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Error Estimates for the Time Discretization of a Semilinear Integrodifferential Parabolic Problem with Unknown Memory Kernel

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Abstract. This paper is devoted to the study of an inverse problem containing a semilinear integrodifferential parabolic equation with an unknown memory kernel. This equation is accompanied by a Robin boundary condition. The missing kernel can be recovered from an additional global measurement in integral form. In this contribution, an error analysis is performed for a time-discrete numerical scheme based on Backward Euler's Method. The theoretical results are supported by some numerical experiments.

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Key words: Parabolic inverse problem, convolution kernel, error estimates.

1. Introduction

The aim of this paper is to derive error estimates for a time-discrete numerical scheme that approximates the solution of an inverse semilinear parabolic integrodifferential problem. This problem contains a Robin boundary condition and an unknown solely time-dependent memory kernel $K(t)$. More exactly, it is mathematically formulated as

$$\begin{cases} \partial_t u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) + K(t)h(\boldsymbol{x},t) - (K * \Delta u(\boldsymbol{x}))(t) = f(u(\boldsymbol{x},t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\boldsymbol{x},t) \in \Omega \times (0,T], \\ \alpha(u(\boldsymbol{x},t)) + \nabla u(\boldsymbol{x},t) \cdot \boldsymbol{\nu} = g(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \Gamma \times [0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad\qquad\qquad\qquad \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

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with $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, a Lipschitz domain with boundary Γ and $[0,T]$, $T>0$, the time frame. The data functions h, f, g, α and u_0 are supposed to be known. The usual convolution in time is denoted by the symbol $*$, i.e.

$$(K * \Delta u(\boldsymbol{x}))(t) = \int_0^t K(t-s)\Delta u(\boldsymbol{x},s)\mathrm{d}s.$$

The convolution kernel $K(t)$ and the function $u(\boldsymbol{x},t)$ need to be reconstructed from the extra given measurement

$$\int_\Omega u(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = m(t), \qquad t \in [0,T]. \tag{1.2}$$

The identification of missing memory kernels in partial integrodifferential equations is relatively new in inverse problems. The first papers on this topic concern abstract parabolic and hyperbolic equations with memory [1–5]. These papers contain some local existence and global uniqueness results applying the contraction mapping principle. Other papers dealing with this topic are [6–14]. For instance, in [14], Colombo and Guidetti derived some local and global in time existence results for the recovery of solely time-dependent memory kernels in semilinear integrodifferential models. More specifically, they studied the evolution equation for materials with memory given by

$$\partial_t u = \Delta u + \int_0^t K(t-s)\Delta u(\boldsymbol{x},s)\,\mathrm{d}s + F(u), \quad \boldsymbol{x} \in \Omega_0 \subset \mathbb{R}^3, \quad t \in [0,T_0],$$

which corresponds with problem (1.1). More recent papers dealing with a similar problem setting are [15] and [16], in which the authors have used the global measurement (1.2) to reconstruct the kernel of a convolution of the form $K * u$ in a semilinear parabolic problem. Such types of integro-differential problems arise in the theory of reactive contaminant transport, cf. [17].

In [18], the development of a numerical algorithm for problems of type (1.1)-(1.2) has been provided under the condition that

$$\min_{t\in[0,T]} \left| \int_\Omega h(t) \right| \geqslant \omega > 0.$$

However, only weak convergence of the numerical approximations to the kernel K has been shown. The first goal of this paper is to slightly change the numerical scheme from [18], such that higher stability results can be obtained. These stability results are needed for the second goal, i.e., to perform an error analysis, from which the strong convergence of the numerical approximations to the kernel K follows. The last goal of this paper is to support the theoretical results with some numerical experiments. The acquired a priori estimates for the error estimates are complicated and deliver a possible solution approach for solving other integrodifferential problems.

The outline of this paper is as follows. First, the numerical scheme of [18] and some corresponding a priori estimates are adapted in Section 2. In the same section, also the convergence of the approximations towards the unique weak solution is proved. Next, in Section 3, some higher stability results are derived, assuming sufficiently regular data. This section also deals with the error analysis. Further, two numerical experiments are conducted in Section 4. Finally, a conclusion is stated in Section 5.

Remark 1.1. The values C, ε and C_ε are considered to be generic and positive constants (independent of the discretization parameter), where ε is arbitrarily small and C_ε arbitrarily large, i.e. $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$. The same notation for different constants is used, but the meaning should be clear from the context.

2. Numerical scheme

In this section, we firstly repeat how the authors from [18] have built up a numerical scheme for problem (1.1). Secondly, we describe why and how this numerical scheme needs to be changed to be able to prove higher stability results. Finally, we repeat some lemmas and theorems that are proved in [18] and that stay valid after the adaption of the scheme.

In [18], it has been shown that the variational formulation of problem (1.1)-(1.2) can be formulated as: find $(K,u) \in L^2(0,T) \times L^2((0,T),H^1(\Omega))$, with $\partial_t u \in L^2\big((0,T), \big(H^1(\Omega)\big)^*\big)$, such that for almost all $t \in (0,T]$ and for all φ in the test space $H^1(\Omega)$, it holds that

$$\begin{aligned}&(\partial_t u,\varphi) + (\nabla u, \nabla\varphi) + K\,(h,\varphi) + (K * \nabla u, \nabla \varphi)\\ &= (f(u),\varphi) + (g-\alpha(u),\varphi)_\Gamma + (K*(g-\alpha(u)),\varphi)_\Gamma\end{aligned} \tag{P}$$

and such that the global measurement (1.2) is satisfied. Putting $\varphi = 1$ in (P), it is clear that

$$m' + K\int_\Omega h = \int_\Omega f(u) + \int_\Gamma (g-\alpha(u)) + \int_\Gamma K*(g-\alpha(u)). \tag{MP}$$

The well-posedness of (P) and (MP) has been studied in [18] by using Rothe's method, cf. [19]: a time-discrete scheme based on Backward Euler's method has been designed and the convergence of the approximations towards the unique weak solution has been proved under appropriate conditions on the data. Accordingly, an equidistant time-partitioning of the time frame $[0,T]$ into $n \in \mathbb{N}$ intervals has been considered. The time step has been denoted by $\tau = T/n < 1$ and the discrete time points by $t_i = i\tau$, $i = 1,\ldots,n$. The notations

$$z_i \approx z(t_i), \quad 0 \leqslant i \leqslant n, \qquad \text{and} \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}, \quad 1 \leqslant i \leqslant n,$$

have been introduced for any function z. In [18], the convolution term $K * \Delta u(\boldsymbol{x})(t_i)$ has been approximated by $\sum_{k=1}^{i} K_k \Delta u_{i-k}\tau$, which led to the following linearized Backward Euler scheme:

$$
\begin{aligned}
&(\delta u_i,\varphi)-(\Delta u_i,\varphi)+K_i\left(h_i,\varphi\right)-\left(\sum_{k=1}^{i}K_k\Delta u_{i-k}\tau,\varphi\right)\\
&=\left(f_{i-1},\varphi\right),\quad \varphi\in H^1(\Omega),
\end{aligned}
\tag{2.1}
$$

in which $f_{i-1}:=f(u_{i-1})$. Next, the following decoupled system for approximating the unknowns (K_i,u_i), $1\leqslant i\leqslant n$, has been proposed

$$
\begin{aligned}
&(\delta u_i,\varphi)+(\nabla u_i,\nabla\varphi)+K_i\left(h_i,\varphi\right)+\left(\sum_{k=1}^{i}K_k\nabla u_{i-k}\tau,\nabla\varphi\right)\\
&\qquad=\left(f_{i-1},\varphi\right)+\left(g_i-\alpha_{i-1},\varphi\right)_\Gamma+\left(\sum_{k=1}^{i}K_k(g_{i-k}-\alpha_{i-k})\tau,\varphi\right)_\Gamma,
\end{aligned}
\tag{2.2}
$$

$$
m_i'+K_i\left(h_i,1\right)=\left(f_{i-1},1\right)+\left(g_i-\alpha_{i-1},1\right)_\Gamma+\left(\sum_{k=1}^{i}K_k(g_{i-k}-\alpha_{i-k})\tau,1\right)_\Gamma,
\tag{2.3}
$$

with $\alpha_i:=\alpha(u_i)$. At every time step t_i, $1\leqslant i\leqslant n$, first (2.3) has been solved for K_i and next (2.2) for u_i. However, (2.2) is not equivalent with (2.1), which is needed to obtain higher stability results of the approximations leading to error estimates of order τ.

