NUMERICAL MATHEMATICS, A Journal of Chinese Universities (English Series)
Numer. Math. J. Chinese Univ. (English Ser.), issue 2, vol. **16**: 147-154, 2007

A (2+1)-Dimensional Dispersive Long Wave Hierarchy and its Integrable Couplings

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Received March 24, 2006; Accepted (in revised version) April 29, 2006

Abstract

Under the frame of the (2+1)-dimensional zero curvature equation and Tu model, the (2+1)-dimensional dispersive long wave hierarchy is obtained. Furthermore, the loop algebra is expanded into a larger one. Moreover, a class of integrable coupling system for dispersive long wave hierarchy and (2+1)-dimensional multi-component integrable system will be investigated.

Keywords: (2+1)-dimensional zero curvature equation; integrable coupling; loop algebra; multi-component hierarchy.

Mathematics subject classification: 35Q51

1. Introduction

Integrable systems and soliton theory have been receiving more recognition in the mathematical and physics communities. A central and very important topic in the study of integrable system is to search for new Lax or Liouville systems which are associated with certain evolution equations with physical meaning. In [1], Tu proposed a new method based on the analysis of loop algebra. By using the loop algebra, some well-known integrable Hamiltonian hierarchies were worked out, such as AKNS hierarchy, KN hierarchy, WKI hierarchy, BPT hierarchy [1-4]. In order to produce multi-component integrable systems, Guo and Zhang [5-9] constructed a class of multi-component loop algebra $\tilde{G}_M$, and proposed some multi-component integrable systems. Zhou expressed the (2+1)-dimensional three-wave equation as a (2+1)-dimensional zero curvature equation whose almost-periodic solutions are obtained [10]. In this paper, a (2+1)-dimensional dispersive long wave hierarchy is presented by using the (2+1)-dimensional zero curvature equation. Further, we expand the loop algebra into a larger one. We will also investigate a class of integrable coupling system and multi-component integrable systems.

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2. A (2+1)-dimensional integrable system

Denote $\frac{\partial}{\partial z}=\frac{\partial}{\partial y}-\frac{\partial}{\partial x}, \frac{\partial}{\partial w}=\frac{\partial}{\partial t}-\frac{\partial}{\partial x}$. Then zero curvature equation

$$U_w - V_z + [U,V] = 0 \tag{2.1}$$

can be written as a (2+1)-dimensional form

$$U_t - V_y + [U,V] + V_x - U_x = 0, \tag{2.2}$$

which is regarded as the compatibility of the Lax pairs

$$\begin{cases} \varphi_y = \varphi_x + U\varphi, \\ \varphi_t = \varphi_x + V\varphi, \quad \lambda_t = 0. \end{cases} \tag{2.3}$$

We wish to use (2.3) or (2.2) to produce a hierarchy of soliton equations systematically, instead of a single equation. In what follows, we consider the dispersive long wave isospectral problem. Take the loop algebra, see, e.g., [1]

$$\begin{cases} e_1(n)=\lambda^n\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad e_2(n)=\lambda^n\begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad e_3(n)=\lambda^n\begin{pmatrix}0&0\\0&1\end{pmatrix}, \\ [e_1(m),e_2(n)]=2e_2(m+n), \quad [e_1(m),e_3(n)]=-2e_3(m+n), \\ [e_2(m),e_3(n)]=e_1(m+n), \\ \deg(e_k(n))=n, \ k=1,2,3. \end{cases} \tag{2.4}$$

Consider an isospectral problem

$$\begin{cases} \varphi_y = \varphi_x + U\varphi, \quad \lambda_t = 0, \\ U=\frac{1}{2}e_1(1)-\frac{1}{2}qe_1(0)-re_2(0)+e_3(0). \end{cases} \tag{2.5}$$

Set

$$V=\sum_{m\geq 0}(a_m e_1(-m)+b_m e_2(-m)+c_m e_3(-m)).$$

From the stationary zero curvature equation

$$V_y - V_x = [U,V], \tag{2.6}$$

one arrives at the recursion relation as follows:

$$\begin{cases} a_{my}-a_{mx}=-b_m-rc_m, \\ b_{my}-b_{mx}=b_{m+1}-qb_m+2ra_m, \\ c_{my}-c_{mx}=-c_{m+1}+qc_m+2a_m, \\ a_0=\alpha=const\neq 0, \quad b_0=c_0=0, \quad a_1=0, \quad b_1=-2\alpha r, \quad c_1=2\alpha, \\ a_2=2\alpha r, \quad b_2=-2\alpha[(r_y-r_x)+qr], \quad c_2=2\alpha q, \\ b_3=-2\alpha[(r_y-r_x+qr)_y-(r_y-r_x+qr)_x+q((r_y-r_x+qr)+2r^2], \\ c_3=2\alpha(q_x-q_y+q^2+2r), \quad a_3=2\alpha(r_y-r_x+2qr)\cdots \end{cases} \tag{2.7}$$

