NUMERICAL MATHEMATICS, A Journal of Chinese Universities (English Series)
Numer. Math. J. Chinese Univ. (English Ser.), issue 2, vol. **16**: 112-130, 2007

Spectral Element Viscosity Methods for Nonlinear Conservation Laws on the Semi-Infinite Interval

Liang Jiang
(*Department of Mathematics, Putian University, Putian 351115, China*
E-mail: ptjiangliang@mail.china.com)

Chuanju Xu*
(*Department of Mathematics, Xiamen University, Xiamen 361005, China*
E-mail: cjxu@xmu.edu.cn)

Received April 23, 2006; Accepted (in revised version) October 29, 2006

Abstract

In this paper we propose a spectral element vanishing viscosity (SEVV) method for the conservation laws on the semi-infinite interval. By using a suitable mapping, the problem is first transformed into a modified conservation law in a bounded interval, then the well-known spectral vanishing viscosity technique is generalized to the multi-domain case in order to approximate this transformed equation more efficiently. The construction details and convergence analysis are presented. Under a usual assumption of boundedness of the approximation solutions, it is proven that the solution of the SEVV approximation converges to the unique entropy solution of the conservation laws. A number of numerical tests is carried out to confirm the theoretical results.

Keywords: Spectral element methods; spectral vanishing viscosity; conservation laws; unbounded domain.
Mathematics subject classification: 35L65, 65M10, 65M15

1. Introduction

Spectral methods represent a relatively new approach to the numerical solution of partial differential equations as compared to some more "classical" methods, such as finite difference and finite element methods. Since its appearance (see, e.g., [2, 6]), spectral methods have been applied with success to a broad variety of mathematical equations, and particularly those modelling fluid dynamics. As a matter of fact, the most attractive property of spectral methods may be that when the solution of the problem is infinitely smooth the rate of convergence is exponential (the so-called spectral accuracy). However, when spectral methods are used to solve hyperbolic problems (for example nonlinear conservation laws), there are some difficulties stemming essentially from the fact that hyperbolic problems feature the presence of discontinuous solutions, arising in nonlinear equations, as well as in linear problems with discontinuous initial data. In the classical spectral approximations of the nonlinear conservation laws, the oscillations produced by the Gibbs

*Corresponding author.

phenomenon often grow up due to the nonlinearities and the calculations fail even if the initial data is smooth. Several approaches have been introduced to overcome this difficulty. The spectral vanishing viscosity (SVV) method, among others, is specially of interest because it allows to stabilize the scheme, while preserving the spectral accuracy. It was first established by Tadmor [17] for the resolution of conservation laws using the Fourier spectral method. The non-periodic case was then considered in the frame of the spectral Legendre approximation in [7, 9–11, 15]. Recently, SVV methods have been also used in the calculation of the incompressible flow of high Reynolds number [18], as well as in large eddy simulations [8, 13]. The analysis of these methods is available only on problems in bounded and single domain.

In this paper we essentially focus on applying the SVV method to conservation laws in a unbounded domain. Precisely, we consider spectral methods for the following problem:

$$\partial_t v(y,t) + \partial_y f(v(y,t)) = 0, \qquad (y,t) \in (0,\infty) \times (0,T].$$

Generally, there are three basic ways to construct global approximations to functions defined on the unbounded domain:

(i) truncate the unbounded domain to $[0, L]$, with a suitably large enough L;

(ii) employ the Laguerre polynomials or the Laguerre functions to expand the functions;

(iii) map the semi-infinite interval into a finite one and then expand the solution by using Legendre polynomials.

Here we follow the third way: employ a suitable mapping to geometrically transform the unbounded interval into a bounded one so that the classical SVV method can be applied.

The contributions of this work are three folds: Firstly, after transformation, a modified conservation law is obtained. Our convergence analysis shows that the additional term in front of the flux will not present fundamental difficulty in maintaining the efficiency of the classical SVV method for conservation laws. Secondly, in order to improve the efficiency, we construct a spectral method with the domain partition, which is based on the so-called C^0 spectral element method. Thirdly, when the multi-domain is concerned, the following point must be fixed: what is the appropriate definition of the SVV operator. We are going to see that due to the weak formulation the global SVV operator can be defined naturally. A detailed convergence analysis and numerical experiments are carried out. It is worthy to mention that with the domain partition, one can expect the corresponding algebraic problem to be better-conditioned because in this case it is not required to use large polynomial degree.

The paper is organized as follows. In Section 2, we formulate the problem and propose a spectral element vanishing viscosity method (SEVV) for conservation laws on the semi-infinite interval. Then, in Section 3 we carry out the convergence analysis. The numerical experiments are presented in Section 4.

2. Spectral vanishing viscosity approximation

We consider the following hyperbolic problem of conservation laws on the semi-infinite interval:

$$\partial_t v(y,t) + \partial_y f(v(y,t)) = 0, \qquad \forall (y,t) \in (0,\infty)\times(0,T], \tag{2.1}$$

subjecting to the boundary conditions prescribed at the inflow boundary point $y=0$:

$$v(0,t) = g(t), \quad \forall t \in (0,T],$$

and the initial condition:

$$v(y,0) = v_0(y), \quad \forall y \in (0,\infty).$$

Let $\Lambda = (-1,1)$ and $\Omega = \Lambda \times (0,T]$. We make the variable transformation

$$y = -\ln\frac{1-x}{2} : x \in \Lambda \to y \in (0,\infty) \tag{2.2}$$

to transform the problem (2.1) into a problem in Λ:

$$\begin{cases} \partial_t u(x,t) + (1-x)\partial_x f(u(x,t)) = 0, & \forall (x,t) \in \Omega, \\ u(-1,t) = g(t), & \forall t \in (0,T], \\ u(x,0) = u_0(x), & \forall x \in \Lambda, \end{cases} \tag{2.3}$$

where $u(x,t) = v(y(x),t)$ and $u_0(x) = v_0(y(x))$.

An entropy solution of (2.3) is sought, i.e., a bounded measurable function $u(x,t)$, which satisfies the prescribed boundary and initial data in the proper sense, and admits the following entropy condition: for all convex entropy pairs, (U,F), $U''(\cdot) \geq 0$, satisfying the compatibility relation $F'(\cdot) = U'f'(\cdot)$, it holds

$$\partial_t U(u(x,t)) + (1-x)\partial_x F(u(x,t)) \leq 0\ , \quad \forall (x,t) \in \Omega. \tag{2.4}$$

The entropy inequality (2.4) is sufficient to single out a unique, physically relevant solution. In fact, this so called entropy solution, can be realized by the vanishing viscosity limit, $u = \lim\limits_{\varepsilon\to 0} u^\varepsilon$, where u^ε is the solution of the regularized vanishing viscosity equation

$$\partial_t u^\varepsilon(x,t) + (1-x)\partial_x f(u^\varepsilon(x,t)) = \varepsilon\partial_x(Q\partial_x u^\varepsilon(x,t))\ , \quad \varepsilon > 0,$$

where Q is a SVV operator. We note in passing that the regularized viscosity equation admits an equivalent weak formulation: for all $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega [u^\varepsilon(x,t)\partial_t\phi(x,t) + f(u^\varepsilon(x,t))\partial_x((1-x)\phi(x,t)) + \varepsilon Q\partial_x u^\varepsilon(x,t)\partial_x\phi(x,t)]dxdt = 0. \tag{2.5}$$

As compared to the traditional conservation laws in bounded domain, we have an extra term $1-x$ in front of the flux f in the problem in unbounded domain after mapping, see (2.3). However, we can prove by using a standard analysis (see, e.g., [7]), without fundamental difficulty arising from this additional term, that the weak entropy solution of (2.3) is unique. Although we do not give the detailed analysis for the above statement, this point will become clear when a discrete counterpart of the analysis is presented in next section.