In this article, we solve the previous issue using the approximations

$$
K*\Delta u(\boldsymbol{x})(t_i)\approx\sum_{k=0}^{i-1}K_k\Delta u_{i-k}\tau,\qquad \nabla u_i\cdot\boldsymbol{\nu}\approx g_i-\alpha_{i-1},
$$

leading to the following linearized Backward Euler scheme

$$
\begin{aligned}
&(\delta u_i,\varphi)-(\Delta u_i,\varphi)+K_i\left(h_i,\varphi\right)-\left(\sum_{k=0}^{i-1}K_k\Delta u_{i-k}\tau,\varphi\right)\\
&=\left(f_{i-1},\varphi\right),\quad \varphi\in H^1(\Omega)
\end{aligned}
\tag{DPi1}
$$

and the equivalent decoupled system

$$
\begin{aligned}
(\delta u_i,\varphi)+(\nabla u_i,\nabla\varphi)&=-K_i\left(h_i,\varphi\right)-\left(\sum_{k=0}^{i-1}K_k\nabla u_{i-k}\tau,\nabla\varphi\right)+\left(g_i-\alpha_{i-1},\varphi\right)_\Gamma\\
&\quad+\left(f_{i-1},\varphi\right)+\left(\sum_{k=0}^{i-1}K_k(g_{i-k}-\alpha_{i-k-1})\tau,\varphi\right)_\Gamma,\quad \varphi\in H^1(\Omega),
\end{aligned}
\tag{DPi2}
$$

$$
m_i'+K_i\left(h_i,1\right)=\left(f_{i-1},1\right)+\left(g_i-\alpha_{i-1},1\right)_\Gamma+\left(\sum_{k=0}^{i-1}K_k(g_{i-k}-\alpha_{i-k-1})\tau,1\right)_\Gamma.
\tag{DMPi}
$$

Eq. (DPi2) is also conveniently written as $B(u_i,\varphi)=F_i(\varphi)$, with

$$B(u_i,\varphi)=\frac{1}{\tau}(u_i,\varphi)+((1+K_0\tau)\nabla u_i,\nabla\varphi)$$

and

$$\begin{aligned}F_i(\varphi)=(f_{i-1},\varphi)+(g_i-\alpha_{i-1},\varphi)_\Gamma-K_i\,(h_i,\varphi)-\left(\sum_{k=1}^{i-1}K_k\nabla u_{i-k}\tau,\nabla\varphi\right)\\+\left(\sum_{k=0}^{i-1}K_k(g_{i-k}-\alpha_{i-k-1})\tau,\varphi\right)_\Gamma+\frac{1}{\tau}(u_{i-1},\varphi).\end{aligned}$$

The resulting **numerical algorithm** is as follows:

Algorithm 2.1. Numerical scheme in pseudo code.

Input: $T>0$, $n\in\mathbb{N}$ and functions f, g, h, α, m, m' and u_0
Output: kernel K and solution u at discrete time steps

Step 1. $\tau\leftarrow T/n$;
Step 2. $\theta\leftarrow[0:\tau:T]$;
Step 3. $K\leftarrow$ zeros$(n+1)$;
Step 4. $u[0]\leftarrow u_0$;
Step 5. $K[0]\leftarrow\dfrac{1}{(h_0,1)}\left((f_0,1)-m_0'+(g_0-\alpha_0,1)_\Gamma\right)$;
Step 6. **For** $i=1$ **to** n **do**

$$K[i]\leftarrow\frac{1}{(h_i,1)}\times\left((f_{i-1},1)+(g_i-\alpha_{i-1},1)_\Gamma+\left(\sum_{k=0}^{i-1}K_k(g_{i-k}-\alpha_{i-k-1})\tau,1\right)_\Gamma-m_i'\right);$$

$u[i]\leftarrow$ **solveEP**$(B(u_i,\varphi)=F_i(\varphi))$.

Remark 2.1. Note that

(i) in step 5, we need $(h_0,1)\neq 0$.

(ii) from step 5 and the assumptions on the data, it follows that $K_0\leqslant C$.

(iii) in step 6, $(h_i,1)\neq 0$ is necessary.

Using the Lax-Milgram lemma, we can prove the existence and uniqueness of a solution $(K_i,u_i)\in\mathbb{R}\times\mathrm{H}^1(\Omega)$ obeying (DMPi) and (DPi2), see the following lemma. The proof is similar as the proof of [18, Proposition 3.1]. Therefore, we omit it.

Lemma 2.1. *Let f and α be bounded. Moreover, assume that $u_0 \in L^2(\Omega)$, $g \in C\big([0,T], L^2(\Gamma)\big)$, $h \in C\left([0,T], L^2(\Omega)\right)$,*

$$\min_{t\in[0,T]} \left| \int_\Omega h(t) \right| \geqslant \omega > 0, \qquad m \in C^1\left([0,T]\right).$$

Then constants $C > 0$ and $\tau_0 > 0$ exist such that for any $\tau < \tau_0$ and each $i \in \{1,\dots,n\}$ a unique couple $(K_i, u_i) \in \mathbb{R} \times \mathrm{H}^1(\Omega)$ exists, solving (DMPi) *and* (DPi2).

Remark 2.2. If in practice $m \notin C^1([0,T])$, then a smooth approximation of m in $[0,T]$ is considered.

In the following lemma, the a priori estimates from [18] are collected, combined with Remark 2.1. They stay valid for the scheme (DPi2)-(DMPi). The proofs follow the same line as the proofs in [18].

Lemma 2.2. *Let the assumptions of Lemma* 2.1 *be satisfied. Then positive constants C and τ_0 exist such that, for any $\tau < \tau_0$, it holds that*

(i) $\max_{0\le i\le n} |K_i| \le C$,

(ii) $\max_{1\le i\le n} \|u_i\|^2 + \sum_{i=1}^n \|\nabla u_i\|^2 \tau + \sum_{i=1}^n \|u_i - u_{i-1}\|^2 \le C$,

(iii) $\sum_{i=1}^n \|\delta u_i\|^2_{(\mathrm{H}^1(\Omega))^*} \tau \le C$.

Now, the discrete solutions are prolonged in time in two ways: piecewise linear and piecewise constant. The piecewise linear in time functions u_τ are defined as

$$u_\tau : [-\tau, T] \to L^2(\Omega) : t \mapsto \begin{cases} u_0 & t \in [-\tau, 0] \\ u_{i-1} + (t - t_{i-1})\delta u_i & t \in (t_{i-1}, t_i], \quad 1 \le i \le n, \end{cases}$$

and the piecewise constant in time functions $\overline{u}_\tau$ as

$$\overline{u}_\tau : [-\tau, T] \to L^2(\Omega) : t \mapsto \begin{cases} u_0 & t \in [-\tau, 0] \\ u_i & t \in (t_{i-1}, t_i], \quad 1 \le i \le n. \end{cases}$$

Similarly, the step functions $\overline{K}_\tau$, $\overline{h}_\tau$, $\overline{g}_\tau$, $\overline{m}_\tau$ and $\overline{m'}_\tau$ are introduced. Using these so-called Rothe's functions, we rewrite (DPi2) and (DMPi) on the whole time frame as

$$\begin{aligned}
&(\partial_t u_\tau(t), \varphi) + (\nabla \overline{u}_\tau(t), \nabla\varphi) \\
=& -\overline{K}_\tau(t)(\overline{h}_\tau(t), \varphi) - \left(\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k) \nabla \overline{u}_\tau(t - t_k)\tau, \nabla\varphi \right) + (f(\overline{u}_\tau(t-\tau)), \varphi) \\
&+ (\overline{g}_\tau(t) - \alpha(\overline{u}_\tau(t-\tau)), \varphi)_\Gamma + \left(\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k) \left[\overline{g}_\tau(t - t_k) - \alpha(\overline{u}_\tau(t - t_{k+1}))\right] \tau, \varphi \right)_\Gamma \qquad \text{(DP)}
\end{aligned}$$

and

$$\begin{aligned} &\overline{m'}_\tau(t) + \overline{K}_\tau(t)\left(\overline{h}_\tau(t), 1\right) \\ &= (f(\overline{u}_\tau(t-\tau)), 1) + (\overline{g}_\tau(t) - \alpha(\overline{u}_\tau(t-\tau)), 1)_\Gamma \\ &\quad + \left(\sum_{k=0}^{\lfloor t\rfloor_\tau} \overline{K}_\tau(t_k)\left[\overline{g}_\tau(t-t_k) - \alpha(\overline{u}_\tau(t-t_{k+1}))\right]\tau, 1\right)_\Gamma, \end{aligned} \tag{DMP}$$

with $\lfloor t\rfloor_\tau = i-1$ for $t \in (t_{i-1}, t_i]$. The convergence of (DP) to (P) and of (DMP) to (MP), as well as the existence of a unique weak solution to (P)-(MP), can be shown in a similar way as for the discrete scheme in [18]. The results are summarized in the following theorem (i)-(ii). Only the main differences with the proof in [18, Theorem 2] are mentioned. The proof of Theorem 2.1(iii) can be found in [18, Proposition 2.2].

Theorem 2.1 (Existence, uniqueness and convergence). *Let the assumptions of Lemma 2.2 be fulfilled. Moreover, assume that $u_0 \in H^1(\Omega)$ and that f and α are Lipschitz continuous. Then*

(i) a couple $(K,u) \in L^2(0,T) \times \left[C\left([0,T], L^2(\Omega)\right) \cap L^2\left((0,T), H^1(\Omega)\right)\right]$ with $\partial_t u \in L^2\left((0,T), \left(H^1(\Omega)\right)^*\right)$ exists such that $\overline{K}_\tau \rightharpoonup K$ in $L^2(0,T)$, $u_\tau \to u$ in $L^2\big((0,T), L^2(\Omega)\big)$, $\overline{u}_\tau \to u$ in $L^2\left((0,T), L^2(\Omega)\right)$ and $\overline{u}_\tau \to u$ in $L^2\left((0,T), L^2(\Gamma)\right)$ as $\tau \to 0$,

(ii) the couple (K,u) from (i) is a unique solution to (P)-(MP),

(iii) a positive constant $C > 0$ exists such that $\max_{t\in[0,T]} |K(t)| \leqslant C$.

