Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2011.m12si06

Vol. **5**, No. 1, pp. 99-109
February 2012

Multigrid Methods for Elliptic Optimal Control Problems with Pointwise State Constraints

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Received 08 December 2010; Accepted (in revised version) 05 May 2011

Available 21 December 2011

Abstract. An elliptic optimal control problem with constraints on the state variable is considered. The Lavrentiev-type regularization is used to treat the constraints on the state variable. To solve the problem numerically, the multigrid for optimization (MGOPT) technique and the collective smoothing multigrid (CSMG) are implemented. Numerical results are reported to illustrate and compare the efficiency of both multigrid strategies.

AMS subject classifications: 49K20, 65N06, 65N55

Key words: Elliptic optimal control problems, Lavrentiev regularization, multigrid methods, pointwise state constraints.

1. Introduction

Different numerical techniques solve elliptic optimal control problems efficiently. Multigrid is considered as one of the most efficient tools for solving elliptic type problems. As evidence, previous results show that multigrid solves optimal control problems with optimal computational complexity. See for example the application of multigrid to unconstrained optimization problems [13, 17], to optimal control problems [4, 5, 7, 12] and to inverse problems [18, 19]. The purpose of this paper is to formulate a fast numerical technique for solving state-constrained optimal control problems. These type of problems are very important in different applications of optimal control of partial differential equations. We focus on two representatives of multigrid methods for solving state-constrained optimal control problems: the multigrid for optimization (MGOPT) technique and the collective smoothing multigrid (CSMG). The CSMG scheme solves optimal control problems by solving the corresponding optimality system. This approach needs to customize the collective smoothing strategy for each individual problem. Nevertheless, an appropriate design of the CSMG components results in a robust algorithm with typical multigrid efficiency [6].

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On the other hand, the MGOPT method was first introduced in [13,17]. In this scheme the multigrid solution process represents the outer loop where the control function is considered as the unique dependent variable. The inner loop consists of a classical one-grid optimization scheme. We consider the application of these multigrid methods for solving state-constrained elliptic optimal control problems. This work is an extension of [22], which is the case of control-constrained elliptic optimal control problems. For the state-constrained case, there are several well-known techniques available. Take for example the Lavrentiev-type regularization and the Moreau-Yosida regularization, together with numerical solvers like the interior point methods and the active set strategies [1, 2, 10, 14–16, 20, 21]. For optimal control problems with state constraints, the corresponding Lagrange multipliers are in general not contained in a function space but only given as measures [3, 8, 16]. In order to overcome this difficulty, a Lavrentiev-type regularization for the solution of state-constrained optimal control problems is used. The Lagrange multipliers associated with the regularized state constraints can be assumed to be functions in L^2 [14, 16, 20]. This type of regularization procedure approximates the state constraints by mixed control-state constraints. The solution of the regularized problem converges to the solution of the original problem for regularization parameters tending to zero [14, 16, 21].

In the next sections, state-constrained optimal control problems are presented together with the discretization scheme and a detailed description of appropriate smoothing and optimization algorithms. In Section 4, the multigrid scheme is formulated. Numerical experiments follow to demonstrate the ability of multigrid in solving state-constrained optimal control problems and a section of conclusion completes this paper.

2. Constrained optimal control problems

In this section, we discuss state-constrained elliptic optimal control problems. The corresponding optimality system is presented and the multigrid solution procedure is given in the next section.