In order to define the spectral element approximation to problem (2.3), we need some notations and definitions. We recall some classical functional spaces, together with some corresponding standard norms and inner products that are used hereafter: $L^2(\Lambda)$ is the space of measurable functions whose square is Lebesgue integrable in Λ; the inner products of $L^2(\Lambda)$ and $L^2(\Omega)$ are defined by

$$(\phi,\psi)=\int_\Lambda \phi\psi\,dx,\quad (\phi,\psi)_\Omega=\int_0^T(\phi,\psi)\,dt.$$

The corresponding norms are denoted respectively by

$$\|\phi\|=\sqrt{(\phi,\phi)},\quad \|\phi\|_\Omega=\sqrt{(\phi,\phi)_\Omega}.$$

Let

$$\begin{aligned}&H^1(\Lambda):=\{\phi\in L^2(\Lambda):\ \partial_x\phi\in L^2(\Lambda)\},\\&H_0^1(\Lambda):=\{\phi\in H^1(\Lambda):\phi(-1)=\phi(1)=0\}.\end{aligned}$$

For spectral approximations, we define the Gauss-Lobatto points, denoted by $\{\xi_j\}_{j=0}^N$, which are the zeros of $(1-x^2)L_N'(x)$ with $-1=\xi_0<\xi_1<\cdots<\xi_N=1$, where L_N is the Legendre polynomial of degree N. In the sequel we shall use the Legendre Gauss-Lobatto quadrature rule, stating that there exist weights ρ_j, $0\le j\le N$, such that

$$\int_{-1}^1\phi(x)dx=\sum_{j=0}^N\rho_j\phi(\xi_j),\quad \forall\phi\in\mathbb{P}_{2N-1}(\Lambda),\tag{2.6}$$

where $\mathbb{P}_N(\Lambda)$ denotes the space of all polynomials of degree not exceeding N defined in Λ. This suggests to define a discrete inner product,

$$(\phi,\varphi)_N=\sum_{j=0}^N\rho_j\phi(\xi_j)\psi(\xi_j),\quad (\phi,\varphi)_{\Omega,N}=\int_0^T(\phi,\psi)_N\,dt.$$

Now let $\mathscr{N}$ be the pair of positive integers (K,N), we divide the interval Λ into some subintervals (spectral elements) such that

$$\begin{aligned}&\bar\Lambda=\bigcup_{k=1}^K\bar\Lambda_k,\quad \Lambda_k\cap\Lambda_l=\emptyset,\ \text{for}\ \ k\neq l,\\&\bar\Omega=\bigcup_{k=1}^K\bar\Omega_k,\quad \Omega_k=\Lambda_k\times[0,T],\end{aligned}$$

where $\Lambda_k = (a_k, a_{k+1}), -1 = a_1 < a_2 < \cdots < a_{K+1} = +1$.

We then define the space of piecewise polynomials of degree $\le N$:

$$\mathbb{P}_{N,K}(\Lambda) := \{\phi \in L^2(\Lambda) : \ \ \phi|_{\Lambda_k} \in \mathbb{P}_N(\Lambda_k), 1 \le k \le K\}.$$

The space of the spectral element approximation for the solution u will consist of the subspace $V_{\mathcal{N}}$ of $H^1(\Lambda)$, defined by

$$V_{\mathcal{N}} = H^1(\Lambda) \cap \mathbb{P}_{N,K}(\Lambda).$$

We will need spectral element numerical integrals. To this end, we first introduce the affine mapping:

$$\Lambda_k \overset{F^k}{\rightarrow} \Lambda,$$

where F^k is given by

$$r = F^k(x) = 2\frac{x - a_k}{a_{k+1} - a_k} - 1.$$

Then the global integration points (mapped Gauss-Lobatto points) ξ_j^k and the associated weights ρ_j^k are defined as follows

$$\xi_j^k = a_k + (\xi_j + 1)h^k/2, \ \ \rho_j^k = \rho_j h^k/2, \ \ 0 \le j \le N, \ \ 1 \le k \le K, \tag{2.7}$$

where h^k is the length of interval Λ_k, i.e., $h^k = a_{k+1} - a_k$. To simplify the notation, we suppose, without lose of generality, $h^k = h, \ \ 1 \le k \le K$. Now we define a discrete inner product for all piecewise continuous functions ϕ and ψ,

$$(\phi, \psi)_{\mathcal{N}} = \sum_{k=1}^{K}\sum_{j=0}^{N} \phi(\xi_j^k)\psi(\xi_j^k)\rho_j^k,$$

and let $\|\cdot\|_{\mathcal{N}}$ denote the corresponding discrete norm.

Throughout this paper, we will use the expression $A \lesssim B$ to mean that there exists a generic positive constant c, independent of any function and discretization parameters such as N etc (but may be dependent of K), such that $A \le cB$.

It is known that the norm $\|\cdot\|_{\mathcal{N}}$ is equivalent to the usual L^2-norm for the space $\mathbb{P}_{N,K}(\Lambda)$ (see, e.g., [1]), i.e.,

$$\|\phi\| \lesssim \|\phi\|_{\mathcal{N}} \lesssim \|\phi\|, \ \ \forall \phi \in \mathbb{P}_{N,K}(\Lambda), \tag{2.8}$$

and we also have

$$(\phi, \psi) = (\phi, \psi)_{\mathcal{N}}, \quad \text{for all } \phi\psi \in \mathbb{P}_{2N-1,K}(\Lambda). \tag{2.9}$$

The spectral element vanishing viscosity (SEVV) approximation of (2.3) reads: for all $t \in (0, T]$, find $u_{\mathcal{N}}(x,t) \in V_{\mathcal{N}}$, such that $u_{\mathcal{N}}(0) = I_{\mathcal{N}} u_0(x)$, and

$$\begin{aligned} &(\frac{\partial}{\partial t}u_{\mathcal{N}} + (1-x)\frac{\partial}{\partial x}\mathcal{I}_{\mathcal{N}} f(u_{\mathcal{N}}), \phi)_{\mathcal{N}} \\ =\ & -\varepsilon_N(\mathcal{Q}\frac{\partial}{\partial x}u_{\mathcal{N}}, \frac{\partial}{\partial x}\phi)_{\mathcal{N}} + (B(u_{\mathcal{N}}), \phi)_{\mathcal{N}}, \quad \forall \phi \in V_{\mathcal{N}}, \end{aligned} \tag{2.10}$$

where $\mathscr{I}_{\mathcal{N}}$ denotes the polynomial interpolation operator onto $V_{\mathcal{N}}$, and ε_N is the viscosity amplitude:

$$\varepsilon_N = \mathscr{O}(1/N^{\alpha}), \quad 0 < \alpha < 1. \tag{2.11}$$

In order to well define the above SEVV approximation, we have to clarify two critical points: boundary condition term $B(u_{\mathcal{N}})$ and the spectral element viscosity operator $\mathscr{Q}$.

First, in (2.10), the inflow boundary condition is imposed in a weak way [4] via use of the boundary operator $B(u_{\mathcal{N}})$, which is defined, for all t, as a piecewise polynomial of $I\!P_{N,K}(\Lambda)$ of the form:

$$B(u_{\mathcal{N}})|_{\Lambda_1} = 2\mu(t)\frac{a_2 - x}{a_2 + 1}L_N'(r) \circ F^k(x),$$
$$B(u_{\mathcal{N}})|_{\Lambda_k} = 0,\ 2 \le k \le K,$$

where $\mu(t)$ is a nonzero free parameter, which should be determined to handle the inflow boundary condition.

Second, for $\mathscr{Q}$, we define it piecewisely via

$$\mathscr{Q}\phi|_{\Lambda_k} = \mathscr{Q}^k\phi^k,\ \ 1 \le k \le K, \tag{2.12}$$

where, for a function ϕ defined in Λ, ϕ^k denotes its restriction in Λ_k, that is $\phi^k := \phi|_{\Lambda_k}$. $\mathscr{Q}^k$ is the local spectral viscosity operator, defined by

$$\mathscr{Q}^k\phi^k := (\tilde{Q}\tilde{\phi^k}) \circ F^k, \tag{2.13}$$

where

$$\tilde{\phi^k} := \phi^k \circ (F^k)^{-1},$$

and $\tilde{Q}$ is the standard spectral viscosity operator defined in the reference interval such that

$$\tilde{Q}\tilde{\phi}(r) := \sum_{i=0}^{N}\hat{Q}_i\hat{\phi}_i L_i(r), \quad \forall\tilde{\phi}, \quad \tilde{\phi}(r) = \sum_{i=0}^{\infty}\hat{\phi}_i L_i(r), \tag{2.14}$$

with $\hat{Q}_i = 0$ if $i \le m_N$ and $1 \ge \hat{Q}_i \ge 0$ if $i > m_N$. Hereafter we use symbol $\tilde{\ }$ to denote the quantities in the standard domain Λ. Typical choices for m_N are $m_N = \mathscr{O}(\sqrt{N})$ or $m_N = N/2$ [8], whereas for the Bürgers equation theoretical studies rather yield $m_N < \mathscr{O}(N^{1/4})$ [11]. For $m_N < i \le N$, the theoretical studies suggest

$$1 \ge \hat{Q}_i \ge 1 - (\frac{m_N}{i})^4,\ \ m_N < i \le N. \tag{2.15}$$

In virtue of (2.14), Eq. (2.13) can be rewritten as

$$\mathscr{Q}^k\phi^k = [\sum_{i=0}^{N}\hat{Q}_i\hat{\phi}_i^k L_i(r)] \circ F^k(x) = \sum_{i=0}^{N}\hat{Q}_i\hat{\phi}_i^k L_i^k(x), \tag{2.16}$$

