spectrums of a square matrix T.

Two kinds of inverse eigenvalue problems for unitary Hessenberg matrices have been considered up to now. One is described in [1] and the methods for constructing a unitary Hessenberg matrix from spectral data are described in [1, 10]. It tells us that $H \in \mathscr{H}_n$ is uniquely determined by its eigenvalues and the eigenvalues of a multiplicative rank-one perturbation of H. Another inverse eigenvalue problem appears in [2] which demonstrates that $H \in \mathscr{H}_n$ can also be determined by its eigenvalues and the eigenvalues of a modified $(n-1) \times (n-1)$ leading principal submatrix of H. All of them are analogous to relevant inverse eigenvalue problems of Jacobi matrices, *i.e.*, real symmetric tridiagonal matrices with positive subdiagonal elements. Recent work by Jiang [9] proves a kind of inverse eigenvalue problem for Jacobi matrices.

Theorem 1.1. [9] *Given two real number sets $\{\lambda_i\}_{i=1}^{n}$ and $\{\mu_i\}_{i=1}^{n-1}$. If there is no common number between $\mu_1, \mu_2, \cdots, \mu_{k-1}$ and $\mu_k, \mu_{k+1}, \cdots, \mu_{n-1}$, and*

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \mu_{j_{k-1}} < \lambda_k < \mu_{j_k} < \lambda_{k+1} < \cdots < \mu_{j_{n-1}} < \lambda_n,$$

where $(j_1, \cdots, j_{n-1})$ is a unique permutation of $(1, 2, \cdots, n-1)$, then there exists a unique Jacobi matrix T, such that $\lambda(T) = \{\lambda_i\}_{i=1}^{n}$, $\lambda(T_{1,k-1}) = \{\mu_i\}_{i=1}^{k-1}$, and $\lambda(T_{k+1,n}) = \{\mu_i\}_{i=k}^{n-1}$, where $T_{1,k-1}$ is the $(k-1) \times (k-1)$ leading principal submatrix of T, and $T_{k+1,n}$ is the complementary submatrix of $T_{1,k}$ ($T_{1,k}$ is the $k \times k$ leading principal submatrix of T).

Because the unitary upper Hessenberg matrices with positive subdiagonal elements have rich mathematical structures which are analogous to Jacobi matrices, we propose a new inverse eigenvalue problem for the matrix $H \in \mathscr{H}_n$ similar to Theorem 1.1. That is, if we know all the eigenvalues of H and all the eigenvalues of matrices $\widehat{H}_{11}$ and $\widehat{H}_{22}$, which are rank-one modifications of H_{11} and H_{22} respectively, can we construct the matrix H uniquely? Note that there is a little difference between Theorem 1.1 and our question: we just modify the last column of H_{11} and the first row of H_{22} instead of deleting the k-th row and the k-th column from H.

The paper is organized as follows. In Section 2, using the notation in (1.1), we introduce two modified submatrices $\widehat{H}_{kk}$, $k = 1, 2$. Then the relations of spectral decompositions between H and $\widehat{H}_{kk}$, $k = 1, 2$, are discussed. At the end of this section, a rank-one modification on unitary diagonal matrix, which has the same eigenvalues with H, is obtained. Here the methods we used are analogous to an eigendecomposition in divide and conquer algorithm for unitary eigenproblem (see, e.g., [3,6,8]). In Section 3, we discuss the strictly interlacing properties between the eigenvalues of H and of $\widehat{H}_{11}$ and $\widehat{H}_{22}$ on the assumption that there is no common number between the eigenvalues of $\widehat{H}_{11}$ and $\widehat{H}_{22}$. Then we describe how to construct H from two sets of spectra uniquely, and obtain the main theorem of this paper. In the final section, a numerical algorithm is proposed.