A state-constrained optimal control problem governed by a partial differential equation can be formulated as follows:

$$
\begin{aligned}
\min_{u\in L^2(\Omega)} J(y,u) := \frac{1}{2}\|y-z\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},& \\
-\Delta y + F(y) + u = f \quad \text{in } \Omega,& \\
y = 0 \quad \text{on } \partial\Omega,& \\
\underline{y} \le y \le \overline{y} \quad \text{on } \partial\Omega,&
\end{aligned}
\tag{2.1}
$$

where $\nu > 0$ is the weight of the cost of the control, $z \in L^2(\Omega)$ is the target function, $f \in L^2(\Omega)$ and the function F is twice continuously differentiable and monotonically increasing [9, 15]. The bounds $\underline{y}$ and $\overline{y}$ are fixed functions in $L^2(\Omega)$, where $\underline{y} \le \overline{y}$ almost everywhere in Ω. The existence and uniqueness of a solution to a state-constrained optimal control problem depend on the nonlinearity and on the given constraints. See for example [9, 15, 20].

Different numerical approaches can be used in solving state-constrained optimal control problems. In this paper, we consider the Lavrentiev-type regularization. The pointwise state constraints $\underline{y}(x) \leq y(x) \leq \overline{y}(x)$ can be approximated by the following mixed control-state constraints

$$\underline{y}(x) \leq y(x) - \lambda u(x) \leq \overline{y}(x) \quad \text{a.e. in } \Omega,$$

where $\lambda > 0$ is a small parameter. With this approximation, the regularized state-constrained optimal control problem is written as

$$\begin{aligned}
&\min_{u\in L^2(\Omega)} J(y,u) := \frac{1}{2}\|y-z\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)},\\
&-\Delta y + F(y) + u = f,\\
&y = 0,\\
&\underline{y}(x) \leq y(x) - \lambda u(x) \leq \overline{y}(x).
\end{aligned}$$

For sufficiently small λ, satisfactory numerical results are obtained in Section 5. The corresponding solution of the regularized problem converges to the solution of the original problem as the regularization parameter λ tends to zero [14, 16, 21].

By introducing an auxiliary variable $v = y - \lambda u$, the control variable u can be expressed in terms of v. Then the regularized state-constrained optimal control problem becomes

$$\begin{aligned}
&\min_{v\in L^2(\Omega)} J(y,v) := \frac{1}{2}\|y-z\|^2_{L^2(\Omega)} + \frac{\nu}{2\lambda^2}\|y-v\|^2_{L^2(\Omega)},\\
&-\Delta y + F(y) + \frac{y}{\lambda} - \frac{v}{\lambda} = f,\\
&y = 0,\\
&\underline{y} \leq v \leq \overline{y}.
\end{aligned}$$

This obtained system is now similar to an optimal control problem having a control-constrained structure with respect to the variable v. To derive the optimality system which is a characterization of the solution to the given optimization problem (2.1), we define the Lagrange functional

$$\mathscr{L}(y,v,p) = J(y,v) + \left\langle -\Delta y + F(y) + \frac{y}{\lambda} - \frac{v}{\lambda} - f, p \right\rangle,$$

where p is the Lagrange multiplier, which is assumed to be a function in L^2 [14,16,20]. We derive the first-order necessary conditions for a minimum by taking the Fréchet derivatives of $\mathscr{L}$ with respect to the triple (y,v,p). We get

$$\begin{aligned}
&-\Delta y + F(y) + \frac{y}{\lambda} - \frac{v}{\lambda} = f && \text{in } \Omega, \quad y = 0 \ \text{ on } \partial\Omega,\\
&-\Delta p + F'(y)p + \frac{p}{\lambda} + (1+\gamma)y - \gamma v = z && \text{in } \Omega, \quad p = 0 \ \text{ on } \partial\Omega,\\
&\left(-\frac{p}{\lambda} - \gamma(y-v), t - v\right) \geq 0 && \text{in } \Omega,
\end{aligned} \tag{2.2}$$

where $\gamma = \nu/\lambda^2$ and the inequality condition must hold for all t in the admissible set V_{ad} defined as

$$V_{ad} = \left\{ v \in L^2(\Omega) \mid \underline{y} \le v \le \overline{y} \text{ a.e. in } \Omega \right\}. \tag{2.3}$$

The first equation is called the state equation and the second is called the adjoint equation. The inequality condition is the optimality condition.

