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The Fundamental Solutions of the Space, Space-Time Riesz Fractional Partial Differential Equations with Periodic Conditions

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Abstract

In this paper, the space-time Riesz fractional partial differential equations with periodic conditions are considered. The equations are obtained from the integral partial differential equation by replacing the time derivative with a Caputo fractional derivative and the space derivative with Riesz potential. The fundamental solutions of the space Riesz fractional partial differential equation (SRFPDE) and the space-time Riesz fractional partial differential equation (STRFPDE) are discussed, respectively. Using methods of Fourier series expansion and Laplace transform, we derive the explicit expressions of the fundamental solutions for the SRFPDE and the STRFPDE, respectively.

Keywords: Fundamental solution; Riesz potential; Caputo derivative; Fourier series; Laplace transform.
Mathematics subject classification: 35C05, 35K30

1. Introduction

Fractional-order partial differential equations are generalizations of classical partial differential equations. Recently, a growing number of works from various fields of science and engineering, such as physics, finance, hydrology, thermodynamics, etc., deal with dynamical systems described by fractional-order equations [1–7]. This is because fractional-order derivatives and integrals provide a powerful instrument for the description of memory and hereditary properties of different substances. Therefore, many authors want to find the fundamental solutions of some fractional-order differential equations by different ways. A Cauchy problem for the Riemann-Liouville fractional differential operator was studied by Luchko et al. [8], who got the solution of the problem using a Mikusiński-type operational calculus. Mainardi [9] considered the fundamental solutions for the time fractional diffusion-wave equation using Laplace transform. Furthermore, Mainardi et al. [10] discussed the fundamental solution of the space-time fractional diffusion equation using the

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Fourier and Laplace integral transforms and Mittag-Leffler functions, in which the fundamental solution can only be expressed as convolution form of the Green function and the initial value function, and then computed difficultly. Elizarraraz et al. [11] studied the solution of homogeneous differential equations associated with a fractional differential operator related to Weyl's operator in a vector space of generalized exponential polynomials, whose methods are based on linear algebra constructions. Duan et al. [12] considered the solution of the mixed problem with time fractional-order derivative and the third kind homogeneous boundary condition using integral transform. Moreover, in [13] they also discussed the problem of time fractional diffusion-wave equations on finite interval and obtained variable separated solution using the Laplace transform and its inverse transform. Schneider and Wyss [14] considered the time fractional diffusion and wave equations and derived the corresponding Green functions in closed form for arbitrary space dimensions in terms of Fox functions.

Gorenflo et al. [15] used the similarity method and the method of Laplace transform to obtain the scale-invariant solution of the time-fractional diffusion-wave equation in terms of the Wright function. However, an explicit representation of the Green functions for the problem in a half-space is difficult to determine, except in the special cases $\alpha = 1$ (i.e., the first-order time derivative) with arbitrary n, or $n = 1$ with arbitrary α (i.e., the fractional-order time derivative). Huang and Liu [16] considered the time-fractional diffusion equations in a n-dimensional whole-space and half-space. They investigated the explicit relationships between the problems in whole-space with the corresponding problems in half-space by the Fourier-Laplace transform. Liu et al. [17] considered time fractional advection dispersion equation and derived the complete solution. The solution of time fractional diffusion-reaction equation was considered by Lu and Liu [18].

As mentioned above, many authors deal with the fractional-order differential equations using integral transforms and the order of derivative was restricted between 0 and 2. The solutions were expressed by convolution form of the Green function and initial value function, which cannot be computed easily. Moreover, sufficient solution regularity is required. In this paper, we consider the Riesz fractional partial differential equations (RFPDE) with periodic functions. The order of derivative can be extended to arbitrary order. Additionally, we only require piecewise smooth of functions about spacial variable and get the analysis solution in the form of series, which can be computed easily.

This paper is organized as follows. Section 2 presents basic definitions and propositions. In Sections 3 and 4, the fundamental solutions of the SRFPDE and the STRFPDE are derived, respectively. The explicit expressions of the fundamental solutions of the SRFPDE and the STRFPDE are obtained.