2. Eigendecomposition of H

It is known that a complex Givens matrix $G_k(\gamma_k)$, $1 \le k \le n-1$, in (1.2) is diagonally uniquely equivalent with a real Givens reflector. More precisely, if we define

$$\gamma_k' = \begin{cases} \dfrac{\gamma_k}{|\gamma_k|}, & \text{if } \gamma_k \neq 0, \\ 1, & \text{if } \gamma_k = 0, \end{cases} \tag{2.1}$$

then $|\gamma_k'| = 1$, and

$$G_k(\gamma_k) = \begin{bmatrix} I_{k-1} & & \\ & \gamma_k' & \\ & & I_{n-k} \end{bmatrix} \begin{bmatrix} I_{k-1} & & & \\ & -|\gamma_k| & \sigma_k & \\ & \sigma_k & |\gamma_k| & \\ & & & I_{n-k-1} \end{bmatrix} \begin{bmatrix} I_k & & \\ & \overline{\gamma_k}' & \\ & & I_{n-k-1} \end{bmatrix}.$$

Consequently, (1.1) can be rewritten as

$$\begin{aligned} H=&G_1 \cdots G_{k-1} \begin{bmatrix} I_{k-1} & & \\ & \gamma_k' & \\ & & I_{n-k} \end{bmatrix} \begin{bmatrix} I_{k-1} & & & \\ & -|\gamma_k| & \sigma_k & \\ & \sigma_k & |\gamma_k| & \\ & & & I_{n-k-1} \end{bmatrix} \begin{bmatrix} I_k & & \\ & \overline{\gamma_k}' & \\ & & I_{n-k-1} \end{bmatrix} G_{k+1} \cdots \widetilde{G}_n \\ =&G_1 \cdots G_{k-1} \begin{bmatrix} \widetilde{G}_k(-\gamma_k') & 0 \\ 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} I_{k-1} & & & \\ & -|\gamma_k| & \sigma_k & \\ & \sigma_k & |\gamma_k| & \\ & & & I_{n-k-1} \end{bmatrix} G_{k+1}(\overline{\gamma}_k' \gamma_{k+1}) \cdots \widetilde{G}_n(\overline{\gamma}_k' \gamma_n) \\ =&\begin{bmatrix} \widehat{H}_{11} & \\ & I_{n-k} \end{bmatrix} (I - 2ww^*) \begin{bmatrix} I_k & \\ & \widehat{H}_{22} \end{bmatrix}, \end{aligned} \tag{2.2}$$

where

$$\widehat{H}_{11} = H(\gamma_1, \cdots, \gamma_{k-1}, -\gamma_k') \in \mathscr{H}_k, \quad \widehat{H}_{22} = H(\overline{\gamma}_k' \gamma_{k+1}, \overline{\gamma}_k' \gamma_{k+2}, \cdots, \overline{\gamma}_k' \gamma_n) \in \mathscr{H}_{n-k}, \tag{2.3}$$

and $w \in \mathbb{R}^n$ satisfies

$$w = \omega_k e_k + \omega_{k+1} e_{k+1},$$

where $\omega_k = (\frac{1+|\gamma_k|}{2})^{\frac{1}{2}}$, $\omega_{k+1} = -(\frac{1-|\gamma_k|}{2})^{\frac{1}{2}}$. For $k = 1, 2$, taking a closer look at $\widehat{H}_{kk}$ reveals that $\widehat{H}_{kk}$ differs from H_{kk} in (0.1) only in the last column or the first row. Therefore, $\widehat{H}_{kk}$ is called as a rank-one modification of H_{kk}.