where

$$L_i^k(x) := L_i(r) \circ F^k(x). \tag{2.17}$$

Let us note that the weak formulation of the SEVV approximation (2.10) can be viewed as a pseudospectral (collocation) problem. This is done by "testing" (2.10) with $\phi = \phi_i$, the standard Lagrangian polynomial of $V_{\mathcal{N}}$ being nonzero at only one global collocation point, then at the interior points of each element we obtain

$$\begin{aligned}&\partial_t u_N^k(\xi_j^k,t) + (1-\xi_j^k)\partial_x \mathscr{I}_{\mathcal{N}} f(u_N^k)(\xi_j^k,t)\\ =\ &\varepsilon_N \partial_x Q(\partial_x u_N^k)(\xi_j^k,t), \quad 1 \le j \le N-1, \ \ 1 \le k \le K,\end{aligned} \tag{2.18}$$

and, at the interface points, we have

$$\begin{aligned}&\rho_0^{k+1}h^{k+1}[\partial_t u_N^{k+1}(\xi_0^{k+1},t) + (1-\xi_0^{k+1})\partial_x \mathscr{I}_N f(u_N^{k+1})(\xi_0^{k+1},t)\\ &+\varepsilon_N \partial_x Q(\partial_x u_N^{k+1})(\xi_0^{k+1},t)]/2 + \rho_N^k h^k[\partial_t u_N^k(\xi_N^k,t)\\ &+(1-\xi_N^k)\partial_x \mathscr{I}_N f(u_N^k)(\xi_N^k,t) + \varepsilon_N \partial_x Q(\partial_x u_N^k)(\xi_N^k,t)]/2\\ =\ &\varepsilon_N((Q\partial_x u_N^{k+1})(\xi_0^{k+1},t) - (Q\partial_x u_N^k)(\xi_N^k,t)), \quad 1 \le k \le K-1.\end{aligned} \tag{2.19}$$

For the inflow boundary condition, we have

$$\begin{aligned}&\partial_t u_N^1(-1,t) + 2\partial_x \mathscr{I}_N f(u_N^1)(-1,t)\\ =\ &\varepsilon_N \partial_x Q(\partial_x u_N^1)(-1,t) + \frac{\varepsilon_N}{\rho_0^1}Q(\partial_x u_N^1)(-1,t) - \frac{2(-1)^N \mu(t)}{\rho_0^1}.\end{aligned} \tag{2.20}$$

The last term on the right hand side of (2.20) is used for preventing the creation of a boundary layer [4].

3. Convergence analysis of the SEVV method

We first derive or recall some a priori estimates associated with interpolation and projection operators, which will be useful in the analysis that follows.

Let $\mathscr{I}_{\mathcal{N}}$ denote the Legendre Gauss-Lobatto interpolation operator based on the global Legendre Gauss-Lobatto points, i.e., $\forall \phi \in C^0(\Lambda)$, $\mathscr{I}_{\mathcal{N}}\phi \in \mathbb{P}_{N,K}(\Lambda)$, such that $\mathscr{I}_{\mathcal{N}}\phi(\xi_j^k) = \phi(\xi_j^k)$, $\ 1 \le k \le K, \ \ 0 \le j \le N$, then a result in [1] provides us with the estimate:

$$\|\partial_x \mathscr{I}_{\mathcal{N}}\phi\| + N\|\phi - \mathscr{I}_{\mathcal{N}}\phi\| \lesssim \|\partial_x \phi\|, \quad \forall \phi \in H_0^1(\Lambda). \tag{3.1}$$

It is worthy to mention that the estimate (3.1) is not optimal. The optimal estimate should depend on h [14]. We will address this point in the future and focus here our attentions on the convergence analysis of the SEVV method under assumption of uniform boundedness of h^k.

Let

$$\mathbb{P}_N^0(\Lambda) := \{\phi_N \in \mathbb{P}_N(\Lambda) : \phi_N(-1) = \phi_N(1) = 0\}.$$

We define the standard L^2-orthogonal projection $\widetilde{\Pi}_N : L^2(\Lambda) \rightarrow I\!P_N(\Lambda)$, by: $\forall \tilde{\phi} \in L^2(\Lambda)$, $\widetilde{\Pi}_N \tilde{\phi} \in I\!P_N(\Lambda)$, such that

$$\int_{-1}^{1} (\tilde{\phi} - \widetilde{\Pi}_N \tilde{\phi}) \psi_N dx = 0, \quad \forall \psi_N \in I\!P_N(\Lambda),$$

and the projection $\widetilde{\Pi}_N^{1,0} : H_0^1(\Lambda) \rightarrow I\!P_N^0(\Lambda)$, by: $\forall \tilde{\phi} \in H_0^1(\Lambda), \widetilde{\Pi}_N^{1,0} \tilde{\phi} \in I\!P_N^0(\Lambda)$, such that

$$\widetilde{\Pi}_N^{1,0} \tilde{\phi} := \int_{-1}^{r} (\widetilde{\Pi}_{N-1} \partial_s \tilde{\phi})(s)\, ds.$$

It is known that $\widetilde{\Pi}_N^{1,0}$ has the following property [17]:

$$\|\partial_r \widetilde{\Pi}_N^{1,0} \tilde{\phi}\| + N \|\tilde{\phi} - \widetilde{\Pi}_N^{1,0} \tilde{\phi}\| \lesssim \|\partial_r \tilde{\phi}\|, \quad \forall \tilde{\phi} \in H_0^1(\Lambda). \tag{3.2}$$

Then we define the projection operator $\widetilde{\Pi}_N^1 : H^1(\Lambda) \rightarrow I\!P_N(\Lambda)$, $\forall \tilde{\phi} \in H^1(\Lambda)$,

$$\widetilde{\Pi}_N^1 \tilde{\phi}(r) = \widetilde{\Pi}_N^{1,0} \tilde{\phi}^* + \tilde{\phi}(-1) \frac{1-r}{2} + \tilde{\phi}(1) \frac{1+r}{2},$$

where

$$\tilde{\phi}^*(r) = \tilde{\phi}(r) - \tilde{\phi}(-1) \frac{1-r}{2} - \tilde{\phi}(1) \frac{1+r}{2}.$$

It can be verified that the operator $\widetilde{\Pi}_N^1$ satisfies:

$$\tilde{\phi} - \widetilde{\Pi}_N^1 \tilde{\phi} = \tilde{\phi}^* - \widetilde{\Pi}_N^{1,0} \tilde{\phi}^*, \quad \partial_r \widetilde{\Pi}_N^1 \tilde{\phi} = \widetilde{\Pi}_{N-1} \partial_r \tilde{\phi}, \quad \widetilde{\Pi}_N^1 \tilde{\phi}|_{\partial\Lambda} = \tilde{\phi}|_{\partial\Lambda}. \tag{3.3}$$

Now we define the projection operator $\Pi_{\mathcal{N}}^1 : H^1(\Lambda) \rightarrow V_{\mathcal{N}}$ as follow: $\forall \phi \in H^1(\Lambda)$, then $\Pi_{\mathcal{N}}^1 \phi \in V_{\mathcal{N}}$ satisfies

$$\Pi_{\mathcal{N}}^1 \phi|_{\Lambda_k} := \widetilde{\Pi}_N^1 \tilde{\phi^k}(r) \circ F^k(x),$$

where we recall that $\tilde{\phi^k} := \phi^k \circ (F^k)^{-1}$, with ϕ^k the restriction of ϕ in Λ_k. In fact, thanks to (3.3), we have

$$\Pi_{\mathcal{N}}^1 \phi|_{\partial\Lambda_k} = \widetilde{\Pi}_N^1 \tilde{\phi^k} \circ F^k|_{\partial\Lambda_k} = \widetilde{\Pi}_N^1 \tilde{\phi^k}|_{\partial\Lambda} = \tilde{\phi^k}|_{\partial\Lambda} = \phi^k|_{\partial\Lambda_k}.$$

Hence $\Pi_{\mathcal{N}}^1 \phi$ is continuous crossing the elemental interfaces, which implies that $\Pi_{\mathcal{N}}^1 \phi$ well defines a function of $V_{\mathcal{N}}$.

The projection operator $\Pi_{\mathcal{N}}^1$ plays an important role in our convergence analysis below. First we have the following result.