2. Basic definitions and propositions

In this section, we first present some basic definitions and propositions. We consider the RFPDE as follows:

$$\mathscr{D}_t^{\beta} u(x,t) = D_x^{\alpha} u(x,t), \quad x \in R, \quad t \in R^+, \tag{2.1}$$

where D_x^α is the Riesz differential operator defined by

$$
\begin{aligned}
&D_x^\alpha \varphi(x) = -\frac{1}{2\cos(\frac{\alpha\pi}{2})}\{I_+^{-\alpha}\varphi(x) + I_-^{-\alpha}\varphi(x)\},\\
&I_\pm^{-\alpha} = (\pm 1)^m \frac{d^m}{dx^m} I_\pm^{m-\alpha} \qquad (m-1<\alpha<m), \\
&\begin{cases} (I_+^\alpha \varphi)(x) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{-\infty}^{x} (x-\xi)^{\alpha-1}\varphi(\xi)d\xi, \\ (I_-^\alpha \varphi)(x) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{x}^{+\infty} (\xi-x)^{\alpha-1}\varphi(\xi)d\xi, \end{cases} \qquad x\in R,
\end{aligned}
\tag{2.2}
$$

and $\mathscr{D}_t^\beta$ the Caputo fractional derivative [2,3] defined by

$$
\mathscr{D}_t^\beta \varphi(t) = \begin{cases} \frac{1}{\Gamma(k-\beta)}\int_0^t \frac{\varphi^{(k)}(\tau)d\tau}{(t-\tau)^{\beta+1-k}}, & k-1<\beta<k, \\ \frac{d^k}{dt^k}\varphi(t), \quad \beta = k. \end{cases}
\tag{2.3}
$$

The Laplace transform formula for the Caputo time-fractional derivative of order $\beta(k-1<\beta<k, k\in N)$ is

$$
\mathscr{L}[\mathscr{D}_t^\beta f(t);s] = s^\beta \tilde{f}(s) - \sum_{j=0}^{k-1} s^{\beta-1-j} f^{(j)}(0^+)
\tag{2.4}
$$

(see [3], p.106).

We denote the function space of the periodic, continuous and piecewise smooth functions by Ω and always suppose that the unknown function u belongs to Ω with respect to the spacial variable x and has k-order smooth derivative with respect to the time variable t in this paper. Without loss of generality, we can also let the period of all functions be 2π. In these cases, we derive the explicit expressions of the solutions for SRFPDE and STRPDE by using methods of Fourier series expansion and Laplace transform, which can be computed easily.

Proposition 2.1. *For $a\in(0,1)$ and $\kappa\in R$, we have*

$$
\begin{aligned}
&(i)\ \int_0^\infty \frac{e^{i\kappa r}}{r^a}dr = (-i\kappa)^{a-1}\Gamma(1-a),\\
&(ii)\ \int_0^\infty \frac{e^{-i\kappa r}}{r^a}dr = (i\kappa)^{a-1}\Gamma(1-a).
\end{aligned}
$$

Proof. (i) We first consider the case $\kappa>0$. Making a closed contour $C=[\varepsilon,R]+\Gamma_R+[iR,i\varepsilon]+\Gamma_\varepsilon^-$ (see Fig. 1), then the function $\frac{e^{i\kappa r}}{r^a}$ is analytic in the inner domain bounded by contour C. So by the Cauchy's residue theorem we have

$$
\int_C \frac{e^{i\kappa r}}{r^a}dr = 0.
$$

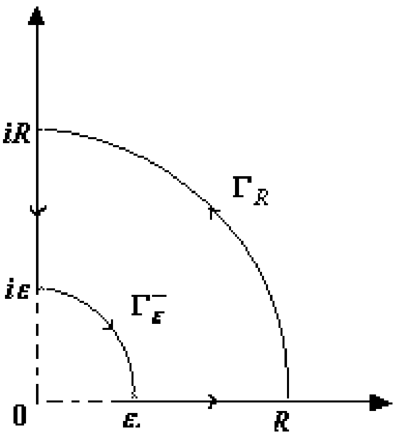

Fig. 2.1. Integral curve C.