Note that

$$\begin{aligned}V_{+}^{(n)} =&(\lambda^n V)_+ = \sum_{m=0}^{n}(a_m e_1(n-m)+b_m e_2(n-m)+c_m e_3(n-m)),\\ V_{-}^{(n)} =&\lambda^n V - V_{+}^{(n)}.\end{aligned}$$

Then Eq. (2.6) can be written as

$$-V_{+y}^{(n)}+[U,V_{+}^{(n)}]+V_{+x}^{(n)}=V_{-y}^{(n)}-[U,V_{-}^{(n)}]-V_{-x}^{(n)}. \tag{2.8}$$

A direct calculation reads that the degree of the terms on the left-hand side of (2.8) is nonnegative, i.e., $(deg)\geq 0$, while that on the right-hand side of (2.8) is non-positive, i.e., $(deg)\leq 0$. Therefore,

$$-V_{+y}^{(n)}+[U,V_{+}^{(n)}]+V_{+x}^{(n)}=-b_{n+1}e_2(0)+c_{n+1}e_3(0).$$

Set $V^{(n)}=V_{+}^{(n)}-\frac{1}{2}c_{n+1}e_1(0)$. Thus the zero curvature equation

$$U_t-U_x-V_y^{(n)}+[U,V^{(n)}]+V_x^{(n)}=0, \tag{2.9}$$

leads to the following (2+1)-dimensional dispersive long wave hierarchy

$$\begin{cases} q_t-q_x=c_{n+1y}-c_{n+1x},\\ r_t-r_x=-b_{n+1}-rc_{n+1}=a_{n+1y}-a_{n+1x}.\end{cases} \tag{2.10}$$

It follows from (2.10) with $n=1$ that

$$\begin{cases} q_t=q_x+2\alpha(q_y-q_x),\\ r_t=r_x+2\alpha(r_y-r_x).\end{cases} \tag{2.11}$$

By letting $n=2$, we obtain the (2+1)-dimensional dispersive long wave equation:

$$\begin{cases} q_t=q_x+2\alpha[(q_x-q_y+q^2+2r)_y-(q_x-q_y)q^2+2r)_x],\\ r_t=r_x+2\alpha[(r_y-r_x+2qr)_y-(r_y-r_x+2qr)_x].\end{cases} \tag{2.12}$$

If we take $\partial_x=\partial_x^{-1}=0$, then the system (2.12) reduces to the dispersive long wave equation.

3. Integrable coupling of (2.10)

By making use of the Lie algebra [22], we obtain

$$\begin{aligned}&[e_1(m),e_2(n)]=2e_2(m+n), \quad [e_1(m),e_3(n)]=-2e_3(m+n),\\ &[e_2(m),e_3(n)]=e_1(m+n), \quad [e_1(m),e_4(n)]=e_4(m+n),\\ &[e_1(m),e_5(n)]=-e_5(m+n), \quad [e_2(m),e_4(n)]=0,\\ &[e_2(m),e_5(n)]=e_4(m+n), \quad [e_3(m),e_4(n)]=e_5(m+n),\\ &[e_3(m),e_5(n)]=[e_4(m),e_5(n)]=0, \quad e_i(n)=e_i\otimes\lambda^n,\\ &[e_i(m),e_j(n)]=[e_i,e_j]\lambda^{m+n}, \quad 1\leq i,j\leq 5, \quad \deg e_i(n)=n.\end{aligned} \tag{3.1}$$

A new loop algebra $\tilde{G}$ is formed if we set

$$\begin{cases} e_i(n)=e_i\lambda^n, \quad i=1,2,3,4,5; \\ [e_i(m),e_j(n)]=[e_i,e_j]\lambda^{m+n}, \quad 1\le i, \quad j\le 5, \\ deg(e_i(n))=n, \quad i=1,2,3,4,5. \end{cases}$$

Consider an isospectral problem

$$\begin{cases} \varphi_y=\varphi_x+U\varphi, \quad \lambda_t=0, \\ U=\frac{1}{2}e_1(1)-\frac{1}{2}u_1e_1(0)-u_2e_2(0)+e_3(0)+u_3e_4(0)+u_4e_5(0). \end{cases} \tag{3.2}$$