Proof. (ii) Integrating (DP) in time over $(0,\eta)$, $\eta \in (0,T]$, we get two different terms as compared to the scheme in [18]:

$$\begin{aligned} T_1 &:= \int_0^\eta \left(\sum_{k=0}^{\lfloor t\rfloor_\tau} \overline{K}_\tau(t_k)\nabla\overline{u}_\tau(t-t_k)\tau, \nabla\varphi\right), \\ T_2 &:= \int_0^\eta \left(\sum_{k=0}^{\lfloor t\rfloor_\tau} \overline{K}_\tau(t_k)\left[\overline{g}_\tau(t-t_k) - \alpha(\overline{u}_\tau(t-t_{k+1}))\right]\tau, \varphi\right)_\Gamma. \end{aligned}$$

We need to prove that: $\forall\varphi \in H^1(\Omega)$,

$$T_1 \to \int_0^\eta (K * \nabla u(t), \nabla\varphi), \quad T_2 \to \int_0^\eta (K * (g(t) - \alpha(t)), \varphi)_\Gamma,$$

as $n \to \infty$. First, we note that

$$\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k)\overline{u}_\tau(t-t_k)\tau = \overline{K}_\tau * \overline{u}_\tau(t) + K_0\overline{u}_\tau(t)\tau - \int_{\tau\lfloor t \rfloor_\tau}^{t} \overline{K}_\tau(s)\overline{u}_\tau(t-s), \tag{2.4}$$

$$\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k)\overline{g}_\tau(t-t_k)\tau = \overline{K}_\tau * \overline{g}_\tau(t) + K_0\overline{g}_\tau(t)\tau - \int_{\tau\lfloor t \rfloor_\tau}^{t} \overline{K}_\tau(s)\overline{g}_\tau(t-s), \tag{2.5}$$

$$\begin{aligned}\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k)\alpha(\overline{u}_\tau(t-t_{k+1}))\tau &= \overline{K}_\tau * \alpha(\overline{u}_\tau)(t) + K_0\alpha(\overline{u}_\tau(t-\tau))\tau \\ &- \int_{\tau\lfloor t \rfloor_\tau}^{t} \overline{K}_\tau(s)\alpha(\overline{u}_\tau(t-s)) + \int_0^{\tau\lfloor t \rfloor_\tau} \overline{K}_\tau(s)\left[\alpha(\overline{u}_\tau(t-s-\tau)) - \alpha(\overline{u}_\tau(t-s))\right]. \end{aligned}\tag{2.6}$$

From the Green Theorem, it follows that, $\forall \varphi \in C^\infty(\overline{\Omega})$,

$$\begin{aligned}T_1 &= -\int_0^\eta \left(\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k)\overline{u}_\tau(t-t_k)\tau, \Delta\varphi\right) + \int_0^\eta \left(\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k)\overline{u}_\tau(t-t_k)\tau, \nabla\varphi\cdot\boldsymbol{\nu}\right)_\Gamma \\ &=: T_{1,1} + T_{1,2}.\end{aligned}$$

Equality (2.4) and standard techniques based on the triangle and Cauchy inequalities and on the results of Lemma 2.2(i) and (ii) then lead to

$$\begin{aligned}&\left|T_{1,1} + \int_0^\eta (K * u(t), \Delta\varphi)\right| \\ \leqslant &\left|\int_0^\eta (\overline{K}_\tau * \overline{u}_\tau(t), \Delta\varphi) - \int_0^\eta (K * u(t), \Delta\varphi)\right| \\ &\quad + \left|\int_0^\eta \left(\int_{\tau\lfloor t \rfloor_\tau}^t \overline{K}_\tau(s)\overline{u}_\tau(t-s), \Delta\varphi\right)\right| + \left|\int_0^\eta (K_0\overline{u}_\tau(t)\tau, \Delta\varphi)\right| \\ \leqslant &\left|\int_0^\eta \int_0^t \left(\overline{K}_\tau(s) - K(s)\right)(\overline{u}_\tau(t-s), \Delta\varphi)\right| \\ &\quad + C\sqrt{\int_0^t |K(s)|^2} \int_0^\eta \sqrt{\int_0^t \|\overline{u}_\tau(t-s) - u(t-s)\|^2} + C\tau.\end{aligned}$$

This converges to 0 as $n \to \infty$, which follows from the results of Theorem 2.1(i) and of Lemma 2.2(ii). Using the same techniques, combined with the trace inequality, we arrive at

$$\left|T_{1,2} - \int_0^\eta (K * u(t), \nabla\varphi\cdot\nu)_\Gamma\right| \to 0$$

as $n \to \infty$. From this, the Green Theorem and the density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$, it then follows that

$$T_1 \to \int_0^\eta (K * \nabla u(t), \nabla\varphi), \quad \forall \varphi \in H^1(\Omega), \quad \text{as } n \to \infty.$$

Next, some simple calculations based on identity (2.5), combined with the same standard techniques as before, the regularity of g and the results of Theorem 2.1(i) and Lemma 2.2(i) and (ii) give

$$\left| \int_0^\eta \left(\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k)\overline{g}_\tau(t-t_k), \varphi \right)_\Gamma - \int_0^\eta (K * g(t), \varphi)_\Gamma \right| \to 0,$$

$\forall \varphi \in H^1(\Omega)$, as $n \to \infty$. Finally, identity (2.6) leads to

$$\begin{aligned}
&\left| \int_0^\eta \left(\sum_{k=0}^{\lfloor t \rfloor_\tau} \overline{K}_\tau(t_k)\alpha\left(\overline{u}_\tau(t-t_{k+1})\right)\tau, \varphi \right)_\Gamma - \int_0^\eta (K * \alpha(u(t)), \varphi)_\Gamma \right| \\
\leqslant &\left| \int_0^\eta (K_0\alpha(\overline{u}_\tau(t-\tau))\tau, \varphi)_\Gamma \right| + \left| \int_0^\eta \left(\int_{\tau\lfloor t \rfloor_\tau}^t \overline{K}_\tau(s)\alpha(\overline{u}_\tau(t-s)), \varphi \right)_\Gamma \right| \\
&+ \left| \int_0^\eta \left(\overline{K}_\tau * \alpha\left(\overline{u}_\tau\right)(t), \varphi \right)_\Gamma - \int_0^\eta (K * \alpha(u)(t), \varphi)_\Gamma \right| \\
&+ \left| \int_0^\eta \left(\int_0^{\tau\lfloor t \rfloor_\tau} \overline{K}_\tau(s)\left[\alpha(\overline{u}_\tau(t-s-\tau)) - \alpha(\overline{u}_\tau(t-s))\right], \varphi \right)_\Gamma \right| \\
=: &\sum_{k=1}^4 A_k.
\end{aligned}$$

Using the same techniques again and applying the Lipschitz continuity and boundedness of α, we then obtain

$$\begin{aligned}
A_1 + A_2 &\leqslant C\tau, \\
A_3 &\leqslant C\int_0^\eta \sqrt{\int_0^t |K(s)|^2}\sqrt{\int_0^t \|\overline{u}_\tau(t-s) - u(t-s)\|_\Gamma^2} \\
&\qquad + \left| \int_0^\eta \int_0^t \left(\overline{K}_\tau(s) - K(s) \right) \left(\alpha\left(\overline{u}_\tau(t-s)\right), \varphi \right)_\Gamma \right|, \\
A_4 &\leqslant C\int_0^\eta \sqrt{\int_0^t \|\overline{u}_\tau(s-\tau) - \overline{u}_\tau(s)\|_\Gamma^2},
\end{aligned}$$

which all converge to 0 if $n \to \infty$. Collecting all previous results gives

$$T_2 \to \int_0^\eta (K * (g(t) - \alpha(t)), \varphi)_\Gamma,$$

$\forall \varphi \in H^1(\Omega)$, as $n \to \infty$.

Finally, integrating (DMP) in time over $(0,\eta)$, $\eta\in(0,T]$, we get one different term in comparison with the scheme in [18]:

$$T_3 := \int_0^\eta \left(\sum_{k=0}^{\lfloor t\rfloor_\tau} \overline{K}_\tau(t_k)\left[\overline{g}_\tau(t-t_k) - \alpha(\overline{u}_\tau(t-t_{k+1}))\right]\tau, 1 \right)_\Gamma ,$$

of which the convergence to $\displaystyle\int_0^\eta (K*(g(t)-\alpha(t)),1)_\Gamma$ can be proved in exactly the same manner as the convergence of T_2. $\square$

A drawback of the latter theorem is that the convergence of $\overline{K}_\tau$ to K is only proven in weak sense. However, a consequence of the error estimates proved in the remainder of this paper is the strong convergence of $\overline{K}_\tau$ to K in $L^2(0,T)$.