Assume that spectral decompositions of H and of the two submatrices $\widehat{H}_{kk}$ $(k = 1, 2)$ are known, *i.e.*,

$$H = W \Lambda W^*,$$

and

$$\widehat{H}_{kk} = W_k \Omega_k W_k^*, \qquad k = 1, 2,$$

where W and W_k are unitary matrices and Λ, Ω_k are diagonal. Define

$$\widetilde{W} = \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix}, \quad \Omega = diag(\mu_1, \cdots, \mu_n) = \begin{bmatrix} \Omega_1 & \\ & \Omega_2 \end{bmatrix},$$

$$z = [z_1, z_2, \cdots, z_n]^T = \begin{bmatrix} W_1^* e_k \omega_k \\ \overline{\Omega}_2 W_2^* e_1 \omega_{k+1} \end{bmatrix}. \tag{2.4}$$

Substituting above into (2.2) renders the following formulation,

$$H = \widetilde{W}\Omega(I - 2zz^*)\widetilde{W}^*.$$

Let

$$A = \Omega(I - 2zz^*).$$

Since $\widetilde{W}$ is unitary and A has the same eigenvalues as H, we introduce the characteristic polynomial of H,

$$\chi(\lambda) \doteq \det(H - \lambda I) = \det(A - \lambda I) = \det(\Omega - \lambda I)(1 - 2z^*(\Omega - \lambda I)^{-1}\Omega z). \tag{2.5}$$

3. Inverse eigenvalue problem

Notice that any eigenvalue of $H \in \mathscr{H}_n$ lies on the unit circle. Then $\lambda \in \lambda(H)$, $\{\mu_j\}_{j=1}^k \in \lambda(\widehat{H}_{11})$ and $\{\mu_j\}_{j=k+1}^n \in \lambda(\widehat{H}_{22})$ can be described as

$$\lambda = \exp(i\theta), \quad \mu_j = \exp(i\nu_j), \quad j = 1, \cdots, n, \tag{3.1}$$

where

$$-\pi < \nu_1 < \nu_2 < \cdots < \nu_k \leq \pi,$$

$$-\pi < \nu_{k+1} < \nu_{k+2} < \cdots < \nu_n \leq \pi.$$

Assume that there is no common number between $\{\mu_j\}_{j=1}^k$ and $\{\mu_j\}_{j=k+1}^n$, *i.e.*, $\nu_i \neq \nu_j$, $i = 1, \cdots, k$, $j = k+1, \cdots, n$. Then there exists a unique permutation $(j_1, j_2, \cdots, j_n)$ of $(1, 2, \cdots, n)$, such that

$$-\pi < \nu_{j_1} < \nu_{j_2} < \cdots < \nu_{j_n} \leq \pi. \tag{3.2}$$

The eigenvalues of $\widehat{H}_{kk}$ and H have the following interlacing property.

Lemma 3.1. *Suppose that $\{\lambda_j\}_{j=1}^n \in \lambda(H)$, $\{\mu_j\}_{j=1}^k \in \lambda(\widehat{H}_{11})$ and $\{\mu_j\}_{j=k+1}^n \in \lambda(\widehat{H}_{22})$, where H is defined by* (1.1)*, $\widehat{H}_{11}$ and $\widehat{H}_{22}$ by* (2.3)*. If there is no common number between $\{\mu_j\}_{j=1}^k$ and $\{\mu_j\}_{j=k+1}^n$, then there exists a unique permutation $(j_1, j_2, \cdots, j_n)$ of $(1, 2, \cdots, n)$, such that $\{\lambda_j\}_{j=1}^n$ and $\{\mu_{j_m}\}_{m=1}^n$ have the strictly interlacing properties on the unit circle.*

Proof. It is known that the eigenvalues of $\widehat{H}_{11}$ interlace with the eigenvalues of H (for detail, see [4]). In fact, $\widehat{H}_{22}$ has the same property as $\widehat{H}_{11}$. Let $\widetilde{H} = JH^TJ$ and $\widetilde{H}_{22} = J\widehat{H}_{22}^TJ$, where $J = [e_n, e_{n-1}, \cdots, e_1]$. Then $\widetilde{H} \in \mathscr{H}_n$ and $\widetilde{H}$ have the same eigenvalues as that of H. So do $\widetilde{H}_{22}$ and $\widehat{H}_{22}$. Using the notion of (1.1), we get that