Set

$$V=\sum_{m\ge 0}(a_me_1(-m)+b_me_2(-m)+c_me_3(-m)+d_me_4(-m)+f_me_5(-m)).$$

Solving the stationary zero curvature equation

$$V_y-V_x=[U,V], \tag{3.3}$$

yields

$$\begin{cases} a_{my}-a_{mx}=-b_m-u_2c_m, \quad b_{my}-b_{mx}=b_{m+1}-u_1b_m+2u_2a_m, \\ c_{my}-c_{mx}=-c_{m+1}+u_1c_m+2a_m, \\ d_{my}-d_{mx}=\frac{1}{2}d_{m+1}-\frac{1}{2}u_1d_m-u_3a_m-u_4b_m-u_2f_m, \\ f_{my}-f_{mx}=-\frac{1}{2}f_{m+1}+\frac{1}{2}u_1f_m+u_4a_m+d_m-u_3c_m, \\ a_0=\alpha=const\ne 0, \quad b_0=c_0=f_0=d_0=0, \quad a_1=0, \quad b_1=-2\alpha r, \quad c_1=2\alpha, \\ d_1=2\alpha u_3, \quad f_1=2\alpha u_4, \quad a_2=2\alpha u_2, \quad b_2=-2\alpha[(u_{2y}-u_{2x})+u_1u_2], \quad c_2=2\alpha u_1, \\ d_2=4\alpha(u_{3y}-u_{3x})+2\alpha u_1u_3, \quad f_2=4\alpha(u_{4x}-u_{4y})+2\alpha u_1u_4, \\ c_3=2\alpha(u_{1x}-u_{1y}+u_1^2+2u_2), \quad a_3=2\alpha(u_{2y}-u_{2x}+2u_1u_2), \\ b_3=-2\alpha[(u_{2y}-u_{2x}+u_1u_2)_y-(u_{2y}-u_{2x}+u_1u_2)_x+u_1((u_{2y}-u_{2x}+u_1u_2)+2u_2^2],\cdots \end{cases} \tag{3.4}$$

Note that

$$\begin{cases} V_+^{(n)}=\sum_{m=0}^{n}(a_me_1(n-m)+b_me_2(n-m)+c_me_3(n-m)+d_me_4(n-m)+f_me_5(n-m)), \\ V_-^{(n)}=\lambda^nV-V_+^{(n)}. \end{cases}$$

A direct calculation reads

$$-V_{+y}^{(n)}+[U,V_+^{(n)}]+V_{+x}^{(n)}=-b_{n+1}e_2(0)+c_{n+1}e_3(0)-\frac{1}{2}d_{n+1}e_4(0)+\frac{1}{2}f_{m+1}e_5(0).$$

Set $V^{(n)}=V_{+}^{(n)}-\frac{1}{2}c_{n+1}e_1(0)$. Then the zero curvature equation

$$U_t-U_x-V_y^{(n)}+[U,V^{(n)}]+V_x^{(n)}=0$$

gives

$$\begin{cases} u_{1t}-u_{1x}=c_{n+1y}-c_{n+1x}, \\ u_{2t}-u_{2x}=-b_{n+1}-rc_{n+1}=a_{n+1y}-a_{n+1x}, \\ u_{3t}-u_{3x}=-\dfrac{1}{2}d_{n+1}+\dfrac{1}{2}u_3c_{n+1}, \\ u_{4t}-u_{4x}=\dfrac{1}{2}f_{n+1}-\dfrac{1}{2}u_4c_{n+1}. \end{cases} \tag{3.5}$$

Taking $u_3=u_4=0, u_1=q, u_2=r$ in (3.5), then system (3.5) is reduced to (2.10). Therefore, we call (3.5) an extending (2+1)-dimensional integrable model of dispersive long wave hierarchy.

4. A corresponding multi-component integrable system

Denoting $G_M=\{a=(a_{ij})_{M\times 3}=(a_1,a_2,a_3)\}$, where a_i $(i=1,2,3)$ stands for the i-th column of the matrix a, and M is an arbitrary positive integer.