Lemma 3.1. *The following estimate holds:*

$$\|\partial_x \Pi_{\mathcal{N}}^1 \phi\| + N \|\phi - \Pi_{\mathcal{N}}^1 \phi\| \lesssim \|\partial_x \phi\|, \quad \forall \phi \in H_0^1(\Lambda). \tag{3.4}$$

Proof. We first prove

$$\|\phi - \Pi^1_{\mathscr{N}}\phi\| \lesssim N^{-1}\|\partial_x\phi\|.$$

By definition and using (3.2), we have

$$\begin{aligned}
\|\phi - \Pi^1_{\mathscr{N}}\phi\|^2 &= \sum_{k=1}^{K}\int_{\Lambda_k}(\phi - \Pi^1_{\mathscr{N}}\phi)^2 dx = \sum_{k=1}^{K}\int_{\Lambda_k}(\tilde{\phi}^k - \tilde{\Pi}^1_N\tilde{\phi}^k)^2(r)\circ F^k(x)dx \\
&= \sum_{k=1}^{K}\int_{\Lambda}(\tilde{\phi}^k - \tilde{\Pi}^1_N\tilde{\phi}^k)^2(r)\frac{h}{2}dr \lesssim \sum_{k=1}^{K}\|\tilde{\phi}^k - \tilde{\Pi}^1_N\tilde{\phi}^k\|^2_\Lambda \\
&\lesssim \sum_{k=1}^{K}\|\tilde{\phi}^{k^*} - \tilde{\Pi}^{1,0}_N\tilde{\phi}^{k^*}\|^2_\Lambda \lesssim N^{-2}\sum_{k=1}^{K}\|\partial_r\tilde{\phi}^{k^*}\|^2_\Lambda \\
&\lesssim N^{-2}\sum_{k=1}^{K}\|\partial_r(\tilde{\phi}^k(r) - \tilde{\phi}^k(-1)\frac{1-r}{2} - \tilde{\phi}^k(1)\frac{1+r}{2})\|^2_\Lambda \\
&\lesssim N^{-2}\sum_{k=1}^{K}(\|\partial_r\tilde{\phi}^k(r)\|^2_\Lambda + |\tilde{\phi}^k(1)|^2 + |\tilde{\phi}^k(-1)|^2) \\
&\lesssim N^{-2}\|\partial_x\phi\|^2 + N^{-2}\sum_{k=1}^{K}\|\partial_r\tilde{\phi}^k\|^2 \lesssim N^{-2}\|\partial_x\phi\|^2.
\end{aligned}$$

Note that in the last estimate above, the Gagliardo-Nirenberg inequality has been used. On the other hand, we also have

$$\begin{aligned}
\|\partial_x\Pi^1_{\mathscr{N}}\phi\|^2 &= \sum_{k=1}^{K}\int_{\Lambda_k}(\partial_x\Pi^1_{\mathscr{N}}\phi)^2dx = \sum_{k=1}^{K}\int_{\Lambda_k}(\partial_r\tilde{\Pi}^1_N\tilde{\phi}^k)^2(r)(\partial_xF^k(x))^2\circ F^k(x)dx \\
&= \sum_{k=1}^{K}\int_{\Lambda}(\partial_r\tilde{\Pi}^1_N\tilde{\phi}^k)^2(r)\frac{2}{h}dr \\
&= \sum_{k=1}^{K}\int_{\Lambda}\left(\partial_r\tilde{\Pi}^{1,0}_N\tilde{\phi}^{k^*}(r) - \frac{1}{2}\tilde{\phi}^{k^*}(-1) + \frac{1}{2}\tilde{\phi}^{k^*}(1)\right)^2\frac{2}{h}dx \\
&\lesssim \sum_{k=1}^{K}(\|\partial_r\tilde{\Pi}^{1,0}_N\tilde{\phi}^{k^*}(r)\|^2 + |\tilde{\phi}^{k^*}(1)|^2 + |\tilde{\phi}^{k^*}(-1)|^2).
\end{aligned}$$

Then, by (3.2) and the Gagliardo-Nirenberg inequality, we obtain

$$\|\partial_x\Pi^1_{\mathscr{N}}\phi\| \lesssim \|\partial_x\phi\|.$$

By combining the above estimates, we complete the proof of the lemma. ■

Concerning the error of the numerical integration, we use (2.9) with $\psi \equiv \mathscr{I}_{\mathscr{N}-1}\psi + (\psi - \mathscr{I}_{\mathscr{N}-1}\psi)$ and (3.1) to obtain (see [12])

$$|(\phi,\psi) - (\phi,\psi)_{\mathscr{N}}| \lesssim \|\psi - \mathscr{I}_{\mathscr{N}-1}\psi\|\|\phi\| \lesssim \frac{1}{N}\|\partial_x\psi\|\|\phi\|, \quad \forall\phi,\psi\in V_{\mathscr{N}}. \tag{3.5}$$

We end the preparation by introducing a low modes filtering operator. Let $\tilde{R} = I - \tilde{Q}$ and $\hat{R}_i = 1 - \hat{Q}_i$. Then clearly $\hat{R}_i = 1$ for $i \le m_N$, and

$$\hat{R}_i \le (\frac{m_N}{i})^4 \quad \text{for} \quad i > m_N.$$

Let $\mathscr{R}$ denote the corresponding low modes filter

$$\mathscr{R}\phi|_{\Lambda_k} := \mathscr{R}^k\phi^k := \tilde{R}\tilde{\phi}^k \circ F^k, \quad 1 \le k \le K. \tag{3.6}$$

Consequently,

$$\mathscr{R}^k\phi^k = \sum_{i=0}^{N} \hat{R}_i \hat{\phi}_i^k L_i^k(x), \quad \phi^k = \sum_{i=0}^{\infty} \hat{\phi}_i^k L_i^k(x), \tag{3.7}$$

where L_i^k is defined in (2.17). Obviously, we have $\mathscr{R} = \mathscr{I} - \mathscr{Q}$. Let $\|\phi\|_{\mathscr{Q}}$ and $\|\phi\|_{\mathscr{R}}$ denote the weighted inner product $(\mathscr{Q}\phi, \phi)$ and $(\mathscr{R}\phi, \phi)$, respectively. Then we have the following estimates.

Lemma 3.2. *Consider the SVV operators $\mathscr{Q}$ and $\mathscr{R}$ defined in* (2.12) *and* (3.6) *respectively, with the parameterizations in* (2.11) *and* (2.15). *Then the following estimates hold:* $\forall \phi \in V_{\mathscr{N}}$,

$$\|\partial_x\phi\|^2 \le \|\partial_x\phi\|_{\mathscr{Q}}^2 + cm_N^4 \ln N \|\phi\|^2, \quad \|\partial_x\phi\|_{\mathscr{R}}^2 \lesssim m_N^4 \ln N \|\phi\|^2. \tag{3.8}$$

Proof. Since $\partial_x\phi \equiv \mathscr{Q}\partial_x\phi + \mathscr{R}\partial_x\phi$, it suffices to prove the second part of (3.8). Using an estimate about the operator $\tilde{Q}$ in the reference domain Λ (see, e.g., [7,17]), we have

$$\begin{aligned}
\|\partial_x\phi\|_{\mathscr{R}}^2 &= \int_{\Lambda} (\mathscr{R}\partial_x\phi)\partial_x\phi dx \\
&= \sum_{k=1}^{K} \int_{\Lambda_k} (\mathscr{R}^k\partial_x\phi^k)\partial_x\phi^k dx = \sum_{k=1}^{K} \int_{\Lambda} (\tilde{R}\partial_r\tilde{\phi}^k)\partial_r\tilde{\phi}^k \frac{2}{h} dr \\
&\lesssim \sum_{k=1}^{K} \int_{\Lambda} (\tilde{R}\partial_r\tilde{\phi}^k)\partial_r\tilde{\phi}^k dr \lesssim \sum_{k=1}^{K} m_N^4 \ln N \|\tilde{\phi}^k\|^2 \\
&\lesssim \sum_{k=1}^{K} m_N^4 \ln N \|\phi\|_{\Lambda_k}^2 \lesssim m_N^4 \ln N \|\phi\|^2.
\end{aligned}$$

Then the desired estimate follows. ■

Lemma 3.3. *Suppose $g(t) \in H^1[0,T]$, and let $u_{\mathscr{N}}$ be a solution of the SEVV approximation* (2.10). *If $u_{\mathscr{N}}$ is uniformly bounded, i.e.,*

$$M := \max_{0 \le t \le T} \|u_{\mathscr{N}}(x,t)\|_{L^\infty(\Lambda)} < \infty, \tag{3.9}$$

then there exists a constant (depending on M, T, and g) such that the following estimate holds:

$$\varepsilon_N(\|\partial_t u_{\mathscr{N}}\|_{\Omega}^2 + \|\partial_x u_{\mathscr{N}}\|_{\Omega}^2) \le C. \tag{3.10}$$