3. Error analysis

The next step is to derive some higher stability results for the approximations. These are needed to obtain a convergence rate of $\mathcal{O}(\tau)$ in the error analysis. They are stated in Lemmas 3.1-3.3. In the proof of Lemma 3.1, it is needed that $\Delta u_i\in L^2(\Omega)$, $1\leqslant i\leqslant n$. From (DPi1) follows that

$$-\Delta u_i = \frac{1}{1+K_0\tau}\left(f_{i-1} - K_i h_i - \delta u_i + \sum_{k=1}^{i-1} K_k \Delta u_{i-k}\tau \right).$$

This equality must be understood in the sense of duality, as a functional on $H^1(\Omega)$. However, if f_{i-1}, h_i, δu_i and $\Delta u_1,\dots,\Delta u_{i-1}$ are elements of $L^2(\Omega)$, it follows that also $\Delta u_i\in L^2(\Omega)$ and

$$-\Delta u_i = \frac{1}{1+K_0\tau}\left(f_{i-1} - K_i h_i - \delta u_i + \sum_{k=1}^{i-1} K_k \Delta u_{i-k}\tau \right) \quad \text{in } L^2(\Omega). \tag{3.1}$$

Hence, using the assumptions that $|f|\leqslant C$ and $h\in C\left([0,T],L^2(\Omega)\right)$ and applying the result of Lemma 2.2(ii), a bootstrap argument gives that $\Delta u_i\in L^2(\Omega)$, $i=1,\cdots,n$.

Lemma 3.1. *Let the assumptions of Theorem* 2.1 *be fulfilled. Moreover, assume that* $g\in C^1\left([0,T],L^2(\Gamma)\right)$. *Then positive constants* C *and* τ_0 *exist such that,* $\forall\,\tau<\tau_0$,

(i) $\displaystyle\sum_{i=1}^n \|\Delta u_i\|^2\tau + \max_{1\leqslant j\leqslant n}\|\nabla u_j\|^2 + \sum_{i=1}^n \|\nabla u_i - \nabla u_{i-1}\|^2 \leqslant C,$

(ii) $\displaystyle\sum_{i=1}^n \|\delta u_i\|^2\tau \leqslant C.$

Proof. (i) We set $\varphi = -\Delta u_i \tau$ in (DPi1) and sum it up for $i = 1, \ldots, j$, keeping $1 \leqslant j \leqslant n$. We obtain

$$\begin{aligned}&\sum_{i=1}^{j} (\delta u_i, -\Delta u_i)\,\tau + \sum_{i=1}^{j} \|\Delta u_i\|^2\,\tau \\ =&\sum_{i=1}^{j} K_i\,(h_i, \Delta u_i)\,\tau - \sum_{i=1}^{j} (f_{i-1}, \Delta u_i)\,\tau - \sum_{i=1}^{j} \left(\sum_{k=0}^{i-1} K_k \Delta u_{i-k}\tau, \Delta u_i \right) \tau. \end{aligned} \tag{3.2}$$

For the terms in the right-hand side (RHS) of (3.2), we need to construct an upper bound. Using the Cauchy and Young inequalities and Lemma 2.2(i), we derive that

$$\begin{aligned}&\left| \sum_{i=1}^{j} \left(\sum_{k=0}^{i-1} K_k \Delta u_{i-k}\tau, \Delta u_i \right) \tau \right| = \left| \sum_{i=1}^{j} \left(\sum_{k=1}^{i} K_{i-k} \Delta u_k \tau, \Delta u_i \right) \tau \right| \\ \leqslant& C_\varepsilon \sum_{i=1}^{j} \sum_{k=1}^{i} \|\Delta u_k\|^2\,\tau^2 + \varepsilon \sum_{i=1}^{j} \|\Delta u_i\|^2\,\tau.\end{aligned}$$

Moreover, the triangle, Cauchy and Young inequalities, the regularity of f and h and Lemma 2.2(i) lead to

$$\left| \sum_{i=1}^{j} K_i\,(h_i, \Delta u_i)\,\tau - \sum_{i=1}^{j} (f_{i-1}, \Delta u_i)\,\tau \right| \leqslant C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\Delta u_i\|^2\,\tau.$$

On the first term in the LHS of (3.2), we apply the Green theorem and Abel's lemma – see [20] – which states that

$$2\sum_{i=1}^{j} a_i(a_i - a_{i-1}) = a_j^2 - a_0^2 + \sum_{i=1}^{j} (a_i - a_{i-1})^2, \quad \forall\, a_i \in \mathbb{R}. \tag{3.3}$$

We successively deduce that

$$\begin{aligned}&\sum_{i=1}^{j} (\delta u_i, -\Delta u_i)\,\tau \\ =&\sum_{i=1}^{j} (\nabla \delta u_i, \nabla u_i)\,\tau - \sum_{i=1}^{j} (\delta u_i, \nabla u_i \cdot \boldsymbol{\nu})_\Gamma\,\tau \\ =&\frac{1}{2}\left(\|\nabla u_j\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2 \right) - \sum_{i=1}^{j} (\delta u_i, g_i - \alpha_{i-1})_\Gamma\,\tau.\end{aligned} \tag{3.4}$$

In what comes next, we focus on the last term in the RHS of (3.4). First, note that for any real sequences $\{z_i\}_{i=0}^{\infty}$ and $\{w_i\}_{i=0}^{\infty}$ the following summation by parts identity takes place

$$\sum_{i=1}^{j} z_i \left(w_i - w_{i-1}\right) = z_j w_j - z_0 w_0 - \sum_{i=1}^{j} \left(z_i - z_{i-1}\right) w_{i-1}. \tag{3.5}$$

Moreover, the inequality below holds true, see [21],

$$\|z\|_\Gamma^2 \leqslant \varepsilon \left\|\nabla z\right\|^2 + C_\varepsilon \left\|z\right\|^2, \quad \forall z \in H^1\left(\Omega\right), \; 0 < \varepsilon < \varepsilon_0. \tag{3.6}$$

Using the Cauchy, triangle, Young and trace inequalities, the Mean Value Theorem (MVT), the regularity of u_0 and g and Lemma 2.2(ii), we then get

$$\begin{aligned}
&\left|\sum_{i=1}^{j} \left(\delta u_i, g_i\right)_\Gamma \tau\right| \overset{(3.5)}{=} \left|\left(g_j, u_j\right)_\Gamma - \left(g_0, u_0\right)_\Gamma - \sum_{i=1}^{j} \left(\delta g_i, u_{i-1}\right)_\Gamma \tau\right| \\
&\leqslant C\Bigg(\|g_j\|_\Gamma^2 + \|u_j\|_\Gamma^2 + \|g_0\|_\Gamma^2 + \|u_0\|_{H^1(\Omega)}^2 + \sum_{i=1}^{j} \|\delta g_i\|_\Gamma^2 \tau + \sum_{i=1}^{j} \|u_{i-1}\|_{H^1(\Omega)}^2 \tau \Bigg) \\
&\overset{(3.6)}{\leqslant} C + C_\varepsilon \left\|u_j\right\|^2 + \varepsilon \left\|\nabla u_j\right\|^2 \leqslant C_\varepsilon + \varepsilon \left\|\nabla u_j\right\|^2.
\end{aligned}$$

Furthermore, it is easy to see that

$$\begin{aligned}
&\sum_{i=1}^{j} \left(\delta u_i, \alpha_{i-1}\right)_\Gamma \tau = \sum_{i=1}^{j} \left(u_i - u_{i-1}, \alpha_i\right)_\Gamma + \sum_{i=1}^{j} \left(u_i - u_{i-1}, \alpha_{i-1} - \alpha_i\right)_\Gamma \\
&=: T_1 + T_2.
\end{aligned}$$

The Cauchy inequality, the Lipschitz continuity of α and Lemma 2.2(ii) then give us

$$\begin{aligned}
\left|T_2\right| &\leqslant C \sum_{i=1}^{j} \left\|u_i - u_{i-1}\right\|_\Gamma^2 \\
&\overset{(3.6)}{\leqslant} C_\varepsilon \sum_{i=1}^{j} \left\|u_i - u_{i-1}\right\|^2 + \varepsilon \sum_{i=1}^{j} \left\|\nabla u_i - \nabla u_{i-1}\right\|^2 \\
&\leqslant C_\varepsilon + \varepsilon \sum_{i=1}^{j} \left\|\nabla u_i - \nabla u_{i-1}\right\|^2.
\end{aligned}$$

Next, using the Lipschitz constant of α, which we denote by L_α, we can rewrite T_1 as

$$T_1 = \sum_{i=1}^{j} \left(u_i - u_{i-1}, L_\alpha u_i + \alpha_i\right)_\Gamma - \sum_{i=1}^{j} \left(u_i - u_{i-1}, L_\alpha u_i\right)_\Gamma. \tag{3.7}$$