$$\widetilde{H} = H(\widetilde{\gamma}_1, \widetilde{\gamma}_2, \cdots, \widetilde{\gamma}_n), \qquad \widetilde{H}_{22} = H(\widetilde{\gamma}_1, \widetilde{\gamma}_2, \cdots, \widetilde{\gamma}_{n-k-1}, \widetilde{\widetilde{\gamma}}_{n-k}),$$

where $\widetilde{\gamma}_i = \overline{\gamma}_{n-i}\gamma_n$, $i = 1, \cdots, n-1$, $\widetilde{\gamma}_n = \gamma_n$, $\widetilde{\widetilde{\gamma}}_{n-k} = \overline{\gamma}'_k\gamma_n$, $k = 1, \cdots, n-1$.

It is obvious that $\widetilde{H}_{22}$ is the modified $(n-k)\times(n-k)$ leading principal submatrix of $\widetilde{H}$. So the eigenvalues of $\widetilde{H}_{22}$ and $\widetilde{H}$ have interlacing property. Recall that $\widetilde{H}_{22}$ has the same eigenvalues as that of $\widehat{H}_{22}$. By (3.1) and let $\lambda_j = \exp(i\theta_j)$, $j = 1, \cdots, n$, we can get that $\theta_i \neq \nu_j$, $1 \leq i, j \leq n$. Hence according to (2.5), we know that all eigenvalues of H include the roots of the function

$$f(\lambda) = 1 - 2z^*(\Omega - \lambda I)^{-1}\Omega z = 1 - 2\sum_{j=1}^{n} \frac{\mu_j}{\mu_j - \lambda}|z_j|^2 = \sum_{j=1}^{n} |z_j|^2 \frac{\lambda + \mu_j}{\lambda - \mu_j}.$$

It follows from (3.1) that

$$f(\lambda) = \sum_{j=1}^{n} |z_j|^2 \frac{\exp(i(\theta - \nu_j)) + 1}{\exp(i(\theta - \nu_j)) - 1} = -i\sum_{j=1}^{n} |z_j|^2 \cot\left(\frac{\theta - \nu_j}{2}\right),$$

where we have used the fact $z^*z = 1$. Define

$$\Phi(\theta) \doteq -if(\lambda) = \sum_{j=1}^{n} |z_j|^2 \cot\left(\frac{\nu_j - \theta}{2}\right).$$

Reordering $\{\nu_j\}_{j=1}^n$ just as that in (3.2), we have

$$\Phi(\theta) = \sum_{m=1}^{n} |z_{j_m}|^2 \cot\left(\frac{\nu_{j_m} - \theta}{2}\right). \tag{3.3}$$

Thus, finding the roots of the spectral function $f(\lambda)$ is equivalent to finding the roots of $\Phi(\theta)$. An inspection of $\Phi(\theta)$ shows that it has n poles on the interval $(-\pi, \pi]$ occurring at each of the ν_{j_m}'s. Assume all of the $\{|z_j|\}_{j=1}^n$ are nonvanishing. Then by (3.3) we have

$$\Phi'(\theta) = \frac{1}{2}\sum_{j=1}^{n} \frac{|z_{j_m}|^2}{\sin^2\left(\frac{\nu_{j_m} - \theta}{2}\right)} \geq \frac{1}{2}.$$

Hence we obtain that $\Phi(\theta)$ has n distinct zeros $\{\theta_j\}_{j=1}^n$ and the sets $\{\nu_{j_m}\}_{m=1}^n$ interlace on the unit circle with $\{\theta_j\}_{j=1}^n$ strictly. That is

$$-\pi < \nu_{j_1} < \theta_1 < \nu_{j_2} < \cdots < \nu_{j_n} < \theta_n < \nu_{j_1} + 2\pi, \tag{3.4}$$

or

$$\nu_{j_n} - 2\pi < \theta_1 < \nu_{j_1} < \theta_2 < \nu_{j_2} < \cdots < \theta_n < \nu_{j_n} \le \pi. \tag{3.5}$$