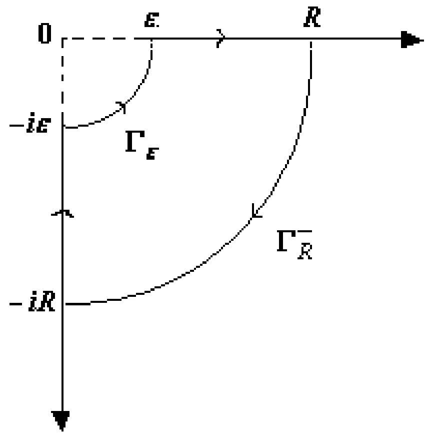

Fig. 2.2. Integral curve L.

Moreover,

$$\lim_{R\to+\infty} r^{-a}=0 \Rightarrow \lim_{R\to+\infty}\int_{\Gamma_R}\frac{e^{i\kappa r}}{r^a}dr=0 \quad (\text{Jordan's Lemma})$$

(see [19], P. 262) and

$$\lim_{\varepsilon\to0} r\cdot r^{-a}e^{i\kappa r}=0 \Rightarrow \lim_{\varepsilon\to0}\int_{\Gamma_\varepsilon^-}\frac{e^{i\kappa r}}{r^a}dr=0.$$

We have

$$\lim_{\varepsilon\to0,R\to+\infty}\Big(\int_\varepsilon^R+\int_{iR}^{i\varepsilon}\Big)\frac{e^{i\kappa r}}{r^a}dr=0,$$

which gives

$$\int_0^\infty\frac{e^{i\kappa r}}{r^a}dr=\int_0^{i\infty}\frac{e^{i\kappa r}}{r^a}dr.$$

Substituting variable $i\kappa r=-y$ in the above formula, then we obtain

$$\int_0^{i\infty}\frac{e^{i\kappa r}}{r^a}dr=\int_0^\infty\Big(\frac{y}{-i\kappa}\Big)^{-a}e^{-y}\frac{dy}{-i\kappa}=(-i\kappa)^{a-1}\int_0^\infty e^{-y}y^{-a}dy=(-i\kappa)^{a-1}\Gamma(1-a),$$

which gives

$$\int_0^\infty\frac{e^{i\kappa r}}{r^a}dr=(-i\kappa)^{a-1}\Gamma(1-a)\ (\kappa>0).$$

As for $\kappa<0$, similar to the case $\kappa>0$, making a closed contour $L=[\varepsilon,R]+\Gamma_R^-+[-iR,-i\varepsilon]+\Gamma_\varepsilon$(see Fig. 2), then the function $\frac{e^{i\kappa r}}{r^a}$ is analytic in the inner domain bounded by contour L. So by the Cauchy's residue theorem we have

$$\int_L \frac{e^{i\kappa r}}{r^a} dr = 0.$$

Moreover,

$$\lim_{R\to+\infty} r^{-a} = 0 \Rightarrow \lim_{R\to+\infty} \int_{\Gamma_R^-} \frac{e^{i\kappa r}}{r^a} dr = 0,$$

$$\lim_{\varepsilon\to 0} r \cdot r^{-a} e^{i\kappa r} = 0 \Rightarrow \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \frac{e^{i\kappa r}}{r^a} dr = 0.$$

We have

$$\lim_{\varepsilon\to 0, R\to+\infty} \left(\int_\varepsilon^R + \int_{-iR}^{-i\varepsilon}\right) \frac{e^{i\kappa r}}{r^a} dr = 0,$$

or equivalently,

$$\int_0^\infty \frac{e^{i\kappa r}}{r^a} dr = \int_0^{-i\infty} \frac{e^{i\kappa r}}{r^a} dr.$$