We construct an upper bound for the second term in the RHS of (3.7) using the triangle and trace inequalities, the regularity of u_0 and Lemma 2.2(ii). We obtain that

$$
\begin{aligned}
&\left|\sum_{i=1}^{j}(u_i-u_{i-1},L_\alpha u_i)_\Gamma\right| \overset{(3.3)}{=} \left|\frac{L_\alpha}{2}\left(\|u_j\|_\Gamma^2-\|u_0\|_\Gamma^2+\sum_{i=1}^{j}\|u_i-u_{i-1}\|_\Gamma^2\right)\right| \\
\overset{(3.6)}{\leqslant}& C_\varepsilon\|u_j\|^2+\varepsilon\|\nabla u_j\|^2+C\|u_0\|_{H^1(\Omega)}^2+C_\varepsilon\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2+\varepsilon\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2 \\
\leqslant& C_\varepsilon+\varepsilon\left(\|\nabla u_j\|^2+\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2\right).
\end{aligned}
$$

Now, we introduce the following notation for any function $\beta:\mathbb{R}\to\mathbb{R}$

$$
\Phi_\beta(z):=\int_0^z\beta(s)\mathrm{d}s, \quad \forall\, z\in\mathbb{R}.
$$

If β is a monotonically increasing function, then Φ_β is convex and it can be easily verified that

$$
\beta(z_1)(z_2-z_1)\leqslant\Phi_\beta(z_2)-\Phi_\beta(z_1)\leqslant\beta(z_2)(z_2-z_1) \tag{3.8}
$$

for any $z_1,z_2\in\mathbb{R}$. If β is Lipschitz continuous with Lipschitz coefficient L_β and $\beta(0)=0$, then

$$
\frac{\beta^2(z)}{2L_\beta}\leqslant\Phi_\beta(z)\leqslant\frac{L_\beta z^2}{2}, \tag{3.9}
$$

see, e.g [22, 23]. It is easy to see that $\theta(s):=L_\alpha s+\alpha(s)$ is Lipschitz continuous and monotone, which follows from

$$
0\leqslant L_\alpha+\alpha'(s)=\frac{\mathrm{d}\theta}{\mathrm{d}s}(s)\leqslant 2L_\alpha=:L_\theta.
$$

Therefore, we obtain for the first term in the RHS of (3.7) that

$$
\begin{aligned}
&\sum_{i=1}^{j}(u_i-u_{i-1},L_\alpha u_i+\alpha_i)_\Gamma \\
\overset{(3.8)}{\geqslant}&\sum_{i=1}^{j}\int_\Gamma(\Phi_\theta(u_i)-\Phi_\theta(u_{i-1}))=\int_\Gamma(\Phi_\theta(u_j)-\Phi_\theta(u_0)).
\end{aligned}
$$

Using lemma 2.2(ii), the trace inequality and the regularity of u_0, we then get

$$
\begin{aligned}
&\left|\int_\Gamma(\Phi_\theta(u_j)-\Phi_\theta(u_0))\right|\overset{(3.9)}{\leqslant}L_\alpha\left(\|u_j\|_\Gamma^2+\|u_0\|_\Gamma^2\right) \\
\overset{(3.6)}{\leqslant}&C_\varepsilon\|u_j\|^2+\varepsilon\|\nabla u_j\|^2+C\|u_0\|_{H^1(\Omega)}^2\leqslant C_\varepsilon+\varepsilon\|\nabla u_j\|^2.
\end{aligned}
$$

Putting everything together and using the regularity of u_0, we arrive at

$$\begin{aligned}&\sum_{i=1}^{j}\|\Delta u_i\|^2\tau+\frac{1}{2}\|\nabla u_j\|^2+\frac{1}{2}\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2\\ &\leqslant C_\varepsilon\left(1+\sum_{i=1}^{j}\sum_{k=1}^{i}\|\Delta u_k\|^2\tau^2\right)+\varepsilon\left(\sum_{i=1}^{j}\|\Delta u_i\|^2\tau+\|\nabla u_j\|^2+\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2\right).\end{aligned}$$

Since ε can be chosen arbitrarily small, we obtain

$$\begin{aligned}&\sum_{i=1}^{j}\|\Delta u_i\|^2\tau+\|\nabla u_j\|^2+\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2\\ \leqslant &C\left(1+\sum_{i=1}^{j}\left(\sum_{k=1}^{i}\|\Delta u_k\|^2\tau\right)\tau\right).\end{aligned}$$

Finally, taking τ smaller than or equal to a suitable fixed $\tau_0>0$ and applying the discrete Grönwall lemma, we get

$$\sum_{i=1}^{j}\|\Delta u_i\|^2\tau+\|\nabla u_j\|^2+\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2\leqslant C,$$

which is valid for all $1\leqslant j\leqslant n$. From this, we conclude the proof.
(ii) From (3.1), we derive that

$$\delta u_i=(1+K_0\tau)\Delta u_i-K_ih_i+f_{i-1}+\sum_{k=1}^{i-1}K_k\Delta u_{i-k}\tau\quad\text{in } L^2(\Omega).$$

Therefore, taking into account the regularity of the data and the stability results from Lemma 2.2(i) and Lemma 3.1(i) and fixing τ sufficiently small, we immediately obtain that

$$\sum_{i=1}^{n}\|\delta u_i\|^2\tau\leqslant C+C\sum_{i=1}^{n}\|\Delta u_i\|^2\tau+C\sum_{i=1}^{n}\sum_{k=1}^{i-1}|K_{i-k}|^2\|\Delta u_k\|^2\tau^2\leqslant C.$$

This completes the proof of the lemma. □

For the next two lemmas, we set

$$\partial_t u(0):=\Delta u_0-K_0h_0+f(u_0) \tag{3.10}$$

in the space where the RHS is defined. Further, we define

$$\delta u_0:=\partial_t u(0)\quad\text{and}\quad u_{-1}:=u_0-\delta u_0\tau. \tag{3.11}$$

From (3.10) and (3.11), it is easy to see that

$$u_0 \in H^2(\Omega),\ |K_0| \leqslant C,\ h_0 \in L^2(\Omega) \text{ and } f \text{ Lipschitz continuous}$$
$$\Rightarrow \delta u_0 \in L^2(\Omega) \text{ and } u_{-1} \in L^2(\Omega)$$

and

$$u_0 \in H^3(\Omega),\ |K_0| \leqslant C,\ h_0 \in H^1(\Omega) \text{ and } f \text{ Lipschitz continuous}$$
$$\Rightarrow \delta u_0 \in H^1(\Omega) \text{ and } u_{-1} \in H^1(\Omega). \tag{3.12}$$

For the proof of Lemma 3.2, we also need the discrete measured problem at $t = 0$. First, multiplying (3.10) by $\varphi \in H^1(\Omega)$, integrating the result over the domain Ω and applying the Green theorem give the discrete variational problem at $t = 0$, i.e.

$$(\delta u_0, \varphi) + (\nabla u_0, \nabla \varphi) + K_0\,(h_0, \varphi) = (f(u_0), \varphi) + (g_0 - \alpha_0, \varphi)_\Gamma\,. \tag{DP0}$$

Next, setting $\varphi = 1$ and using the measurement (1.2) yield the discrete measured problem at $t = 0$:

$$m_0' + K_0\,(h_0, 1) = (f_0, 1) + (g_0 - \alpha_0, 1)_\Gamma\,. \tag{DMP0}$$

Lemma 3.2. *Let the assumptions of Lemma* 3.1 *be fulfilled. Moreover, assume that* $u_0 \in H^3(\Omega)$, $h \in C^1\left([0,T], L^2(\Omega)\right)$, $h_0 \in H^1(\Omega)$ *and* $m \in C^2([0,T])$. *Then positive constants* C *and* τ_0 *exist such that,* $\forall\, \tau < \tau_0$, $\forall\, j \in \{1, \cdots, n\}$,

$$\sum_{i=1}^{j} |\delta K_i|^2 \tau \leqslant C\left(1 + \sum_{i=1}^{j} \|\nabla \delta u_i\|^2\, \tau\right).$$

Proof. First, we subtract (DMP0) from (DMPi) for $i = 1$ and divide the result by τ. We obtain

$$\delta K_1\,(h_1, 1) = (\delta g_1, 1)_\Gamma + K_0\,(g_1 - \alpha_0, 1)_\Gamma - K_0\,(\delta h_1, 1) - \delta m_1'. \tag{3.13}$$

Using the triangle and Cauchy inequalities, the MVT, the assumptions on the data and Lemma 2.2 (i), we then immediately derive from (3.13) that

$$|\delta K_1|\,|(h_1, 1)| \leqslant C,$$

from which

$$|\delta K_1| \leqslant C. \tag{3.14}$$

Next, we apply the δ-operator to (DMPi) for $i \geqslant 2$ and use the rule

$$\delta\,(a_i b_i) = \delta a_i b_i + a_{i-1} \delta b_i,$$

which is valid for any real sequences $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$. This gives us

$$\begin{aligned}\delta K_i\,(h_i,1) &= (\delta f_{i-1},1) + (\delta g_i,1)_\Gamma - (\delta\alpha_{i-1},1)_\Gamma - K_{i-1}\,(\delta h_i,1) - \delta m_i' \\ &\quad + \sum_{k=0}^{i-2} K_k\,(\delta g_{i-k} - \delta\alpha_{i-k-1},1)_\Gamma\,\tau + K_{i-1}\,(g_1-\alpha_0,1)_\Gamma\,.\end{aligned}$$