Definition 4.1. *If $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_M)^T$, $\beta=(\beta_1,\beta_2,\dots,\beta_M)^T$ are two vectors, we define their vector product $\alpha*\beta$ and vector quotient α/β as*

$$\begin{aligned} &\alpha*\beta=\beta*\alpha=(\alpha_1*\beta_1,\alpha_2*\beta_2,\dots,\alpha_M*\beta_M)^T, \\ &\frac{\alpha}{\beta}=\alpha*\frac{1}{\beta}=\left(\frac{\alpha_1}{\beta_1},\frac{\alpha_2}{\beta_2},\dots,\frac{\alpha_M}{\beta_M}\right)^T. \end{aligned}$$

Definition 4.2. *Let $a=(a_1,a_2,a_3)$, $b=(b_1,b_2,b_3)\in G_M$. Then a commutation operation $[a,b]$ is defined as*

$$[a,b]=(a_2*b_3-a_3*b_2,2(a_1*b_2-a_2*b_1),2(a_3*b_1-a_1*b_3)). \tag{4.1}$$

Definition 4.3. *Set*

$$\tilde{G}_M=\{a\lambda^n|a\in G_M,n=0,\pm1,\pm2,\dots\} \tag{4.2}$$

along with a commutation operation presented by

$$[a\lambda^m,b\lambda^n]=[a,b]\lambda^{m+n},\quad \forall a,b\in G_M. \tag{4.3}$$

In terms of (4.1)-(4.3), we conclude that $\tilde{G}_M$ is a loop algebra.

Consider an isospectral problem

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_M\end{pmatrix}_y-\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_M\end{pmatrix}_x=\left[\left(\frac{1}{2}\lambda I_M-\frac{q}{2},-r,I_M\right),\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_M\end{pmatrix}\right], \tag{4.4}$$

where $I_M=(1,1,\ldots,1)^T_{1\times M}, q=(q_1,q_2,\ldots,q_M)^T, r=(r_1,r_2,\ldots,r_M)^T$. Set

$$V = \sum_{m\geq 0}(a_m,b_m,c_m)\lambda^{-m},$$

where $a_m=(a_{m1},a_{m2},\ldots,a_{mM})^T,\ b_m=(b_{m1},b_{m2},\ldots,b_{mM})^T,\ c_m=(c_{m1},c_{m2},\ldots,c_{mM})^T$. Solving the equation as follows

$$\begin{aligned}&\sum_{m\geq 0}\left[\begin{pmatrix}a_{m1}\\a_{m2}\\\vdots\\a_{mM}\end{pmatrix}_y-\begin{pmatrix}a_{m1}\\a_{m2}\\\vdots\\a_{mM}\end{pmatrix}_x,\begin{pmatrix}b_{m1}\\b_{m2}\\\vdots\\b_{mM}\end{pmatrix}_y-\begin{pmatrix}b_{m1}\\b_{m2}\\\vdots\\b_{mM}\end{pmatrix}_x,\begin{pmatrix}c_{m1}\\c_{m2}\\\vdots\\c_{mM}\end{pmatrix}_y-\begin{pmatrix}c_{m1}\\c_{m2}\\\vdots\\c_{mM}\end{pmatrix}_x\right]\lambda^{-m}\\&=\sum_{m\geq 0}\left[\left(\frac{1}{2}\lambda I_M-\frac{q}{2},-r,I_M\right),(a_m,b_m,c_m)\right]\lambda^{-m},\end{aligned}\tag{4.5}$$

yields

$$\begin{aligned}&\begin{pmatrix}a_{m1}\\a_{m2}\\\vdots\\a_{mM}\end{pmatrix}_y=-\begin{pmatrix}r_1\\r_2\\\vdots\\r_M\end{pmatrix}*\begin{pmatrix}c_{m,1}\\c_{m,2}\\\vdots\\c_{m,M}\end{pmatrix}-\begin{pmatrix}b_{m1}\\b_{m2}\\\vdots\\b_{mM}\end{pmatrix}+\begin{pmatrix}a_{m1}\\a_{m2}\\\vdots\\a_{mM}\end{pmatrix}_x,\\&\begin{pmatrix}b_{m+1,1}\\b_{m+1,2}\\\vdots\\b_{m+1,M}\end{pmatrix}=\begin{pmatrix}b_{m1}\\b_{m2}\\\vdots\\b_{mM}\end{pmatrix}_y+\begin{pmatrix}q_1\\q_2\\\vdots\\q_M\end{pmatrix}*\begin{pmatrix}b_{m,1}\\b_{m,2}\\\vdots\\b_{m,M}\end{pmatrix}-2\begin{pmatrix}r_1\\r_2\\\vdots\\r_M\end{pmatrix}*\begin{pmatrix}a_{m,1}\\a_{m,2}\\\vdots\\a_{m,M}\end{pmatrix}-\begin{pmatrix}b_{m1}\\b_{m2}\\\vdots\\b_{mM}\end{pmatrix}_x,\\&\begin{pmatrix}c_{m+1,1}\\c_{m+1,2}\\\vdots\\c_{m+1,M}\end{pmatrix}=\begin{pmatrix}c_{m1}\\c_{m2}\\\vdots\\c_{mM}\end{pmatrix}_x+\begin{pmatrix}q_1\\q_2\\\vdots\\q_M\end{pmatrix}*\begin{pmatrix}c_{m1}\\c_{m2}\\\vdots\\c_{mM}\end{pmatrix}+2\begin{pmatrix}a_{m,1}\\a_{m,2}\\\vdots\\a_{m,M}\end{pmatrix}-\begin{pmatrix}c_{m1}\\c_{m2}\\\vdots\\c_{mM}\end{pmatrix}_y,\\&\begin{pmatrix}a_{01}\\a_{02}\\\vdots\\a_{0M}\end{pmatrix}=\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_M\end{pmatrix},\qquad\begin{pmatrix}b_{01}\\b_{02}\\\vdots\\b_{0M}\end{pmatrix}=\begin{pmatrix}c_{01}\\c_{02}\\\vdots\\c_{0M}\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},\\&b_1=-2\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_M\end{pmatrix}*\begin{pmatrix}r_1\\r_2\\\vdots\\r_M\end{pmatrix},\quad c_1=2\begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_M\end{pmatrix}\cdots\end{aligned}\tag{4.6}$$