Proof. Recall that ξ_j^k are the zeros of $\partial_x L_N^k(x), 1 \le j \le N-1$, so that the boundary operator $B(u_{\mathcal{N}})|_{\Lambda_1}$ vanishes at all but the inflow boundary point $x=-1$, where it involves the corresponding values of $L_N'(-1)=(-1)^{N+1}N(N+1)/2$. Consequently, the contribution of the boundary operator $B(u_{\mathcal{N}})|_{\Lambda_1}$ to (2.10) amounts to

$$(B(u_{\mathcal{N}})|_{\Lambda_1},\phi)_{\mathcal{N}} \equiv -2(-1)^N \mu(t)\phi(-1,t).$$

The role of $\mu(t)$ can be observed by setting $\phi \equiv 1$ in (2.10):

$$\partial_t(u_{\mathcal{N}}(t),1)+(\mathcal{I}_{\mathcal{N}} f(u_{\mathcal{N}}),1)-2f(u_{\mathcal{N}}(-1,t))=-2(-1)^N\mu(t). \tag{3.11}$$

This means that $\mu(t)$ measures the rate of change of total mass over the whole $[-1,1]$ interval. The above equality also leads to

$$|\mu(t)| \le \frac{1}{\sqrt{2}}\|\partial_t u_{\mathcal{N}}\|+2|f|_\infty, \quad \text{with } |f|_\infty \equiv \max_{|u|\le M}|f(u)|. \tag{3.12}$$

Further, let $\eta(t):=\int_0^t \mu(s)ds$. Then integrating (3.11) yields for $t \le T$,

$$|\eta(t)| \le \frac{1}{\sqrt{2}}\|u_{\mathcal{N}}(t)\|+\frac{1}{\sqrt{2}}\|u_{\mathcal{N}}(0)\|+2t|f|_\infty. \tag{3.13}$$

Let $F(u) \equiv \int^u wf'(w)dw$ be the entropy flux corresponding to the quadratic entropy, $U(u)=\frac{u^2}{2}$. Take $\phi=u_{\mathcal{N}}$ in (2.10). Then using (3.5) and Lemma 3.2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{\mathcal{N}}(t)\|_{\mathcal{N}}^2-2F(u_{\mathcal{N}}(-1,t))+\int_{-1}^1 F(u_{\mathcal{N}}(x,t))dx \\
&\quad+\Big[\varepsilon_N(\mathcal{Q}\partial_x u_{\mathcal{N}}(t),\partial_x u_{\mathcal{N}}(t))_{\mathcal{N}}+2(-1)^N\mu(t)u_{\mathcal{N}}(-1,t)\Big] \\
=&\ (\partial_x f(u_{\mathcal{N}}(t)),(1-x)u_{\mathcal{N}}(t))-((1-x)\partial_x\mathcal{I}_{\mathcal{N}} f(u_{\mathcal{N}}(t)),u_{\mathcal{N}}(t))_{\mathcal{N}} \\
\le&\ |((1-x)(\mathcal{I}-\mathcal{I}_{\mathcal{N}})f(u_{\mathcal{N}}(t)),\partial_x u_{\mathcal{N}}(t))| \\
&\quad+|((1-x)\partial_x\mathcal{I}_{\mathcal{N}} f(u_{\mathcal{N}}(t)),u_{\mathcal{N}}(t))-((1-x)\partial_x\mathcal{I}_{\mathcal{N}} f(u_{\mathcal{N}}(t)),u_{\mathcal{N}}(t))_{\mathcal{N}}| \\
\le&\ \frac{c_M}{N}\|\partial_x f(u_{\mathcal{N}}(t))\|\|\partial_x u_{\mathcal{N}}(t)\|+\frac{c_M}{N}\|\partial_x f(u_{\mathcal{N}}(t))\|(\|u_{\mathcal{N}}(t)\|+2\|\partial_x u_{\mathcal{N}}(t)\|) \\
\le&\ \frac{c_M}{N}(\frac{1}{2}M^2+\frac{7}{2}\|\partial_x u_{\mathcal{N}}(t)\|_{\mathcal{Q}}^2+m_N^4\ln N\|u_{\mathcal{N}}(t)\|^2),
\end{aligned}$$

where c_M stands for generic constants depending on the uniform bound M. Thus for any $t \le T$,

$$\begin{aligned}
&\|u_{\mathcal{N}}(t)\|_{\mathcal{N}}^2+2(\varepsilon_N-\frac{7c_M}{2N})\int_0^t\|\partial_x u_{\mathcal{N}}(s)\|_{\mathcal{Q}}^2ds \\
\le&\ c_0+2\int_0^t(2(-1)^{N+1}\mu(s)g(s)+4\max_{|u|\le M}|F(u)|+\frac{c_M}{N}(\frac{1}{2}M^2+cm_N^4\ln N\|u_{\mathcal{N}}(s)\|^2))ds.
\end{aligned}$$

By (3.13), we have

$$\left|\int_0^t \mu(s)g(s)ds\right| = \left|g(t)\eta(t) - g(0)\eta(0) - \int_0^t g'(s)\eta(s)ds\right|$$
$$\leq |g(t)|(\frac{1}{\sqrt{2}}\|u_{\mathcal{N}}(t)\| + c_M) + \int_0^t |g'(s)|(\frac{1}{\sqrt{2}}\|u_{\mathcal{N}}(t)\| + c_M)ds \lesssim \|g(t)\|^2_{H^1[0,t]}.$$

Combining the two estimates above gives

$$\|u_{\mathcal{N}}(t)\|^2_{\mathcal{N}} + \varepsilon_N \int_0^t \|\partial_x u_{\mathcal{N}}(s)\|^2_{\mathcal{Q}} ds \lesssim (\|g(t)\|^2_{H^1[0,t]} + t + 1).$$

The above estimate, together with Lemma 3.2, gives

$$\varepsilon_N \|\partial_x u_{\mathcal{N}}\|^2_\Omega \lesssim (\|g(t)\|^2_{H^1[0,T]} + T + 1). \tag{3.14}$$

Next, we set $\phi = \partial_t u_{\mathcal{N}}$ in the SEVV weak formulation (2.10). By (3.1) and (3.12), we have

$$\|\partial_t u_{\mathcal{N}}(t)\|^2 + \frac{\varepsilon_N}{2}\frac{d}{dt}\|\partial_x u_{\mathcal{N}}(t)\|^2_{\mathcal{Q}} \leq c_M \|\partial_x u_{\mathcal{N}}(t)\|^2 + \frac{1}{2}\|\partial_t u_{\mathcal{N}}(t)\|^2 + c_M|\frac{d}{dt}g(t)|^2.$$

Temporal integration of the above inequality followed by (3.14) yields

$$\frac{1}{2}\|\partial_t u_{\mathcal{N}}\|^2_\Omega \lesssim \|\partial_x u_{\mathcal{N}}(t)\|^2_\Omega + \|g\|^2_{H^1[0,T]} \lesssim \frac{1}{\varepsilon_N}(\|g\|^2_{H^1[0,T]} + T + 1), \quad \forall t \in [0,T].$$

Thus the proof is completed by combining (3.14) and the above inequality together. ■

Theorem 3.1. *Let $u_{\mathcal{N}}$ be a solution of SEVV problem* (2.10)*, with spectral viscosity parameters (ε_N, m_N) satisfying*

$$\varepsilon_N \sim N^{-\alpha}, \quad m_N \sim N^\beta, \quad \textit{with} \quad 0 < 4\beta < \alpha \leq 1. \tag{3.15}$$

Then $u_{\mathcal{N}}$ converges strongly to a weak solution of (2.3)*. Moreover, if the viscosity amplitude is set to be $\varepsilon_N \sim N^{-\alpha}$ with $\alpha < 1$, then $u_{\mathcal{N}}$ converges strongly to the unique entropy solution of* (2.3).