Substituting variable $i\kappa r = -y$ in the above formula gives

$$\begin{aligned}\int_0^{-i\infty} \frac{e^{i\kappa r}}{r^a} dr &= \int_0^\infty \left(\frac{y}{-i\kappa}\right)^{-a} e^{-y} \frac{dy}{-i\kappa} \\ &= (-i\kappa)^{a-1} \int_0^\infty e^{-y} y^{-a} dy = (-i\kappa)^{a-1} \Gamma(1-a),\end{aligned}$$

which gives

$$\int_0^\infty \frac{e^{i\kappa r}}{r^a} dr = (-i\kappa)^{a-1} \Gamma(1-a)\ (\kappa < 0).$$

Consequently,

$$\int_0^\infty \frac{e^{i\kappa r}}{r^a} dr = (-i\kappa)^{a-1} \Gamma(1-a)\ (\kappa \in R).$$

(ii) Similar to the proof of (i), for $\kappa > 0$ we have

$$\begin{aligned}\int_0^\infty \frac{e^{-i\kappa r}}{r^a} dr &= \int_0^{-i\infty} \frac{e^{-i\kappa r}}{r^a} dr = \int_0^\infty \left(\frac{y}{i\kappa}\right)^{-a} e^{-y} \frac{dy}{i\kappa} \\ &= (i\kappa)^{a-1} \int_0^\infty e^{-y} y^{-a} dy = (i\kappa)^{a-1} \Gamma(1-a).\end{aligned}$$

For $\kappa < 0$, we have

$$\int_0^\infty \frac{e^{-i\kappa r}}{r^a}dr = \int_0^{i\infty} \frac{e^{-i\kappa r}}{r^a}dr = \int_0^\infty (\frac{y}{i\kappa})^{-a}e^{-y}\frac{dy}{i\kappa}$$
$$= (i\kappa)^{a-1}\int_0^\infty e^{-y}y^{-a}dy = (i\kappa)^{a-1}\Gamma(1-a).$$

Therefore,

$$\int_0^\infty \frac{e^{-i\kappa r}}{r^a}dr = (i\kappa)^{a-1}\Gamma(1-a),\ \kappa \in R. \quad \blacksquare$$

3. The fundamental solution of the SRFPDE

In the section we always assume $\beta = k$ in the RFPDE (2.1), namely,

$$\frac{\partial^k u(x,t)}{\partial t^k} = D_x^\alpha u(x,t), \quad x \in R, \quad t \in R^+, \quad k \in N, \quad m-1 < \alpha < m, \tag{3.1a}$$
$$\frac{\partial^{j-1} u(x,0)}{\partial t^{j-1}} = f_j(x), \quad f_j(x) \in \Omega, \quad j = 1,2,\cdots,k. \tag{3.1b}$$

Firstly, we solve the SRFPDE (3.1) with $k = 1$. Since $u(\cdot,t) \in \Omega$, we can suppose

$$u(x,t) = \sum_{n=-\infty}^{+\infty} d_n(t)e^{inx}. \tag{3.2}$$

By Proposition 2.1, we have

$$\begin{aligned}
I_+^{-\alpha}u(x,t) &= \frac{d^m}{dx^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^x \frac{u(\xi,t)}{(x-\xi)^{1+\alpha-m}}d\xi\right) \\
&= \frac{d^m}{dx^m}\left(\frac{1}{\Gamma(m-\alpha)}\sum_{n=-\infty}^{+\infty} d_n(t)\int_{-\infty}^x \frac{e^{in\xi}}{(x-\xi)^{1+\alpha-m}}d\xi\right) \\
&= \sum_{n=-\infty}^{+\infty} \frac{d_n(t)}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\left(e^{inx}\int_0^{+\infty} \frac{e^{-inr}}{r^{1+\alpha-m}}dr\right) \\
&= \sum_{n=-\infty}^{+\infty} d_n(t)(in)^\alpha e^{inx};
\end{aligned}$$

$$
\begin{aligned}
I_{-}^{-\alpha}u(x,t) &= (-1)^m \frac{d^m}{dx^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_x^{+\infty}\frac{u(\xi,t)}{(\xi-x)^{1+\alpha-m}}d\xi\right)\\
&= (-1)^m \frac{d^m}{dx^m}\left(\frac{1}{\Gamma(m-\alpha)}\sum_{n=-\infty}^{+\infty} d_n(t)\int_x^{+\infty}\frac{e^{in\xi}}{(\xi-x)^{1+\alpha-m}}d\xi\right)\\
&= (-1)^m \sum_{n=-\infty}^{+\infty}\frac{d_n(t)}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\left(e^{inx}\int_0^{+\infty}\frac{e^{inr}}{r^{1+\alpha-m}}dr\right)\\
&= \sum_{n=-\infty}^{+\infty} d_n(t)(-in)^\alpha e^{inx}.
\end{aligned}
$$