This completes the proof of Lemma 3.1. ■

Now according to (2.5), we have

$$\begin{aligned}\chi(\lambda) &= \det(\Omega - \lambda I)\left(1 - 2\sum_{j=1}^{n}\frac{\mu_j}{\mu_j - \lambda}|z_j|^2\right)\\ &= \prod_{j=1}^{n}(\mu_j - \lambda) - 2\sum_{j=1}^{n}\left(\frac{\mu_j|z_j|^2\prod_{l=1}^{n}(\mu_l - \lambda)}{\mu_j - \lambda}\right).\end{aligned} \tag{3.6}$$

Recall that for $H \in \mathscr{H}_n$, the geometric multiplicity of its eigenvalues is one (for details, see [5]). This implies that

$$\chi(\lambda) = \prod_{j=1}^{n}(\lambda_j - \lambda), \tag{3.7}$$

where $\lambda_i \neq \lambda_j$, for $i \neq j$, $1 \le i, j \le n$. Then combining (3.6) and (3.7), and setting $\lambda = \mu_t$, for $t = 1, \cdots, n$, we obtain

$$\prod_{j=1}^{n}(\lambda_j - \mu_t) = -2\mu_t|z_t|^2\prod_{j=1, j\neq t}^{n}(\mu_j - \mu_t).$$

Assume $\{\lambda_j\}_{j=1}^n$ and $\{\mu_j\}_{j=1}^n$ have strictly interlacing properties on the unit circle, or, $\{\nu_j\}_{j=1}^n$and $\{\theta_j\}_{j=1}^n$ satisfy the condition (3.4) or (3.5). By (3.1), the above result can be rewritten as

$$\begin{aligned}|z_t|^2 &= -\frac{\prod_{j=1}^{n}(\exp(i(\theta_j - \nu_t) - 1)}{2\prod_{j\neq t}(\exp(i(\nu_j - \nu_t) - 1)},\\ &= -i\exp\left(i\sum_{j=1}^{n}\frac{\theta_j - \nu_j}{2}\right)\frac{\prod_{j=1}^{n}\sin(\frac{\theta_j - \nu_t}{2})}{\prod_{j\neq t}\sin(\frac{\nu_j - \nu_t}{2})}, \quad t = 1, \cdots, n.\end{aligned} \tag{3.8}$$

Since $\{\lambda_j\}_{j=1}^n$ are the eigenvalues of H, and $z^*z = 1$, we have

$$\begin{aligned}\exp\left(i\sum_{j=1}^{n}\theta_j\right) &= \prod_{j=1}^{n}\lambda_j = \det(H) = \det(\Omega(I - 2zz^*))\\ &= -\det(\Omega) = -\prod_{j=1}^{n}\mu_j = -\exp\left(i\sum_{j=1}^{n}\nu_j\right).\end{aligned}$$

It follows that if $\{\lambda_j\}_{j=1}^n \in \lambda(H)$, $\{\mu_j\}_{j=1}^k \in \lambda(\widehat{H}_{11})$ and $\{\mu_j\}_{j=k+1}^n \in \lambda(\widehat{H}_{22})$, then

$$\exp\left(i\sum_{j=1}^{n}(\theta_j-\nu_j)\right)=-1.$$

Hence in order to make sure that (3.8) is positive for $j=1,2,\cdots,n$, we know that when $\{\theta_j\}_{j=1}^n$ and $\{\nu_j\}_{j=1}^n$ satisfy (3.4), and

$$\sum_{j=1}^{n}\frac{\theta_j-\nu_j}{2}=\frac{\pi}{2}, \tag{3.9}$$

Eq. (3.8) can be rewritten as

$$|z_t|=\left[\frac{\prod_{j=1}^n \sin(\frac{\theta_j-\nu_t}{2})}{\prod_{j\neq t}\sin(\frac{\nu_j-\nu_t}{2})}\right]^{\frac{1}{2}}, \quad t=1,\cdots,n. \tag{3.10}$$