Proof. For arbitrary $\psi \in H^1_0(\Omega)$ and $\psi_{\mathcal{N}} \in H^1((0,T);V_{\mathcal{N}})$ with $\psi_N(-1,t) = 0$, we have the identity

$$(\partial_t u_{\mathcal{N}} + (1-x)\partial_x f(u_{\mathcal{N}}), \psi)_\Omega \equiv \sum_{j=1}^4 G_j(\psi), \tag{3.16}$$

where

$$\begin{aligned}
G_1(\psi) &= ((\partial_t u_{\mathcal{N}} + (1-x)\partial_x f(u_{\mathcal{N}}), \psi - \psi_{\mathcal{N}})_\Omega;\\
G_2(\psi) &= ((1-x)\partial_x f(u_{\mathcal{N}}) - (1-x)\partial_x \mathscr{I}_{\mathcal{N}} f(u_{\mathcal{N}}), \psi_{\mathcal{N}})_\Omega;\\
G_3(\psi) &= (\partial_t u_{\mathcal{N}} + (1-x)\partial_x \mathscr{I}_{\mathcal{N}} f(u_{\mathcal{N}}), \psi_{\mathcal{N}})_\Omega\\
&\quad -(\partial_t u_{\mathcal{N}} + (1-x)\partial_x \mathscr{I}_{\mathcal{N}} f(u_{\mathcal{N}}), \psi_{\mathcal{N}})_{\Omega,\mathcal{N}};\\
G_4(\psi) &= (\partial_t u_{\mathcal{N}} + (1-x)\partial_x \mathscr{I}_{\mathcal{N}} f(u_{\mathcal{N}}), \psi_{\mathcal{N}})_{\Omega,\mathcal{N}}.
\end{aligned}$$

We now estimate $G_j(\psi)$ term by term. From Lemma 3.3, we have

$$|G_1(\psi)| \lesssim \frac{1}{\sqrt{\varepsilon_N}}\|\psi - \psi_{\mathcal{N}}\|_\Omega.$$

According to (3.1) and (3.10)

$$\begin{aligned}
|G_2(\psi)| &= |((\mathscr{I} - \mathscr{I}_{\mathcal{N}})f(u_{\mathcal{N}}), \partial_x((1-x)\psi_{\mathcal{N}}))_\Omega| \\
&\lesssim \frac{1}{N}\|\partial_x f(u_{\mathcal{N}})\|_\Omega(\|\psi_{\mathcal{N}}\|_\Omega + 2\|\partial_x\psi_{\mathcal{N}}\|_\Omega) \\
&\lesssim \frac{1}{N\sqrt{\varepsilon_N}}(\|\psi_{\mathcal{N}}\|_\Omega + 2\|\partial_x\psi_{\mathcal{N}}\|_\Omega).
\end{aligned}$$

By virtue of (3.1), (3.5) and Lemma 3.2,

$$|G_3(\psi)| \lesssim \frac{1}{N}\|\partial_t u_{\mathcal{N}} + (1-x)\partial_x\mathscr{I}_{\mathcal{N}}f(u_{\mathcal{N}})\|_\Omega\|\partial_x\psi_{\mathcal{N}}\|_\Omega \lesssim \frac{1}{N\sqrt{\varepsilon_N}}\|\partial_x\psi_{\mathcal{N}}\|_\Omega.$$

We are left with the fourth expression, $G_4(\psi)$, which can be rewritten by using (2.10)

$$\begin{aligned}
G_4(\psi) &= (\partial_t u_{\mathcal{N}} + (1-x)\partial_x\mathscr{I}_{\mathcal{N}}f(u_{\mathcal{N}}), \psi_{\mathcal{N}})_{\Omega,\mathcal{N}} = -\varepsilon_N(\mathscr{Q}\partial_x u_{\mathcal{N}}, \partial_x\psi)_{\Omega,\mathcal{N}} \\
&= \varepsilon_N(\mathscr{R}\partial_x u_{\mathcal{N}}, \partial_x\psi_{\mathcal{N}})_\Omega - \varepsilon_N(\partial_x u_{\mathcal{N}}, \partial_x\psi_{\mathcal{N}})_\Omega = J_4(\psi) + J_5(\psi).
\end{aligned}$$

Using Lemma 3.2, we find

$$\begin{aligned}
|J_4(\psi)| &= |\varepsilon_N(\mathscr{R}\partial_x u_{\mathcal{N}}, \partial_x\psi_{\mathcal{N}})_\Omega| \\
&\lesssim \varepsilon_N\|\partial_x u_{\mathcal{N}}\|_{\mathscr{R}}\|\partial_x\psi_{\mathcal{N}}\|_\Omega \lesssim \varepsilon_N m_N^2\sqrt{\ln N}\|\partial_x\psi_{\mathcal{N}}\|_\Omega, \\
|J_5(\psi)| &= |-\varepsilon_N(\partial_x u_{\mathcal{N}}, \partial_x\psi_{\mathcal{N}})_\Omega| \\
&\lesssim \varepsilon_N\|\partial_x u_{\mathcal{N}}\|_\Omega\|\partial_x\psi_{\mathcal{N}}\|_\Omega \lesssim \sqrt{\varepsilon_N}\|\partial_x\psi_{\mathcal{N}}\|.
\end{aligned}$$

Now, for each $\psi \in H_0^1(\Omega)$ we assign $\psi_{\mathcal{N}} = \Pi_{\mathcal{N}}^1\psi$, and from Lemma 3.1 we have

$$\|\frac{\partial}{\partial x}\psi_{\mathcal{N}}\| + N\|\psi - \psi_{\mathcal{N}}\| \lesssim \|\frac{\partial}{\partial x}\psi\|, \quad \psi_{\mathcal{N}}(-1,t) = 0. \tag{3.17}$$

Using (3.17), we find that any assignment of such $\psi_{\mathcal{N}}$ gives us

$$\begin{aligned}
&|(\partial_t u_{\mathcal{N}} + (1-x)\partial_x f(u_{\mathcal{N}}), \psi)_\Omega| \lesssim \sum_{j=1}^{3}|G_j\psi| + \sum_{j=1}^{2}|J_j(\psi)| \\
&\lesssim (\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m_N^2\sqrt{\ln N} + \sqrt{\varepsilon}_N)\|\partial_x\psi\| + \frac{\|\psi\|_\Omega}{N\sqrt{\varepsilon_N}},
\end{aligned}$$

from which we can conclude that the solution of the SEVV approximation (2.10), parameterized according to (2.11) and (2.15), satisfies

$$\|\partial_t u_{\mathcal{N}} + (1-x)\partial_x f(u_{\mathcal{N}})\|_{H^{-1}(\Omega)} \to 0.$$

On the other side, the previous statements imply that for all $\phi \in H_0^1(\Omega)$,

$$
\begin{aligned}
&(\partial_t U(u_{\mathcal{N}}) + (1-x)\partial_x F(u_{\mathcal{N}}), \phi) = \sum_{j=1}^{4} G_j(U'(u_{\mathcal{N}})\phi) \\
\lesssim\ &(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m_N^2 \sqrt{\ln N} + \sqrt{\varepsilon_N}) \\
&(\|\partial_x(U'(u_{\mathcal{N}})\phi)\|_\Omega + \|(U'(u_{\mathcal{N}})\phi)\|_\Omega) \\
\lesssim\ &(\frac{1}{N\sqrt{\varepsilon_N}} + \varepsilon_N m_N^2 \sqrt{\ln N} + \sqrt{\varepsilon_N})(\|\partial_x u_{\mathcal{N}}\|_\Omega \|\phi\|_{L^\infty(\Omega)} \\
&+\|\partial_x \phi\|_\Omega \|U'(u_{\mathcal{N}})\|_{L^\infty(\Omega)} + \sqrt{2T}\|U'(u_{\mathcal{N}})\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}).
\end{aligned}
$$

Thus, the entropy production $\partial_t U(u_{\mathcal{N}}) + (1-x)\partial_x F(u_{\mathcal{N}})$ can be written as a sum of two terms which belong, respectively, to a compact subset of $H^{-1}(\Omega)$ and a bounded set of $L^1(\Omega)$, and hence by Murat's lemma [3], to a compact subset of $H^{-1}(\Omega)$. Using the div-curl lemma [16], it follows that the SEVV solution, $u_{\mathcal{N}}(x,t)$, converges strongly to a weak solution, $u(x,t)$, of (2.3).

It remains to show that for $\varepsilon_N \sim N^{-\alpha}, 0 < \alpha < 1$, $u(x,t)$ is indeed the unique entropy solution. This turns out to verify the entropy condition (2.4). In fact, $\forall \phi \in H_0^1(\Omega)$,

$$
\begin{aligned}
&\left|\sum_{j=1}^{3} G_j(U'(u_{\mathcal{N}})\phi)\right| \le \sum_{j=1}^{3} |G_j(U'(u_{\mathcal{N}})\phi)| \\
\lesssim\ &\frac{1}{N\sqrt{\varepsilon_N}}(4\parallel \partial_x(U'(u_{\mathcal{N}})\phi)_{\mathcal{N}} \parallel_\Omega + \|(U'(u_{\mathcal{N}})\phi)_{\mathcal{N}}\|_\Omega),
\end{aligned}
$$

where $(U'(u_{\mathcal{N}})\phi)_{\mathcal{N}} \in H^1((0,T); V_{\mathcal{N}})$ is an approximation of $U'(u_{\mathcal{N}})\phi$. Now let $(U'(u_{\mathcal{N}})\phi)_{\mathcal{N}} = \Pi_{\mathcal{N}}^1(U'(u_{\mathcal{N}})\phi)$. Then, by (3.17),

$$
\begin{aligned}
&\left|\sum_{j=1}^{3} G_j(U'(u_{\mathcal{N}})\phi)\right| \\
\lesssim\ &\frac{1}{N\sqrt{\varepsilon_N}}(4\|\partial_x u_{\mathcal{N}}\|_\Omega \|\phi\|_{L^\infty(\Omega)} + 4\|\partial_x \phi\|_\Omega \|u_{\mathcal{N}}\|_{L^\infty(\Omega)} + \sqrt{2T}\|\phi\|_{L^\infty(\Omega)}\|u_{\mathcal{N}}\|_{L^\infty(\Omega)}) \\
\lesssim\ &(4N^{\alpha-1} \parallel \phi \parallel_{L^\infty(\Omega)} + N^{\frac{\alpha}{2}-1}(4\|\partial_x \phi\|_\Omega + \sqrt{2T}\|\phi\|_{L^\infty(\Omega)}) \to 0,
\end{aligned}
$$

provided that $\varepsilon_N \sim N^{-\alpha}, 0 < \alpha < 1$.