Furthermore,

$$\frac{\partial u}{\partial t} = \sum_{n=-\infty}^{+\infty} d_n^{'}(t)e^{inx}.$$

Let

$$d_n^{'}(t) = -\frac{1}{2\cos(\frac{\alpha\pi}{2})}[(in)^\alpha + (-in)^\alpha]d_n(t).$$

Taking the main value branch gives

$$d_n^{'}(t) = -\mid n\mid^\alpha d_n(t),$$

which is a ordinary differential equation. Solving this equation we can get

$$d_n(t) = C_n exp(-\mid n\mid^\alpha t).$$

To obtain C_n we must use the initial value condition in problem (3.1). Since $f_1(x)\in\Omega$, it can be expanded into a fourier series

$$f_1(x) = \sum_{n=-\infty}^{+\infty} f_{1,n}(0)e^{inx}, \tag{3.3}$$

where the coefficients

$$f_{1,n}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x)e^{-inx}dx, \tag{3.4}$$

then we have $C_n = f_{1,n}(0)$.

From the above discussions we get the following theorem.

Theorem 3.1. *The fundamental solution of the SRFPDE* (3.1) *with* $k=1$ *in the function space* Ω *is*

$$u(x,t) = \sum_{n=-\infty}^{+\infty} f_{1,n}(0)\exp(-\mid n\mid^\alpha t)e^{inx}, \tag{3.5}$$

where the $f_{1,n}(0)$ *is same as* (3.4).

Similarly, we can obtain

Theorem 3.2. *For $k=2$, the fundamental solution of the SRFPDE* (3.1) *can be expressed*

$$u(x,t) = \sum_{n=-\infty}^{+\infty} f_{1,n}(0)\cos(|n|^{\frac{\alpha}{2}} t)e^{inx} + \sum_{n=-\infty,n\neq 0}^{+\infty} f_{2,n}(0)|n|^{-\frac{\alpha}{2}}\sin(|n|^{\frac{\alpha}{2}} t)e^{inx}, \tag{3.6}$$

in the function space Ω, where the coefficients $f_{1,n}(0)$ and $f_{2,n}(0)$ are the coefficients of Fourier series expansion of $f_1(x)$ and $f_2(x)$, respectively.

Proof. Similar to the above discussions, let

$$d_n''(t) = -|n|^{\alpha} d_n(t).$$

Then

$$d_n(t) = A_n \cos(|n|^{\frac{\alpha}{2}} t) + B_n \sin(|n|^{\frac{\alpha}{2}} t).$$

We expand the functions $f_1(x)$ and $f_2(x)$ as follows

$$f_j(x) = \sum_{n=-\infty}^{+\infty} f_{j,n}(0)e^{inx} \quad (j=1,2), \tag{3.7}$$

where the coefficients

$$f_{j,n}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_j(x)e^{-inx}dx \quad (j=1,2). \tag{3.8}$$

According to the initial value conditions of the problem (3.1), we get $A_n = f_{1,n}(0), B_n = |n|^{-\frac{\alpha}{2}} f_{2,n}(0)(n\neq 0)$ and $B_0 = 0$. Therefore the stated result of Theorem 3.2 is valid. ∎