The same techniques as before, combined with the Lipschitz continuity of f and α and with the trace inequality, then lead to

$$|\delta K_i|^2 \leqslant C\left(1 + \|\delta u_{i-1}\|^2 + \|\nabla\delta u_{i-1}\|^2 + \sum_{k=0}^{i-2}\|\delta u_{i-k-1}\|^2\tau + \sum_{k=0}^{i-2}\|\nabla\delta u_{i-k-1}\|^2\tau\right).$$

Summing this up for $i=2,\dots,j$, keeping $2\leqslant j\leqslant n$, combining the result with (3.14) and multiplying it by τ, we get

$$\begin{aligned}\sum_{i=1}^{j}|\delta K_i|^2\tau \leqslant C\Bigg(&1 + \sum_{i=1}^{j}\|\delta u_i\|^2\tau + \sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau \\ &+ \sum_{i=2}^{j}\sum_{k=1}^{i-1}\|\delta u_k\|^2\tau^2 + \sum_{i=2}^{j}\sum_{k=1}^{i-1}\|\nabla\delta u_k\|^2\tau^2\Bigg).\end{aligned}$$

From this, together with the result of Lemma 3.1(ii), we conclude the proof. □

Lemma 3.3. *Let the assumptions of Lemma 3.2 be fulfilled. Moreover, assume that $h\in C^1\left([0,T],L^2(\Omega)\right)\cap C\left([0,T],H^1(\Omega)\right)$. Then positive constants C and τ_0 exist such that, $\forall\,\tau<\tau_0$,*

(i) $\displaystyle\sum_{i=1}^{n}\|\nabla\delta u_i\|^2\tau + \max_{1\leqslant j\leqslant n}\|\Delta u_j\|^2 + \sum_{i=1}^{n}\|\Delta u_i - \Delta u_{i-1}\|^2 \leqslant C,$

(ii) $\displaystyle\sum_{i=1}^{n}|\delta K_i|^2\tau \leqslant C,$

(iii) $\displaystyle\max_{1\leqslant i\leqslant n}\|\delta u_i\| \leqslant C.$

Proof. (i) We set $\varphi = -\Delta\delta u_i\tau$ in (DPi1) and sum it up for $i=1,\dots,j$, keeping $1\leqslant j\leqslant n$. We get

$$\begin{aligned}&-\sum_{i=1}^{j}(\delta u_i,\Delta\delta u_i)\,\tau + \sum_{i=1}^{j}(\Delta u_i,\Delta\delta u_i)\,\tau \\ &= -\sum_{i=1}^{j}\left(\sum_{k=0}^{i-1}K_k\Delta u_{i-k}\tau,\Delta\delta u_i\right)\tau + \sum_{i=1}^{j}K_i\,(h_i,\Delta\delta u_i)\,\tau - \sum_{i=1}^{j}(f_{i-1},\Delta\delta u_i)\,\tau \\ &=: \sum_{k=1}^{3}S_k. \qquad (3.15)\end{aligned}$$

If we now apply the Green theorem on the first term in the LHS of (3.15), we obtain

$$-\sum_{i=1}^{j}(\delta u_i, \Delta\delta u_i)\,\tau = \sum_{i=1}^{j}\|\nabla\delta u_i\|^2\,\tau - \sum_{i=1}^{j}(\delta u_i, \delta g_i - \delta\alpha_{i-1})_\Gamma\,\tau.$$

A simple calculation, based on the Cauchy and Young inequalities, on (3.6), on the result from Lemma 3.1(ii), on the MVT and on the regularity of g, then gives us

$$\left|\sum_{i=1}^{j}(\delta u_i, \delta g_i)_\Gamma\,\tau\right| \leqslant C_\varepsilon + \varepsilon\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\,\tau.$$

Moreover, the same techniques, combined with the Lipschitz continuity of α, the regularity of the data and (3.12), lead to

$$\begin{aligned}
&\left|\sum_{i=1}^{j}(\delta u_i, \delta\alpha_{i-1})_\Gamma\,\tau\right| \leqslant C\left(\sum_{i=1}^{j}\|\delta u_i\|_\Gamma^2\,\tau + \sum_{i=1}^{j}\|\delta\alpha_{i-1}\|_\Gamma^2\,\tau\right)\\
&\overset{(3.6)}{\leqslant} \varepsilon\left(\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\,\tau + \|\nabla\delta u_0\|^2\,\tau\right) + C_\varepsilon\left(\sum_{i=1}^{j}\|\delta u_i\|^2\,\tau + \|\delta u_0\|^2\,\tau\right)\\
&\leqslant C_\varepsilon + \varepsilon\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\,\tau.
\end{aligned}$$

For the second term on the LHS of (3.15) we easily get

$$\sum_{i=1}^{j}(\Delta u_i, \Delta\delta u_i)\,\tau \overset{(3.3)}{=} \frac{1}{2}\left(\|\Delta u_j\|^2 - \|\Delta u_0\|^2 + \sum_{i=1}^{j}\|\Delta u_i - \Delta u_{i-1}\|^2\,\tau\right).$$

Now, note that

$$\delta\left(\sum_{k=0}^{i-1}K_k\Delta u_{i-k}\tau\right) = \delta\left(\sum_{k=1}^{i}K_{i-k}\Delta u_k\tau\right) = \sum_{k=1}^{i-1}\delta K_{i-k}\Delta u_k\tau + K_0\Delta u_i.$$

Using this, together with summation rule (3.5), we get

$$|S_1| = \left|\left(\sum_{k=0}^{j-1}K_k\Delta u_{j-k}\tau, \Delta u_j\right) - \sum_{i=1}^{j}\left(\sum_{k=1}^{i-1}\delta K_{i-k}\Delta u_k\tau + K_0\Delta u_i, \Delta u_{i-1}\right)\tau\right|.$$

The Cauchy and Young inequalities, Lemma 2.2(i) and Lemma 3.1(i) allow us to deduce that

$$\left|\left(\sum_{k=0}^{j-1}K_k\Delta u_{j-k}\tau, \Delta u_j\right)\right| \leqslant C_\varepsilon\sum_{k=1}^{j}\|\Delta u_k\|^2\,\tau + \varepsilon\,\|\Delta u_j\|^2 \leqslant C_\varepsilon + \varepsilon\,\|\Delta u_j\|^2.$$

In a similar way, using also the result of Lemma 3.2 and the regularity of u_0, we derive that

$$\begin{aligned}
&\left|\sum_{i=1}^{j}\left(\sum_{k=1}^{i-1}\delta K_{i-k}\Delta u_k\tau, \Delta u_{i-1}\right)\tau\right| \\
\leqslant &\sum_{i=1}^{j}\left(\sum_{k=1}^{i-1}|\delta K_{i-k}|^2\tau\right)^{\frac{1}{2}}\left(\sum_{k=1}^{i-1}\|\Delta u_k\|^2\tau\right)^{\frac{1}{2}}\|\Delta u_{i-1}\|\tau \\
\leqslant &\,\varepsilon\sum_{i=1}^{j}\sum_{k=1}^{i}|\delta K_k|^2\tau^2 + C_\varepsilon\sum_{i=1}^{j}\|\Delta u_{i-1}\|^2\tau \\
\leqslant &\, C_\varepsilon + \varepsilon\sum_{i=1}^{j}\left(1+\sum_{k=1}^{i}\|\nabla\delta u_k\|^2\tau\right)\tau \leqslant C_\varepsilon + \varepsilon\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau.
\end{aligned}$$

Furthermore, using the Cauchy inequality, the regularity of u_0, Lemma 2.2(i) and Lemma 3.1(i), we find out that

$$\left|\sum_{i=1}^{j}(K_0\Delta u_i, \Delta u_{i-1})\tau\right| \leqslant C\left(\sum_{i=1}^{j}\|\Delta u_i\|^2\tau + \sum_{i=1}^{j}\|\Delta u_{i-1}\|^2\tau\right) \leqslant C.$$

Thus, S_1 can be bounded above as

$$|S_1| \leqslant \varepsilon\|\Delta u_j\|^2 + \varepsilon\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau + C_\varepsilon.$$

To find an upper bound for S_2, we first apply the Green theorem. Next, we use the triangle and Cauchy inequalities, Lemma 2.2(i), the Young and trace inequalities, the Lipschitz continuity of α, the MVT, the regularity of the data, (3.12) and the result of Lemma 3.1(ii). We observe that

$$\begin{aligned}
|S_2| &= \left|-\sum_{i=1}^{j}K_i(\nabla h_i, \nabla\delta u_i)\tau + \sum_{i=1}^{j}K_i(h_i, \delta g_i - \delta\alpha_{i-1})_\Gamma\tau\right| \\
&\leqslant \sum_{i=1}^{j}|K_i|\,\|\nabla h_i\|\,\|\nabla\delta u_i\|\tau + C\sum_{i=1}^{j}|K_i|\,\|h_i\|_{H^1(\Omega)}\Big(\|\delta g_i\|_\Gamma + \|\delta u_{i-1}\|_\Gamma\Big)\tau \\
&\overset{(3.6)}{\leqslant} C_\varepsilon\left(1+\sum_{i=1}^{j}\|\delta u_{i-1}\|^2\tau\right) + \varepsilon\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau \\
&\leqslant C_\varepsilon + \varepsilon\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau.
\end{aligned}$$