In the next section we discuss the finite difference discretization scheme together with the smoothing algorithms associated to CSMG and MGOPT methods.

3. Discretization scheme and smoothing algorithms

We now consider the discrete version of the optimality system (2.2). By the finite difference discretization, $-\Delta_k$ denotes the minus five-point stencil for the Laplacian and hence we have

$$\begin{aligned}
&-\Delta_k y_k + F(y_k) + \frac{1}{\lambda} y_k - \frac{1}{\lambda} v_k = f_k,\\
&-\Delta_k p_k + F'(y_k) p_k + \frac{1}{\lambda} p_k + (1+\gamma) y_k - \gamma v_k = z_k,\\
&(-\frac{1}{\lambda} p_k - \gamma(y_k - v_k), t_k - v_k) \ge 0.
\end{aligned}$$

Let $x \in \Omega_k$ where $x = (ih_k, jh_k)$ and i, j are the indices of the grid points arranged lexicographically. We first set

$$\begin{aligned}
A &= -(y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1}) - h^2 f_{i,j},\\
B &= -(p_{i-1,j} + p_{i+1,j} + p_{i,j-1} + p_{i,j+1}) - h^2 z_{i,j}.
\end{aligned}$$

The values A and B are considered constant during the update of the variables at i, j. Then

$$\begin{aligned}
&A + c y_{i,j} + h^2 F(y_{i,j}) - \frac{1}{\lambda} h^2 v_{i,j} = 0,\\
&B + c p_{i,j} + h^2 F'(y_{i,j}) p_{i,j} + (1+\gamma) h^2 y_{i,j} - \gamma h^2 v_{i,j} = 0,\\
&\left(-\frac{1}{\lambda} p_{i,j} - \gamma y_{i,j} + \gamma v_{i,j}, t_{i,j} - v_{i,j} \right) \ge 0,
\end{aligned} \tag{3.1}$$

where $c = \left(4 + h^2/\lambda\right)$ and the inequality holds for all $t \in V_{adk} = \{v \in L_k^2(\Omega_k) \mid \underline{y} \le v \le \overline{y} \text{ in } \Omega_k\}$.

We can easily compute the updates for the variables $y_{i,j}$ and $p_{i,j}$ by using a Newton method and hence we obtain an update for $v_{i,j}$. In the presence of constraints, a new value for $v_{i,j}$ is obtained by its projection onto the admissible set. Consider the Jacobian J of the state and the adjoint equations with respect to $y_{i,j}$ and $p_{i,j}$,

$$J_{ij} = \begin{pmatrix} c + h^2 F'(y_{i,j}) & 0 \\ h^2(F''(y_{i,j}) p_{i,j} + (1+\gamma)) & c + h^2 F'(y_{i,j}) \end{pmatrix},$$

and the inverse of the Jacobian denoted by J^{-1} is

$$J_{ij}^{-1} = \frac{1}{\det J_{ij}} \begin{pmatrix} c + h^2 F'(y_{i,j}) & 0 \\ -h^2(F''(y_{i,j})p_{i,j} + (1+\gamma)) & c + h^2 F'(y_{i,j}) \end{pmatrix},$$

where $\det J_{ij} = \left(c + h^2 F'(y_{i,j})\right)^2$. The local Newton update for $y_{i,j}$ and $p_{i,j}$ is

$$\begin{pmatrix} y_{i,j} \\ p_{i,j} \end{pmatrix} = \begin{pmatrix} y_{i,j} \\ p_{i,j} \end{pmatrix} + J_{ij}^{-1} \begin{pmatrix} r_y(v_{i,j}) \\ r_p(v_{i,j}) \end{pmatrix},$$

where

$$\begin{aligned} r_y(v_{i,j}) &= -\left[A + c y_{i,j} + h^2 F(y_{i,j}) - \frac{1}{\lambda} h^2 v_{i,j}\right], \\ r_p(v_{i,j}) &= -[B + c p_{i,j} + h^2 F'(y_{i,j}) p_{i,j} + (1+\gamma) h^2 y_{i,j} - \gamma h^2 v_{i,j}], \end{aligned}$$