Remark 3.1. *According to the above discussions, the method of Fourier series expansion adopts to arbitrary integer order k times derivative for problem* (3.1). *In the case, the corresponding $d_n(t)$ are of the form*

$$d_n(t) = \sum_{j=1}^{k} t^{j-1} f_{j,n}(0)E_{k,k-j}(-|n|^{\alpha} t^k).$$

4. The fundamental solution of STRFPDE

In the section we extend the time integer-order to time fractional-order. Assume $k-1 < \beta < k, m-1 < \alpha < m (k,m \in Z^+)$ in the RFPDE (2.1). Then the problem (2.1) is a STRFPDE, i.e.,

$$\mathscr{D}_t^{\beta} u(x,t) = D_x^{\alpha} u(x,t), \quad x\in R, \quad t\in R^+, \tag{4.1a}$$

$$\frac{\partial^{j-1}u(x,0)}{\partial t^{j-1}} = f_j(x), \quad f_j(x)\in\Omega, \quad j=1,2,\cdots,k. \tag{4.1b}$$

Firstly, we consider the problem (4.1) with $k = 1$. The Laplace transform of the first formula of the problem (4.1) leads to

$$s^{\beta}\tilde{u}(x,s)-s^{\beta-1}u(x,0)=D_x^{\alpha}\tilde{u}(x,s). \tag{4.2}$$

Due to the initial value condition and the result of $D_x^{\alpha}u(x,t)$ given in Section 3, we have

$$\tilde{u}(x,s)=\sum_{n=-\infty}^{+\infty}\tilde{d}_n(s)e^{inx},$$

$$u(x,0)=f_1(x)=\sum_{n=-\infty}^{+\infty}f_{1,n}(0)e^{inx}.$$

Consequently,

$$\sum_{n=-\infty}^{+\infty}s^{\beta}\tilde{d}_n(s)e^{inx}-\sum_{n=-\infty}^{+\infty}s^{\beta-1}f_{1,n}(0)e^{inx}=\sum_{n=-\infty}^{+\infty}(-\mid n\mid^{\alpha})\tilde{d}_n(s)e^{inx}. \tag{4.3}$$

Therefore,

$$\tilde{d}_n(s)=\frac{s^{\beta-1}f_{1,n}(0)}{s^{\beta}+\mid n\mid^{\alpha}}.$$

Applying the inverse Laplace transform to the above formula

$$d_n(t)=f_{1,n}(0)E_{\beta,1}(-\mid n\mid^{\alpha}t^{\beta}), \tag{4.4}$$

where $E_{\beta,1}$ is the Mittag-Leffler type function with two-parameter defined by

$$E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)} \tag{4.5}$$

whose k-th derivative is given by

$$E_{\alpha,\beta}^{(k)}(z)=\frac{d^k}{dz^k}E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{(j+k)!z^j}{j!\Gamma(\alpha j+\alpha k+\beta)}$$

(see [2,3]). Eq. (4.4) is deduced according to the following formula

$$\int_0^{\infty}e^{-st}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})dt=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}},\quad (Re(s)>\mid a\mid^{1/\alpha}), \tag{4.6}$$

(see [3], p.21).

So we can have the following result.

Theorem 4.1. *For $k=1$ and $m-1<\alpha<m$, the fundamental solution of the initial value problem of the STRFPDE* (4.1) *in the function space Ω is*

$$u(x,t)=\sum_{n=-\infty}^{+\infty} f_{1,n}(0)E_{\beta,1}(-\mid n\mid^{\alpha} t^{\beta})e^{inx}, \tag{4.7}$$

where the $f_{1,n}(0)$ is given by (3.4).

Remark 4.1. *According to the definition of the Mittag-Leffler type function* (4.5) *we know*

$$E_{1,1}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(j+1)}=\sum_{j=0}^{\infty}\frac{z^j}{j!}=e^z,$$

which gives $E_{1,1}(-\mid n\mid^{\alpha} t^{\beta})=\exp(-\mid n\mid^{\alpha} t^{\beta})$. So if we let $\beta=1$ in (4.7) *then the solution* (4.7) *is consistent with the solution* (3.5).