Finally, the triangle, Cauchy and Young inequalities, even as the MVT, the boundedness and Lipschitz continuity of f and the results of Lemma 3.1 imply that

$$\begin{aligned}|S_3| &\stackrel{(3.5)}{=} \left|(f_j,\Delta u_j)-(f_0,\Delta u_0)-\sum_{i=1}^{j}(\Delta u_i,\delta f_i)\,\tau\right| \\ &\leqslant C_\varepsilon\left(1+\sum_{i=1}^{j}\|\Delta u_i\|^2\tau+\sum_{i=1}^{j}\|\delta u_i\|^2\tau\right)+\varepsilon\|\Delta u_j\|^2 \\ &\leqslant C_\varepsilon+\varepsilon\|\Delta u_j\|^2 .\end{aligned}$$

Collecting all partial results and using the regularity of u_0 yield

$$\begin{aligned}&\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau+\frac{1}{2}\left(\|\Delta u_j\|^2+\sum_{i=1}^{j}\|\Delta u_i-\Delta u_{i-1}\|^2\tau\right) \\ \leqslant &C_\varepsilon+\varepsilon\left(\sum_{i=1}^{j}\|\nabla\delta u_i\|^2\tau+\|\Delta u_j\|^2\right).\end{aligned}$$

Now, fixing ε sufficiently small, we conclude the proof.
(ii) This is a direct consequence of Lemmas 3.2 and 3.3(i).
(iii) Following more or less the same lines as the proof of Lemma 3.1(ii) and using the result of lemma 3.3(i), we immediately obtain

$$\|\delta u_i\|\leqslant C+C\|\Delta u_i\|+\sum_{k=1}^{i-1}|K_k|\,\|\Delta u_{i-k}\|\,\tau\leqslant C.$$

This completes the proof of the lemma. □

Some simple calculations, combined with the result of Lemma 3.3(iii), give

$$\begin{aligned}&\|\overline{u}_\tau(t-\tau)-\overline{u}_\tau(t)\|+\|\overline{u}_\tau(t)-u_\tau(t)\| \\ \leqslant &\|\partial_t u_\tau(t)\|\,\tau\leqslant C\tau, \qquad \forall t\in(0,T].\end{aligned} \tag{3.16}$$

Note that this inequality is also valid for $t=0$ if $\delta u_0\in L^2(\Omega)$. Analogously, it holds that

$$\|\nabla\overline{u}_\tau(t-\tau)-\nabla\overline{u}_\tau(t)\|+\|\nabla\overline{u}_\tau(t)-\nabla u_\tau(t)\|\leqslant\|\partial_t\nabla u_\tau(t)\|\,\tau, \qquad \forall t\in(0,T].$$

Based on the result of Lemma 3.3(i), this yields

$$\int_0^T\left(\|\nabla\overline{u}_\tau(t-\tau)-\nabla\overline{u}_\tau(t)\|^2+\|\nabla\overline{u}_\tau(t)-\nabla u_\tau(t)\|^2\right)\leqslant C\tau^2. \tag{3.17}$$

From the trace theorem then follows that

$$\int_0^T\left(\|\overline{u}_\tau(t-\tau)-\overline{u}_\tau(t)\|_\Gamma^2+\|\overline{u}_\tau(t)-u_\tau(t)\|_\Gamma^2\right)\leqslant C\tau^2. \tag{3.18}$$

After these preliminary remarks, the error analysis can be performed. First, the following notations are introduced:

$$e_u := u_\tau - u, \qquad e_{\overline{K}} := \overline{K}_\tau - K.$$

Analogously, $e_{\overline{g}}$, $e_{\overline{h}}$ and $e_{\overline{m'}}$ are defined. The following frequently used estimates for the convolution term are needed, see [18, Proposition 2.1]:

Lemma 3.4. *Set $I = (0,\eta)$, $\eta > 0$. Suppose $\kappa \in L^2(I)$ and $\upsilon \in L^2(I, L^2(\Omega))$, then it holds that*

(i) $\|\kappa * \upsilon\|^2 \leqslant \kappa^2 * \|\upsilon\|^2$,

(ii) $\displaystyle\int_0^\eta \|\kappa * \upsilon\|^2 \leqslant \int_0^\eta |\kappa|^2 \int_0^\eta \|\upsilon\|^2 .$

Before the derivation of the error estimates, a bound on $e_{\overline{K}}$ in $L^2(0,T)$ is proved in the following Lemma.

Lemma 3.5. *Let the conditions of Lemma 3.3 be fulfilled. Then positive constants C and τ_0 exist such that, $\forall\, \tau < \tau_0$, $\forall\, \eta \in [0,T]$ and for every $\varepsilon > 0$,*

$$\int_0^\eta \left|\overline{K}_\tau - K\right|^2 \leqslant C_\varepsilon \left(\tau^2 + \int_0^\eta \|u_\tau - u\|^2\right) + \varepsilon \int_0^\eta \|\nabla u_\tau - \nabla u\|^2 .$$

Proof. First, we subtract (MP) from (DMP) and we use (2.5) en (2.6) to get

$$\begin{aligned} e_{\overline{K}} \left(\overline{h}_\tau, 1\right) =& \left(f(\overline{u}_\tau(t-\tau)) - f(u(t)), 1\right) - \left(\alpha(\overline{u}_\tau(t-\tau)) - \alpha(u(t)), 1\right)_\Gamma \\ & - e_{\overline{m'}} + \left(e_{\overline{g}}, 1\right)_\Gamma + \left(e_{\overline{K}} * \overline{g}_\tau, 1\right)_\Gamma + \left(K * e_{\overline{g}}, 1\right)_\Gamma \\ & - \left(e_{\overline{K}} * \alpha(\overline{u}_\tau), 1\right)_\Gamma - \left(K * \left(\alpha(\overline{u}_\tau) - \alpha(u)\right), 1\right)_\Gamma \\ & - \left(\int_0^{\tau\lfloor t\rfloor_\tau} \overline{K}_\tau(s) \left[\alpha(\overline{u}_\tau(t-s-\tau)) - \alpha(\overline{u}_\tau(t-s))\right], 1\right)_\Gamma \\ & + \left(\int_{\tau\lfloor t\rfloor_\tau}^t \overline{K}_\tau(s) \left[\overline{g}_\tau(t-s) - \alpha(\overline{u}_\tau(t-s))\right], 1\right)_\Gamma \\ & + \left(K_0 \left(\overline{g}_\tau(t) - \alpha(\overline{u}_\tau(t-\tau))\right) \tau, 1\right)_\Gamma - K \left(e_{\overline{h}}, 1\right) . \end{aligned} \tag{3.19}$$

Next, we estimate all terms in the RHS of (3.19) from above. Clearly,

$$\left|e_{\overline{m'}}\right| + \left|\left(e_{\overline{g}}, 1\right)_\Gamma\right| + \left|K \left(e_{\overline{h}}, 1\right)\right| \leqslant C\tau,$$

by the Cauchy inequality, the MVT, Theorem 2.1(iii) and the regularity of m, g and h. Moreover, using the triangle and Cauchy inequalities, the Lipschitz continuity of f and α and (3.16), we obtain

$$\begin{aligned} & \left| \left(f(\overline{u}_\tau(t-\tau)) - f(u(t)), 1\right) - \left(\alpha(\overline{u}_\tau(t-\tau)) - \alpha(u(t)), 1\right)_\Gamma \right| \\ \leqslant & C\Big(\tau + \|e_u(t)\| + \|\overline{u}_\tau(t-\tau) - u(t)\|_\Gamma \Big). \end{aligned}$$

Now, applying the triangle and Cauchy inequalities, the regularity of g and the boundedness of α, we immediately derive that

$$\left|\left(e_{\overline{K}} * \overline{g}_\tau, 1\right)_\Gamma - \left(e_{\overline{K}} * \alpha(\overline{u}_\tau), 1\right)_\Gamma\right| \leqslant C \int_0^t \left|e_{\overline{K}}\right|.$$

Furthermore, The Cauchy inequality, Theorem 2.1(iii), the MVT and the regularity of g allow us to deduce that

$$\left|(K * e_{\overline{g}}, 1)_\Gamma\right| \leqslant C \int_0^t |K(t-s)| \, \|e_{\overline{g}}(s)\|_\Gamma \,\mathrm{d}s \leqslant C\tau.$$

Next, applying the Cauchy inequality, Theorem 2.1(iii) and the Lipschitz continuity of α once more, we may write that

$$\begin{aligned}
&\left|\left(K * (\alpha(\overline{u}_\tau) - \alpha(u)), 1\right)_\Gamma\right| \\
\leqslant & C \int_0^t |K(t-s)| \, \|\overline{u}_\tau(s) - u(s)\|_\Gamma \,\mathrm{d}s \leqslant C \int_0^t \|\overline{u}_\tau - u\|_\Gamma .
\end{aligned}$$