denote the residuals of the state and the adjoint equations, respectively. The update above shows that $y_{i,j}$ and $p_{i,j}$ can be defined as functions of $v_{i,j}$. To solve an update for $v_{i,j}$, replace $y_{i,j}$ and $p_{i,j}$ in the inequality condition in (3.1). The reduced cost functional is given by $\hat{J}(v) = J(y(v), v)$ and the optimal control solution of (2.1) is characterized by the optimality condition

$$\hat{J}'(v, t - v) = (-p/\lambda - \gamma(y - v), t - v) \geq 0$$

for all $t \in V_{ad}$. In the absence of constraints, we have

$$\hat{J}'(v, t - v) = (-p/\lambda - \gamma(y - v), t - v) = 0.$$

Hence we define an auxiliary variable $\widetilde{v}_{i,j}$ to be the solution of

$$-p_{i,j}/\lambda - \gamma(y_{i,j} - \widetilde{v}_{i,j}) = 0$$

and we get

$$\widetilde{v}_{i,j} = \frac{1}{\gamma}\left(\frac{p_{i,j}}{\lambda} + \gamma y_{i,j}\right).$$

Since the update for $v_{i,j}$ must be within the admissible set (2.3), then the new value for $v_{i,j}$ is obtained by projecting $\widetilde{v}_{i,j}$ onto V_{adk} given by

$$v_{i,j} = \begin{cases} \underline{y}_{i,j} & \text{if } \widetilde{v}_{i,j} \leq \underline{y}_{i,j}, \\ \widetilde{v}_{i,j} & \text{if } \underline{y}_{i,j} < \widetilde{v}_{i,j} < \overline{y}_{i,j}, \\ \overline{y}_{i,j} & \text{if } \widetilde{v}_{i,j} \geq \overline{y}_{i,j}. \end{cases}$$

With this value of $v_{i,j}$, the updates for the state variable $y(v_{i,j})$ and the adjoint variable $p(v_{i,j})$ are computed by the local Newton update. This completes the smoothing algorithm for the CMSG method.

For the MGOPT method, the gradient projection method [11] is utilized as the 'smoothing' algorithm in order to treat the bound constraints. First we introduce the reduced cost functional

$$\hat{J}(v) = J(y(v), v),$$

together with $\nabla \hat{J}(v) = -p/\lambda - \gamma y + \gamma v$ which is the gradient with respect to v. For the unconstrained case, we want to find a solution v of $\min_v \hat{J}(v)$ which is equivalent to solving $\nabla \hat{J}(v) = 0$. For the constrained case, we use a projection to find an update $v \in V_{ad}$, where V_{ad} is given by (2.3). Define the projection $\mathscr{P}$ onto V_{ad} by

$$\mathscr{P}_{V_{ad}}(v) = \begin{cases} \underline{y} & \text{if } v \le \underline{y}, \\ v & \text{if } \underline{y} < v < \overline{y}, \\ \overline{y} & \text{if } v \ge \overline{y}. \end{cases}$$

Given the current iterate v^ℓ, the new iterate $v^\ell(\alpha)$ is defined as

$$v^\ell(\alpha) = \mathscr{P}_{V_{ad}}(v^\ell + \alpha d^\ell),$$

where d^ℓ is a search direction and α satisfies the sufficient decrease condition [11]

$$\hat{J}(v^\ell(\alpha)) - \hat{J}(v^\ell) \le -\frac{\sigma}{\alpha} \|v^\ell - v^\ell(\alpha)\|^2,$$

for bound constrained problems. In the numerical experiments in Section 5, the value of σ is chosen to be 10^{-4}.