At the same time, when $\{\theta_j\}_{j=1}^n$ and $\{\nu_j\}_{j=1}^n$ satisfy (3.5), and

$$\sum_{j=1}^{n}\frac{\theta_j-\nu_j}{2}=-\frac{\pi}{2}, \tag{3.11}$$

(3.8) can be expressed as

$$|z_t|=\left[-\frac{\prod_{j=1}^n \sin(\frac{\theta_j-\nu_t}{2})}{\prod_{j\neq t}\sin(\frac{\nu_j-\nu_t}{2})}\right]^{\frac{1}{2}}, \quad t=1,\cdots,n. \tag{3.12}$$

Hence, $\{|z_i|\}_{i=1}^n$ can be constructed by $\{\theta_j\}_{j=1}^n$ and $\{\nu_j\}_{j=1}^n$ if they have strictly interlacing properties and satisfy (3.9) or (3.11). In fact, using the polynomial division and partial fraction decomposition, the uniqueness of $\{|z_j|\}_{j=1}^n$ is obvious by using (3.6) and (3.7).

Now we introduce a well-known implicit Q theorem (see [5] or [2]) which is useful in proving the uniqueness of H.

Lemma 3.2. *Let $\Lambda=diag(\lambda_1,\lambda_2,\cdots,\lambda_n)$ be a unitary diagonal matrix with distinct λ_i, and let $u_1=[\xi_i]_{i=1}^n\in\mathbb{C}^n$ be a unit vector with each $|\xi_i|\neq 0$. Then there exists a unitary matrix U and a unique $H\in\mathscr{H}_n$ such that $Ue_1=u_1$, and $\Lambda U=UH$.*

From Lemma 3.2, we see that any $H\in\mathscr{H}_n$ can be uniquely determined by its eigenvalues and the moduli space of the first components of its normalized eigenvectors. It is easily seen that H is also uniquely determined by its eigenvalues and the last components of its normalized eigenvectors (for detail, see [2]).

Theorem 3.1. *Let two sets* $\{\lambda_j\}_{j=1}^n$ *and* $\{\mu_j\}_{j=1}^n$ *be on the unit circle in the complex plane which can be expressed by* (3.1)*. If they satisfy the conditions* (3.4) *and* (3.9) *or* (3.5) *and* (3.11)*, then there exists a unique* $H = H(\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathscr{H}_n$ *such that* $\lambda(H) = \{\lambda_j\}_{j=1}^n$, $\lambda(\widehat{H}_{11}) = \{\mu_j\}_{j=1}^k$ *and* $\lambda(\widehat{H}_{22}) = \{\mu_j\}_{j=k+1}^n$, *where* $\widehat{H}_{11}$, $\widehat{H}_{22}$ *are defined by* (2.3) *and* γ_k' *by* (2.1).

Proof. Because there is a one-to-one correspondence between $\{\gamma_j\}_{j=1}^n$ and $H \in \mathscr{H}_n$ (see [1]), we only need to obtain $\{\gamma_j\}_{j=1}^n$ uniquely in order to construct H.

In fact, when $\{\theta_j\}_{j=1}^n$ and $\{\nu_j\}_{j=1}^n$ satisfy the assumption of Theorem 3.1, $\{|z_j|\}_{j=1}^n$ can be calculated easily by (3.10) or (3.12). Now, look back on z in (2.4). Partition it as follows,

$$z = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},$$

where $\eta_1 = W_1^* e_k \omega_k$, and $\eta_2 = \overline{\Omega}_2 W_2^* e_1 \omega_{k+1}$. Notice that η_1 is the last column of W_1^* by a real number ω_k, while η_2 is the first column of W_2^* by a unitary diagonal matrix $\overline{\Omega}_2$ and ω_{k+1}. Hence the last column of W_1^* and the first column of W_2^* can be obtained easily by matrix-vector calculations and normalization. In fact,

$$W_1^* e_k = \begin{bmatrix} x_{k1} \\ x_{k2} \\ \vdots \\ x_{kk} \end{bmatrix}, \qquad W_2^* e_1 = \begin{bmatrix} x_{1,k+1} \\ x_{1,k+2} \\ \vdots \\ x_{1,n} \end{bmatrix}.$$