By the choice of the SEVV parameters in (2.11) and (2.15), together with (3.8) and (3.9),

$$
|J_4(U'(u_{\mathcal{N}})\phi)| \lesssim \sqrt{\varepsilon_N} m_N^2 \sqrt{\ln N}\|\phi\|_\Omega \lesssim N^{2\beta - \frac{\alpha}{2}}\sqrt{\ln N}\|\phi\|_\Omega \to 0.
$$

It remains now to estimate $J_5(U'(u_{\mathcal{N}})\phi)$. This will be done by taking advantage of the properties of the projection $\Pi_{\mathcal{N}}^1$. Indeed, for any nonnegative test function, $\phi \ge 0$, we find, by using (3.3),

$$
\begin{aligned}
&J_5(U'(u_{\mathcal{N}})\phi) = -\varepsilon_N(\partial_x u_{\mathcal{N}}, \partial_x \Pi^1_{\mathcal{N}}(U'(u_{\mathcal{N}})\phi))_\Omega \\
&= -\varepsilon_N \sum_{k=1}^{K} (\partial_x u_{\mathcal{N}}, \partial_x \Pi^1_{\mathcal{N}}(U'(u_{\mathcal{N}})\phi))_{\Omega_k} \\
&= -\varepsilon_N \frac{2}{h} \sum_{k=1}^{K} (\partial_r \tilde{u}^k_N, \partial_r \tilde{\Pi}^1_N(U'(\tilde{u}^k_N)\tilde{\phi}^k))_\Omega \\
&= -\varepsilon_N \frac{2}{h} \sum_{k=1}^{K} (\partial_r \tilde{u}^k_N, \tilde{\Pi}_{N-1}\partial_r(U'(\tilde{u}^k_N)\tilde{\phi}^k))_\Omega \\
&= -\varepsilon_N \frac{2}{h} \sum_{k=1}^{K} (\partial_r \tilde{u}^k_N, U''(\tilde{u}^k_N)\tilde{\phi}^k \partial_r \tilde{u}^k_N)_\Omega - \varepsilon_N \frac{2}{h} \sum_{k=1}^{K} (\partial_r \tilde{u}^k_N, U'(\tilde{u}^k_N)\partial_r \tilde{\phi}^k)_\Omega \\
&\leq -\varepsilon_N \frac{2}{h} \sum_{k=1}^{K} (\partial_r \tilde{u}^k_N, U'(\tilde{u}^k_N)\partial_r \tilde{\phi}^k)_\Omega \lesssim -\varepsilon_N(\partial_x u_{\mathcal{N}}, U'(u_{\mathcal{N}})\partial_x \phi)_\Omega.
\end{aligned}
$$

It follows from Lemma 3.3 that

$$J_5(U'(u_{\mathcal{N}})\phi) \lesssim \sqrt{\varepsilon_N} \parallel \partial_x \phi \parallel_\Omega \lesssim N^{-\frac{\alpha}{2}} \|\partial_x \phi\|_\Omega \to 0.$$

We conclude that $u_{\mathcal{N}}$ satisfies that $\forall \phi \in H^1_0(\Omega)$ with $\phi \geq 0$,

$$(\partial_t U(u_{\mathcal{N}}) + (1-x)\partial_x F(u_{\mathcal{N}}),\ \phi) \leq 0, \quad \text{as} \ \ N \to \infty,$$

which implies the following entropy inequality in the sense of distribution

$$\partial_t U(u) + (1-x)\partial_x F(u) \leq 0.$$

That is, u, the limit of $u_{\mathcal{N}}$, satisfies the entropy condition (2.4). ■

4. Numerical results

We carry out in this section a number of numerical experiments and present some results to confirm our theoretical statements. For time discretization, we use the third-order Adams-Bashforth scheme to treat the nonlinear terms. In all calculations, we fix the time step $\Delta t = 10^{-4}$. All SEVV solutions are computed with SEVV parameters, $\varepsilon_N = 1/KN$ and $m_N = \sqrt[4]{N}$. To demonstrate that the SEVV method possesses the classical spectral accuracy for smooth solutions, we first give some numerical results for the linear problem:

$$
\begin{cases}
\partial_t v(y,t) + \partial_y v(y,t) = 0, & (y,t) \in (0,\infty) \times (0,T], \\
v(y,0) = \sin(\pi(1-2e^{-y})), & y \in (0,\infty), \\
v(0,t) = \sin(\pi(1-2e^{t})), & t \in (0,T].
\end{cases}
\tag{4.1}
$$

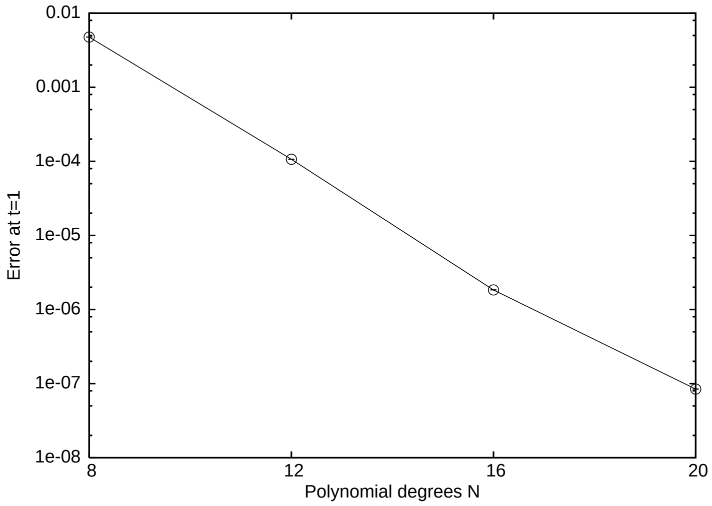

Fig. 4.1. Problem (4.1) at $t = 1$. Errors in the L^2-norm at time $T = 1.0$ for the SEVV solution of (4.2).

Applying the transformation (2.2) to (4.1) gives

$$\begin{cases} \partial_t u(x,t) + (1-x)\partial_x u(x,t) = 0, & (x,t) \in \Omega, \\ u(x,0) = \sin(\pi x), & x \in \Lambda, \\ u(-1,t) = \sin(\pi(1-2e^t)), & t \in (0,T], \end{cases} \tag{4.2}$$

with $u(x,t) = v(y(x),t)$. It is verified that the exact solution of (4.2) is $\sin(\pi((x-1)e^t + 1))$.

In this first test, the domain Λ is divided into 4 equi-elements, i.e., $K = 4$. Fig. 4.1 shows the errors in the L^2-norm at $T = 1.0$ by using $N = 8, 12, 16, 20$. As expected, the errors decay exponentially when N increases, indicating that the SEVV method preserves the spectral accuracy for smooth solutions.

We now consider the inviscid Bürgers equation

$$\begin{cases} \partial_t v(y,t) + \frac{1}{2}\partial_y v^2(y,t) = 0, & (y,t) \in (0,\infty) \times (0,T], \\ v(y,0) = -\sin(\pi(1-2e^{-y})), & y \in (0,\infty), \\ v(0,t) = 0, & t \in (0,T]. \end{cases} \tag{4.3}$$

The corresponding transformed problem reads

$$\begin{cases} \partial_t u(x,t) + \frac{1-x}{2}\partial_x u^2(x,t) = 0, & (x,t) \in \Omega, \\ u(x,0) = -\sin(\pi x), & x \in \Lambda, \\ u(-1,t) = 0, & t \in (0,T]. \end{cases} \tag{4.4}$$

Numerical test shows that the solution of problem (4.4) develops a discontinuity at time $t_s = 0.31$. Hence the standard spectral element method is inconsistent with the

Table 4.1: L^1−Errors of $u_{\mathcal{N}}$ and $v_{\mathcal{N}}$. $K=2$.