4. The multigrid method

In this section we present the the multigrid procedure for solving state-constrained elliptic optimal control problems. A typical multigrid method uses a sequence of nested discretization grids of increasing fineness $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_L = \Omega$. Associated to the sequence of grids is a sequence of finite difference spaces $V_1 \subset V_2 \subset \cdots \subset V_L = V$. This means that at each grid level k, the discrete problem is represented by

$$A_k(w_k) = g_k, \tag{4.1}$$

where $A_k(\cdot)$ represents a discrete nonlinear operator on Ω_k. In the CSMG case, we solve (2.2) and define $w := (y, v, p)$. On the other hand, the MGOPT method is applied to solve $\min_{v_k} \left(\hat{J}_k(v_k) - (g_k, v_k)_k\right)$ which is equivalent to solving $\nabla \hat{J}_k(v_k) = g_k$ in Ω_k. The term g_k is introduced to give a recursive formulation, where $g_k = 0$ at the finest resolution $k = L$. Hence in this case, $w := v$ and $A_k(v_k) = \nabla \hat{J}_k(v_k)$. We need transfer operators between finer and coarser grids. We choose a full-weighting restriction operator $I_k^{k-1} : V_k \to V_{k-1}$ and a bilinear interpolation operator $I_{k-1}^k : V_{k-1} \to V_k$.

Let the smoothing algorithm be represented by S_k such that we get an update $w_k^\ell = S_k(w_k^{\ell-1}, g_k)$. For the constrained case, the projection procedure is incorporated within the smoothing strategy. Starting with an initial approximation w_k^0, we apply γ_1 times of the smoothing procedure and obtain $w_k^{\gamma_1}$. Now the desired solution w_k can be written as $w_k = w_k^{\gamma_1} + e_k$ for some error e_k. Therefore (4.1) can be written as $A_k(w_k^{\gamma_1} + e_k) = g_k$ or equivalently as

$$A_k(w_k^{\gamma_1} + e_k) - A_k(w_k^{\gamma_1}) = g_k - A_k(w_k^{\gamma_1}). \tag{4.2}$$

Next we represent (4.2) on a coarser grid Ω_{k-1} and define

$$w_{k-1} := I_k^{k-1} w_k^{\gamma_1} + e_{k-1}$$

as a coarse-grid approximation to w_k. On the left hand side of (4.2) we represent A_k by A_{k-1} and $w_k^{\gamma_1}$ by $I_k^{k-1} w_k^{\gamma_1}$. On the other side we apply the restriction operator and we get $I_k^{k-1}(g_k - A_k(w_k^{\gamma_1}))$. Hence we have the following equation

$$A_{k-1}(w_{k-1}) = I_k^{k-1}(g_k - A_k(w_k^{\gamma_1})) + A_{k-1}(I_k^{k-1} w_k^{\gamma_1}). \tag{4.3}$$

We define

$$\tau_{k-1} = A_{k-1}(I_k^{k-1} w_k^{\gamma_1}) - I_k^{k-1} A_k(w_k^{\gamma_1})$$

then (4.3) can simply be written as

$$A_{k-1}(w_{k-1}) = I_k^{k-1} g_k + \tau_{k-1}. \tag{4.4}$$

The term τ_{k-1} is called the fine-to-coarse residual/gradient correction. The solution of (4.3) gives the error

$$e_{k-1} := w_{k-1} - I_k^{k-1} w_k^{\gamma_1}.$$

Therefore we have a correction to the fine grid approximation as

$$w_k^{\gamma_1+1} = w_k^{\gamma_1} + \alpha I_{k-1}^k (w_{k-1} - I_k^{k-1} w_k^{\gamma_1}).$$

For CSMG $\alpha = 1$, while for MGOPT α is the step length obtained after a line search procedure in the direction $I_{k-1}^k(w_{k-1} - I_k^{k-1} w_k^{\gamma_1})$. Finally, we apply γ_2 iterations of the smoothing algorithm to damp possible high frequency errors that may arise from the coarse grid correction process. One cycle of the multigrid method is presented in the following algorithm.