Note that

$$\begin{cases} V_+^{(n)} = \sum\limits_{m=0}^{n} (a_m, b_m, c_m)\lambda^{n-m}, \\ V_-^{(n)} = \lambda^n V - V_+^{(n)}. \end{cases}$$

Then a direct calculation gives

$$V_{+y}^{(n)} - V_{+x}^{(n)} - [U, V_+^{(n)}] = (-b_{n+1}, c_{n+1}).$$

Taking $V^{(n)} = V_+^{(n)} + (-\frac{1}{2}c_{n+1}, 0, 0)$, then the zero curvature equation

$$U_t - U_x - V_y^{(n)} + [U, V^{(n)}] + V_x^{(n)} = 0$$

admits the following (2+1)-dimensional multi-component integrable system

$$\begin{aligned} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{pmatrix}_t - \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{pmatrix}_x &= \begin{pmatrix} c_{n+1,1} \\ c_{n+1,2} \\ \vdots \\ c_{n+1,M} \end{pmatrix}_y - \begin{pmatrix} c_{n+1,1} \\ c_{n+1,2} \\ \vdots \\ c_{n+1,M} \end{pmatrix}_x, \\ \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_M \end{pmatrix}_t - \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_M \end{pmatrix}_x &= - \begin{pmatrix} b_{n+1,1} \\ b_{n+1,2} \\ \vdots \\ b_{n+1,M} \end{pmatrix} - \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_M \end{pmatrix} * \begin{pmatrix} c_{n+1,1} \\ c_{n+1,2} \\ \vdots \\ c_{n+1,M} \end{pmatrix} \\ &= \begin{pmatrix} a_{n+1,1} \\ a_{n+1,2} \\ \vdots \\ a_{n+1,M} \end{pmatrix}_y - \begin{pmatrix} a_{n+1,1} \\ a_{n+1,2} \\ \vdots \\ a_{n+1,M} \end{pmatrix}_x. \end{aligned} \tag{4.7}$$

Acknowledgments The author is very grateful to Prof. Zhang Yu-feng for his guidance and help.

References

[1] Tu G Z. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. J. Math. Phys., 1989, 30(2): 330-338.
[2] Ma W X. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. Chinese J. Contemp. Math., 1992, 13(1): 79-89.
[3] Tu G Z, Ma W X. An algebraic approach for extending Hamilton operators. J. Part. Diff. Eq., 1992, 3(2): 53-56.
[4] Ma W X. An approach for constructing nonisospectra hierarchies of evolution equations. J. Phys. A, 1992, 25: 719-726.

[5] Fan E G. A Liouville integrable Hamiltonian system associated with a generalized Kaup-Newell spectral problem. Physica A, 2001, 301: 105-113.

[6] Hu X B. An approach to generate supper extensions of integrable system. J. Phys. A, 1997, 30: 619-632.

[7] Zhang Y F, Zhang Y S. A type of multi-component integrable hierarchy. Chinese Phys., 2004, 13(8): 1183-1186.

[8] Dong H H, Zhang N. A multi-component matrix loop algebra and its application. Commun. Theor. Phys., 2005, 44(6): 997-1001.

[9] Zhou Z X. Finite dimensional Hamiltonian and almost-periodic solutions for (2+1)-dimensional three equations. J. Phys. Soc. Jpn., 2002, 71(8): 1857-1863.

[10] Zhang Y F, Yan Q Y, Wang X P. Two new loop algebras and their applications. Math. Practice Theory, 2003, 33(8): 109-115 (in Chinese).