Similarly, we obtain

Theorem 4.2. *For $k=2$, the fundamental solution of the initial value problem of the STRF-PDE* (4.1) *in the function space Ω can be expressed as*

$$\begin{aligned} u(x,t) = & \sum_{n=-\infty}^{+\infty} f_{1,n}(0)E_{\beta,1}(-\mid n\mid^{\alpha} t^{\beta})e^{inx} \\ & +\sum_{n=-\infty}^{+\infty} f_{2,n}(0)tE_{\beta,2}(-\mid n\mid^{\alpha} t^{\beta})e^{inx}, \end{aligned} \tag{4.8}$$

where the coefficients $f_{j,n}(0)$ $(j=1,2)$ are the coefficients of the Fourier series expansion of $f_j(x)$ (see (3.8)).

Proof. The Laplace transform of the first formula of problem (4.1) leads to

$$s^{\beta}\tilde{u}(x,s)-s^{\beta-1}u(x,0)-s^{\beta-2}u_t^{'}(x,0)=D_x^{\alpha}\tilde{u}(x,s). \tag{4.9}$$

Let $u(x,t)=\sum_{n=-\infty}^{+\infty} d_n(t)e^{inx}$. Using the initial conditions and the result of $D_x^{\alpha}u(x,t)$ in Section 3 we obtain

$$\begin{aligned} & \sum_{n=-\infty}^{+\infty} s^{\beta}\tilde{d}_n(s)e^{inx}-\sum_{n=-\infty}^{+\infty} s^{\beta-1}f_{1,n}(0)e^{inx}-\sum_{n=-\infty}^{+\infty} s^{\beta-2}f_{2,n}(0)e^{inx} \\ = & \sum_{n=-\infty}^{+\infty}(-\mid n\mid^{\alpha})\tilde{d}_n(s)e^{inx}. \end{aligned} \tag{4.10}$$

Therefore,

$$\tilde{d}_n(s)=\frac{s^{\beta-1}f_{1,n}(0)}{s^{\beta}+\mid n\mid^{\alpha}}+\frac{s^{\beta-2}f_{2,n}(0)}{s^{\beta}+\mid n\mid^{\alpha}}.$$

Applying the inverse Laplace transform to the above formula

$$d_n(t) = f_{1,n}(0)E_{\beta,1}(-\mid n\mid^{\alpha} t^{\beta}) + tf_{2,n}(0)E_{\beta,2}(-\mid n\mid^{\alpha} t^{\beta}), \tag{4.11}$$

leads to the desired result of Theorem 4.2. ■

Remark 4.2. *According to the definition of the Mittag-Leffler type function* (4.5)*, we know*

$$E_{2,1}(-\mid n\mid^{\alpha} t^2) = \sum_{j=0}^{\infty} \frac{(-\mid n\mid^{\alpha} t^2)^j}{\Gamma(2j+1)} = \sum_{j=0}^{\infty}(-1)^j \frac{(\mid n\mid^{\alpha/2} t)^{2j}}{(2j)!} = \cos(\mid n\mid^{\alpha/2} t).$$

If $n \neq 0$,

$$tE_{2,2}(-\mid n\mid^{\alpha} t^2) = \mid n\mid^{-\alpha/2} \sum_{j=0}^{\infty}(-1)^j \frac{(\mid n\mid^{\alpha/2} t)^{2j+1}}{(2j+)!} = \mid n\mid^{-\alpha/2} \sin(\mid n\mid^{\alpha/2} t);$$

and if $n = 0$*, then* $tE_{2,2}(-\mid n\mid^{\alpha} t^2) = 0$*. Let* $\beta = 2$ *in* (4.8)*. Then the solution* (4.8) *is consistent with the solution* (3.6)*.*

Finally, we point out that the method of Fourier series expansion and the Laplace transform can be extended to arbitrary fractional-order $k-1 < \beta < k$ $(k \in N)$ time-derivative for the problem (4.1). In this case, the corresponding $d_n(t)$ are of the form

$$d_n(t) = \sum_{j=1}^{k} t^{j-1} f_{j,n}(0)E_{\beta,j}(-\mid n\mid^{\alpha} t^{\beta}).$$

5. Conclusions

In the paper, we present the explicit expression of the fundamental solutions of partial differential equations with β $(k-1 < \beta \leq k)$ order time-derivative and α $(m-1 < \alpha < m)$ order space Riesz derivative under the periodic condition. The explicit expressions can be computed easily.

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