Algorithm 4.1. *Multigrid algorithm*
Initialize w_k^0. *If* $k=1$, *solve* $A_k(w_k)=g_k$. *Else*

1. *Apply* γ_1 *iterations of a smoothing algorithm to the problem at resolution* k.

$$w_k^\ell = S_k(w_k^{\ell-1}), \quad \ell = 1,2,\cdots,\gamma_1.$$

2. *Apply* γ *cycles of MG* (γ_1,γ_2) *to the coarse grid problem*

$$A_{k-1}(w_{k-1}) = g_{k-1}$$

to obtain w_{k-1}, *where*

$$g_{k-1} = I_k^{k-1} g_k + \tau_{k-1},$$
$$\tau_{k-1} = A_{k-1}(I_k^{k-1} w_k^{\gamma_1}) - I_k^{k-1} A_k(w_k^{\gamma_1}).$$

3. *For a given step length* α,

$$w_k^{\gamma_1+1} = w_k^{\gamma_1} + \alpha I_{k-1}^k (w_{k-1} - I_k^{k-1} w_k^{\gamma_1}).$$

4. *Apply* γ_2 *iterations of a smoothing algorithm to the problem at resolution* k.

$$w_k^\ell = S_k(w_k^{\ell-1}), \quad \ell = \gamma_1+2,\cdots,\gamma_1+\gamma_2+1.$$

The parameter γ characterizes the type of multigrid cycle being used. Typical values are $\gamma=1$ which is called the V-cycle and $\gamma=2$ is W-cycle.

5. Numerical results

We now present some numerical results on the computational performance of the multigrid schemes for solving state-constrained elliptic optimal control problems. For the results of the experiments, we use $\gamma_1=\gamma_2=2$ pre- and post- smoothing steps. This means that one multigrid cycle uses $\gamma_1+\gamma_2=4$ iterations of the smoothing algorithm on the finest level. All computations were performed in Matlab on a PC with a 2.67 GHz processor.

We consider problem (2.1) on a unit square domain $\Omega=(0,1)^2\subset\mathbb{R}^2$ with $F(y)=y^3$ and f, $z\in L^2(\Omega)$ given by

$$f(x_1,x_2)=0, \quad z(x_1,x_2)=\sin(2\pi x_1)\sin(2\pi x_2).$$

We also consider box-constraints $-0.5\le y\le 0.5$ and all unknown variables are initialized to zero. The target function z is shown in Fig. 1.

For the application of the CSMG algorithm, the numerical results are shown in Table 1. In this case, the CPU time in seconds are noted until the L^2-norm of the state and adjoint

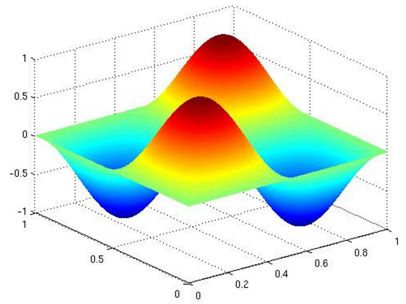

Figure 1: The target function z.

residuals, $\|r_y\|_{L^2}$ and $\|r_p\|_{L^2}$, satisfy a stopping tolerance of $tol = 10^{-5}$. The Lavrentiev regularization parameter λ are chosen to be $\lambda = \sqrt{\nu}$, where $\nu = 10^{-8}$ and 10^{-10}. The CSMG method exhibits an almost independence of the number of iterations on ν and on the size of the mesh where the problem is being solved. The convergence behavior of the CSMG method deteriorates as the parameter ν becomes too small. For MGOPT, the results are reported in Table 2. The CPU time in seconds are noted until the L^2-norm of the difference $\|v^\ell - v^\ell(1)\|_{L^2}$ satisfies a stopping tolerance of $tol = 10^{-5}$. The L^2-norm of the state and adjoint residuals are also reported on Table 2. The results show that the number of iterations is independent on the parameter ν and on the mesh size. The MGOPT method converges to the same solution as the CSMG after one iteration. As the parameter ν goes to zero, the MGOPT method shows a faster convergence behavior. This motivates a further investigation on bang-bang control problems with $\nu = 0$. The numerical solutions for the state variable y using $\lambda = 10^{-4}$ and $\lambda = 10^{-5}$ are shown in Fig. 2.