N	$\|u-u_{\mathcal{N}}\|_{L^1(\Lambda)}$	$\|v-v_{\mathcal{N}}\|_{L^1(0,\infty)}$
20	0.12146e-2	0.12179e-2
40	0.15089e-3	0.15096e-3
80	0.68235e-4	0.68268e-4

Table 4.2: Same as in Table 4.1, but with $N=20$.

K	$\|u-u_{\mathcal{N}}\|_{L^1(\Lambda)}$	$\|v-v_{\mathcal{N}}\|_{L^1(0,\infty)}$
2	0.12146e-2	0.12179e-2
4	0.11190e-2	0.12039e-2
8	0.81372e-3	0.82821e-3

entropy condition, and will fail to converge to the entropy solution once the discontinuity is formed.

Now we employ the SEVV method to compute the numerical solution $u_{\mathcal{N}}$ of (4.4), which will be mapped back to $v_{\mathcal{N}}$ to obtain a numerical solution to problem (4.3). All the numerical results presented were recorded at time $t=0.62$. Fig. 4.2(a) display the computed $v_{\mathcal{N}}$ with $N=40$, $K=2$, while in Fig. 4.2(b) the pointwise errors, $v_{\mathcal{N}}(y)-v(y)$, are plotted. It is shown that the numerical solution is in a good agreement with the exact one in the region where the solution is smooth. In the neighbor of the discontinuity, the numerical solution exhibits a visible oscillation. This oscillation is due to the well known Gibbs phenomenon, which is inevitable in any spectral approximation of discontinuous solutions. It has been proven [5] that the numerical solution of the standard SVV approximation converges exponentially to the L^2−projection of the exact solution, rather than the exact solution itself. As a result, the oscillation near the shock point reflects only the low-order convergence rate of the L^2−projection of v to v (when v is not smooth), rather than the lack of spectral accuracy of the SEVV solution $v_{\mathcal{N}}$. In fact, the spectral accuracy can be recovered by using a reconstruction post-procedure, see, e.g., [5] and the references therein.

Now we carry out the accuracy investigation for the SEVV method. Since the exact solution for problem (4.3) is unknown, we use numerical results with very fine resolution, say $K=20, N=20$, as the "exact" solution. In Tables 4.1 and 4.2, we list the errors of $u_{\mathcal{N}}$ and $v_{\mathcal{N}}$, quantified in terms of the $L^1(\Lambda)$-norm and the $L^1([0,\log(1/(1-\xi^K_{N-1}))])$-norm. Table 4.1 shows the errors for different N with $K=2$ fixed, while Table 4.2 shows the errors for different K with $N=20$ fixed. It is observed, as expected in a spectral-type method, that the convergence is better by increasing the order of polynomial degree with fixed element number, than by resorting to a further partitioning with fixed polynomial degree.

It is worthwhile to notice that, as compared to the single element spectral method, the advantage of the SEVV is obvious. One of the reasons is that with element partition technique, we can locate the elemental interfaces in the neighbor of the shock, such that there is a grid accumulation in the region where the solution exhibits large gradient. Doing

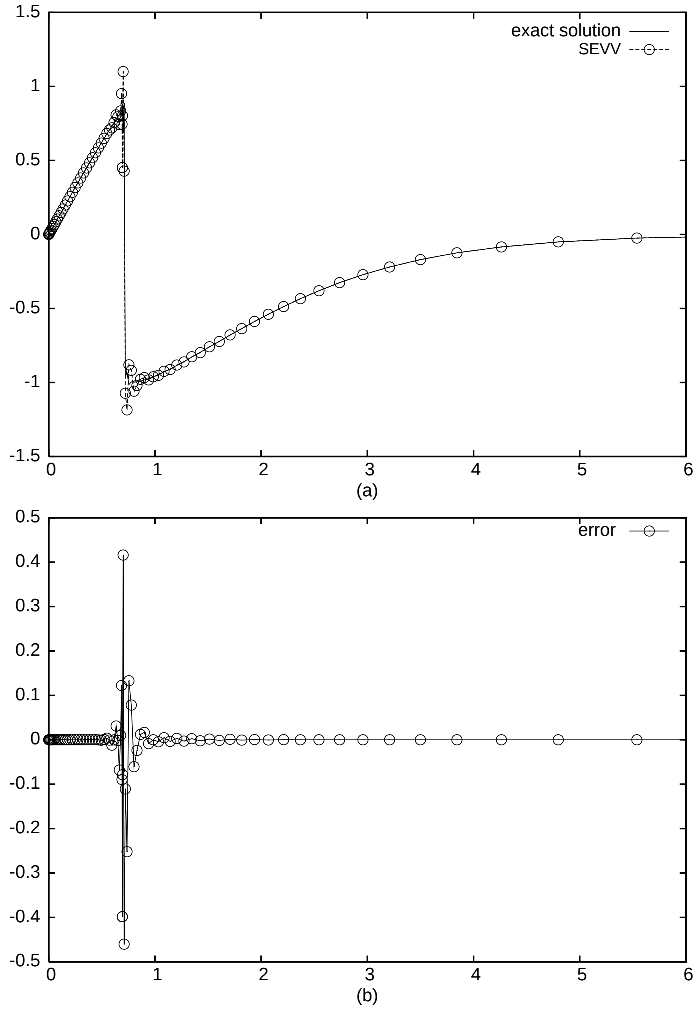

Fig. 4.2. Problem (4.3) at $t = 0.62$. (a) Comparison of the exact solution and numerical solution; (b) Pointwise errors of the numerical solution.

so allows improvement of the accuracy near the discontinuities.

Acknowledgment The research of the corresponding author was partially supported by National NSF of China under Grant 10531080, the Excellent Young Teachers Program by the Ministry of Education of China, and 973 High Performance Scientific Computation Research Program 2005CB321703.

References

[1] Bernardi C, Maday Y. Approximations Spectrales de Problèmes aux Limites Elliptiques. Springer, Berlin, 1992.

[2] Canuto C, Hussaini M Y, Quarteroni A, Zang T. Spectral Method in Fluid Dynamics. Springer-Verlag, Berlin, 1988.

[3] Chen G-Q, Du Q, Tadmor E. Spectral viscosity approximations to multidimensional scalar conservation laws. Math. Comput., 1993, 61: 629-643.
[4] Funaro D, Gottlieb D. A new method of imposing boundary conditions in pseudospectral approximation of hyperbolic equations. Math. Comput., 1998, 51: 599-613.
[5] Gottlieb D, Hesthaven J S. Spectral methods for hyperbolic problems. J. Comput. Appl. Math., 2001, 128: 83-131.
[6] Gottlieb D, Orszag S A. Numerical Analysis of Spectral Methods: Theory and Applications. SIAM-CBMS, Philadelphia, 1977.
[7] Guo B Y, Ma H P, Tadmor E. Spectral vanishing viscosity method for nonlinear conservation laws. SIAM J. Numer. Anal., 2001, 39(4): 1254-1268.
[8] Karamanos G S, Karniadakis G E. A spectral vanishing viscosity method for large-eddy simulations. J. Comput. Phys., 2000, 163: 22-50.
[9] Kaber S M. A legendre pesudospectral viscosity method. J. Comput. Phys., 1996, 128: 165-180.
[10] Maday Y, Tadmor E. Analysis of the spectral viscosity method for periodic conservation laws. SIAM J. Numer. Anal., 1989, 26: 854-870.
[11] Maday Y, Kaber S M, Tadmor E. Legendre pseudospectral viscosity method for nonlinear conservation laws. SIAM J. Numer. Anal., 1993, 30: 321-342.
[12] Noor A K, Oden J T. Surveys on Computational Mechanics. American Society of Mechanical Engineers, 1989.
[13] Pasquetti R, Xu C J. High-order algorithms for large-eddy simulation of incompressible flows. J. Sci. Comput., 2002, 17(1-3): 273-284.
[14] Quarteroni A, Valli A. Numerical Approximation of Partial Differential Equations. Springer-Verlag, Berlin, 1994.
[15] Spalart P R. Spectral method for external viscous flows. Contemp. Math., 1984, 28: 315-335.
[16] Tartar L. Compensated compactness and applications to partial differential equaitons. in Research Notes in Mathematics 39. Nonlinear Analysis and Mechanics, Geriot-Watt Symposium, ed. by Knopps J, 4, 1975, 136-211.
[17] Tadmor E. The convergence of spectral methods for nonlinear conservation laws. SIAM J. Numer. Anal., 1989, 26: 30-40.
[18] Xu C J, Pasquetti R. Stabilized spectral element computations of high Reynolds number incompressible flows. J. Comput. Phys., 2004, 196(2): 680-704.
[19] Zhuang Q Q, Xu C J. A spectral element/Laguerre coupled method to the elliptic Helmholtz problem on the half line. Numer. Math. J. Chinese Univ. (English Ser.), 2006, 15: 193-208.