Using the fact $|\overline{\mu}_j| = 1$ for $1 \le j \le n$ and the definition of ω_k and ω_{k+1}, we get

$$\begin{cases} |z_j| = \omega_k |x_{k,j}|, & j = 1, \cdots, k, \\ |z_j| = -\omega_{k+1} |x_{1,j}|, & j = k+1, \cdots, n. \end{cases} \tag{3.13}$$

Since W_1 is unitary, we have $\sum_{j=1}^k |x_{k,j}|^2 = 1$. So $\omega_k = (\sum_{j=1}^k |z_j|^2)^{\frac{1}{2}}$ and $|x_{k,j}| = |z_j|/\omega_k$, for $j = 1, \cdots, k$. Similarly, ω_{k+1} and $\{|x_{1,j}|\}_{j=k+1}^n$ can be obtained, *i.e.*, $\omega_{k+1} = -(\sum_{j=k+1}^n |z_j|^2)^{\frac{1}{2}}$ and $|x_{1,j}| = -|z_j|/\omega_{k+1}$, $j = k+1, \cdots, n$. Thus, $\gamma_1, \cdots, \gamma_k', \cdots, \gamma_n$ can be calculated by **IUQR** algorithm of [1] and γ_k can be obtained by γ_k' and ω_k easily.

The uniqueness of the construction of H is obvious. In fact, $\gamma_1, \gamma_2, \cdots, \gamma_k', \cdots, \gamma_n$ can be obtained uniquely according to Lemma 3.2. Also since $\{|z_j|\}_{j=1}^n$ is unique, we can obtain $|\gamma_k|$ uniquely. It follows that γ_k can be obtained uniquely. ∎

4. Algorithm

Summarizing the above discussions, we give the following algorithm for constructing the matrix $H = H(\gamma_1, \cdots, \gamma_n)$:

1. Compute $\{|z_j|\}_{j=1}^n$ from (3.10) or (3.12) for $j = 1, \cdots, n$, to get $|x_{k,j}|_{j=1}^k$, $|x_{1,j}|_{j=k+1}^n$ and ω_k.

2. Let $\widetilde{H}_{11} = J\widehat{H}_{11}^T J$, where $J = [e_k, e_{k-1}, \cdots, e_1]$. Use the **IUQR** algorithm of [1] to compute the *Schur parameters* $\{\widetilde{\gamma}_j\}_{j=1}^n$ of $\widetilde{H}_{11}$ such that $\Omega_1 \widetilde{W}_1^* = \widetilde{W}_1^* \widetilde{H}_{11}$, where

$$\widetilde{\gamma}_j = \overline{\gamma}_{k-j}\gamma_k', \quad j = 1, \cdots, k, \quad \widetilde{\gamma}_k = \gamma_k',$$
$$\widetilde{W}_1^* = \overline{W_1}^* J.$$

   Then determine $\gamma_1, \gamma_2, \cdots, \gamma_k'$.

3. If $\gamma_k' = 1$, then $\gamma_k = 0$; otherwise $\gamma_k = \gamma_k'|\gamma_k|$, where $|\gamma_k|$ is obtained by $\omega_k = (\frac{1+|\gamma_k|}{2})^{\frac{1}{2}}$.

4. Use the **IUQR** algorithm of [1] to compute the *Schur parameters* $\{\overline{\gamma}_k'\gamma_i\}_{i=k+1}^n$ and such that $\Omega_2 W_2^* = W_2^* \widehat{H}_{22}$, then get $\gamma_{k+1}, \gamma_{k+2}, \cdots, \gamma_n$.

5. Construct $H \doteq H(\gamma_1, \gamma_2, \cdots, \gamma_n) = G_1(\gamma_1)G_2(\gamma_2)\cdots\widetilde{G}_n(\gamma_n)$.

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