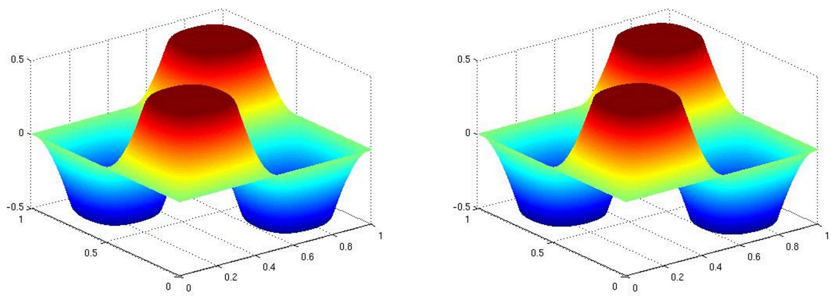

Figure 2: Numerical solutions by CSMG for the state variable y using $\lambda = 10^{-4}$ (left) and $\lambda = 10^{-5}$ (right).

6. Conclusions

Multigrid schemes for solving elliptic optimal control problems with constraints on the state variable are presented. We consider the Lavrentiev-type regularization to treat the

Table 1: Numerical results using CSMG method.

ν	Mesh	Iter	$\|r_y\|_{L^2}$	$\|r_p\|_{L^2}$	Time (sec)
	257×257	7	7.607e-06	7.473e-10	8.2
10^{-8}	513×513	7	6.975e-06	6.944e-10	34.3
	1025×1025	7	6.769e-06	6.852e-10	141.5
	257×257	8	3.225e-06	3.226e-11	9.3
10^{-10}	513×513	7	1.395e-06	1.530e-11	34.6
	1025×1025	7	2.549e-06	2.423e-11	142.4

Table 2: Numerical results using MGOPT with gradient projection method.

ν	Mesh	Iter	$\|r_y\|_{L^2}$	$\|r_p\|_{L^2}$	$\|v^\ell - v^\ell(1)\|_{L^2}$	Time (sec)
	257×257	1	1.073e-10	2.421e-12	1.215e-06	29.0
10^{-8}	513×513	1	3.903e-09	3.767e-10	3.157e-06	183.3
	1025×1025	1	2.320e-07	1.161e-08	7.641e-07	1232.4
	257×257	1	1.938e-12	2.586e-16	1.022e-06	17.0
10^{-10}	513×513	1	3.630e-12	1.739e-16	4.340e-06	102.1
	1025×1025	1	3.418e-11	1.099e-12	1.222e-06	595.1

constraints on the state variable. The results of the numerical experiments show that CSMG and MGOPT multigrid strategies provide a multigrid computational efficiency. It also shows that the CSMG scheme is faster compared to the MGOPT method. The CPU time approximately increase as a factor of four by halving the mesh size, which is a typical characteristic of an efficient multigrid solver. However, CSMG requires a carefully designed smoothing algorithm for each individual problem, while MGOPT does not require any adaptation to the problem. Different optimization techniques can be used as a smoothing procedure for MGOPT method. Since MGOPT accelerates the one grid optimization scheme [23], a topic which can be considered for future research is the appropriate use of other types of optimization algorithms as smoothers in order to achieve faster convergence results.

Acknowledgments Supported by the Office of the Chancellor, in collaboration with the Office of the Vice-Chancellor for Research and Development, of the University of the Philippines Diliman through the Ph.D. Incentive Award.

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