Moreover, Lemma 2.2(i), the regularity of g and the boundedness of α immediately give that

$$\begin{aligned}
&\left|\left(\int_{\tau\lfloor t \rfloor_\tau}^t \overline{K}_\tau(s)\left[\overline{g}_\tau(t-s) - \alpha(\overline{u}_\tau(t-s))\right], 1\right)_\Gamma + \left(K_0\left(\overline{g}_\tau(t) - \alpha(\overline{u}_\tau(t-\tau))\right)\tau, 1\right)_\Gamma\right| \\
\leqslant & C\tau.
\end{aligned}$$

Finally, he Cauchy inequality, Lemma 2.2(i), the Lipschitz continuity of α and (3.18) lead to

$$\begin{aligned}
&\left|\left(\int_0^{\tau\lfloor t \rfloor_\tau} \overline{K}_\tau(s)\left[\alpha(\overline{u}_\tau(t-s-\tau)) - \alpha(\overline{u}_\tau(t-s))\right], 1\right)_\Gamma\right| \\
\leqslant & \left|\left(\int_{t-\tau\lfloor t \rfloor_\tau}^t \overline{K}_\tau(t-s)\left[\alpha(\overline{u}_\tau(s-\tau)) - \alpha(\overline{u}_\tau(s))\right], 1\right)_\Gamma\right| \\
\leqslant & C\sqrt{\int_{t-\tau\lfloor t \rfloor_\tau}^t \left|\overline{K}_\tau(t-s)\right|^2} \sqrt{\int_{t-\tau\lfloor t \rfloor_\tau}^t \|\overline{u}_\tau(s-\tau) - \overline{u}_\tau(s)\|_\Gamma^2} \\
\leqslant & C\tau.
\end{aligned}$$

Gathering all estimates, we obtain

$$\left|e_{\overline{K}}\left(\overline{h}_\tau, 1\right)\right| \leqslant C\left(\tau + \|e_u\| + \|\overline{u}_\tau(t-\tau) - u(t)\|_\Gamma + \int_0^t \|\overline{u}_\tau - u\|_\Gamma + \int_0^t \left|e_{\overline{K}}\right|\right).$$

The Grönwall lemma now yields

$$\left|e_{\overline{K}}\right| \leqslant C\left(\tau + \|e_u\| + \|\overline{u}_\tau(t-\tau) - u(t)\|_\Gamma + \int_0^t \|\overline{u}_\tau - u\|_\Gamma\right) + C\left(\int_0^t \|e_u\| + \int_0^t \|\overline{u}_\tau(t-\tau) - u(t)\|_\Gamma\right).$$

Squaring this result, integrating it in time and using (3.6), (3.16) and (3.17), we easily deduce that

$$\int_0^\eta \left|e_{\overline{K}}\right|^2 \leqslant C_\varepsilon\left(\tau^2 + \int_0^\eta \|e_u\|^2\right) + \varepsilon\int_0^\eta \|\nabla e_u\|^2.$$

Finally, the following theorem contains the error estimates. □

Theorem 3.1. *Let the conditions of Lemma* 3.5 *be fulfilled. Then positive constants C and τ_0 exist such that,* $\forall\, \tau < \tau_0$,

$$\int_0^T \left|\overline{K}_\tau - K\right|^2 + \int_0^T \|\nabla u_\tau - \nabla u\|^2 + \max_{\eta\in[0,T]} \|u_\tau(\eta) - u(\eta)\|^2 \leqslant C\tau^2.$$

Proof. First, we subtract (P) from (DP). Next, we put $\varphi = e_u$ and we integrate the result over $(0,\eta)$, $\eta \in [0,T]$. After some rearrangements in the terms and using (2.5), (2.6) and

$$\sum_{k=0}^{\lfloor t\rfloor_\tau} \overline{K}_\tau(t_k)\nabla\overline{u}_\tau(t-t_k)\tau = \overline{K}_\tau * \nabla\overline{u}_\tau + K_0\nabla\overline{u}_\tau(t)\tau - \int_{\tau\lfloor t\rfloor_\tau}^t \overline{K}_\tau(s)\nabla\overline{u}_\tau(t-s),$$

we obtain

$$\begin{aligned}
&\int_0^\eta (\partial_t e_u, e_u) + \int_0^\eta \|\nabla e_u\|^2 + \int_0^\eta \left(\nabla\left(\overline{u}_\tau - u_\tau\right), \nabla e_u\right) + \int_0^\eta e_{\overline{K}}\left(\overline{h}_\tau, e_u\right) \\
&\quad + \int_0^\eta K\left(e_{\overline{h}}, e_u\right) + \int_0^\eta \left(\overline{K}_\tau * \nabla\left(\overline{u}_\tau - u_\tau\right), \nabla e_u\right) + \int_0^\eta \left(\overline{K}_\tau * \nabla e_u, \nabla e_u\right) \\
&\quad + \int_0^\eta \left(e_{\overline{K}} * \nabla u, \nabla e_u\right) + \int_0^\eta \left(K_0\nabla\overline{u}_\tau(t)\tau, \nabla e_u\right) - \int_0^\eta \left(\int_{\tau\lfloor t\rfloor_\tau}^t \overline{K}_\tau(s)\nabla\overline{u}_\tau(t-s), \nabla e_u\right) \\
&= \int_0^\eta \left(f\left(\overline{u}_\tau(t-\tau)\right) - f\left(u(t)\right), e_u\right) + \int_0^\eta \left(e_{\overline{g}}, e_u\right)_\Gamma - \int_0^\eta \left(\alpha\left(\overline{u}_\tau(t-\tau)\right) - \alpha\left(u(t)\right), e_u\right)_\Gamma \\
&\quad + \int_0^\eta \left(e_{\overline{K}} * \overline{g}_\tau, e_u\right)_\Gamma + \int_0^\eta \left(K * e_{\overline{g}}, e_u\right)_\Gamma - \int_0^\eta \left(e_{\overline{K}} * \alpha(\overline{u}_\tau), e_u\right)_\Gamma \\
&\quad - \int_0^\eta \left(K * \left(\alpha(\overline{u}_\tau) - \alpha(u)\right), e_u\right)_\Gamma + \int_0^\eta \left(K_0\left(\overline{g}_\tau(t) - \alpha(\overline{u}_\tau(t-\tau))\right)\tau, e_u\right)_\Gamma \\
&\quad - \int_0^\eta \left(\int_0^{\tau\lfloor t\rfloor_\tau} \overline{K}_\tau(s)\left[\alpha(\overline{u}_\tau(t-s-\tau)) - \alpha(\overline{u}_\tau(t-s))\right], e_u\right)_\Gamma \\
&\quad - \int_0^\eta \left(\int_{\tau\lfloor t\rfloor_\tau}^t \overline{K}_\tau(s)\left[\overline{g}_\tau(t-s) - \alpha(\overline{u}_\tau(t-s))\right], e_u\right)_\Gamma =: \sum_{k=1}^{10} T_k.
\end{aligned} \tag{3.20}$$

For the first term in the LHS of (3.20), we immediately find that

$$\int_0^\eta (\partial_t e_u, e_u) = \frac{1}{2}\|e_u(\eta)\|^2.$$

In what's next, we estimate the absolute value of the other terms in the LHS of (3.20), with exception of the second one, and every term in the RHS of (3.20) from above. The Cauchy inequality, the regularity of h, the Young inequality and the result of Lemma 3.5 give us

$$\left|\int_0^\eta e_{\overline{K}}\left(\overline{h}_\tau, e_u\right)\right| \leqslant C_\varepsilon\left(\tau^2+\int_0^\eta \|e_u\|^2\right)+\varepsilon\int_0^\eta \|\nabla e_u\|^2.$$

Moreover, the Cauchy and Young inequalities, Theorem 2.1(iii), the MVT and the regularity of h and g lead to

$$\begin{aligned}\left|\int_0^\eta K\left(e_{\overline{h}}, e_u\right)\right|+|T_2| &\leqslant C\left(\int_0^\eta \left(K\left\|e_{\overline{h}}\right\|\right)^2+\int_0^\eta \|e_u\|^2+\int_0^\eta \|e_{\overline{g}}\|_\Gamma^2+\int_0^\eta \|e_u\|_\Gamma^2\right)\\ &\overset{(3.6)}{\leqslant} C_\varepsilon\left(\tau^2+\int_0^\eta \|e_u\|^2\right)+\varepsilon\int_0^\eta \|\nabla e_u\|^2.\end{aligned}$$

Furthermore, we have

$$\left|\int_0^\eta \left(\nabla\left(\overline{u}_\tau-u_\tau\right), \nabla e_u\right)\right| \leqslant C_\varepsilon\tau^2+\varepsilon\int_0^\eta \|\nabla e_u\|^2,$$

$$\begin{aligned}&\left|\int_0^\eta \left(\overline{K}_\tau * \nabla\left(\overline{u}_\tau-u_\tau\right), \nabla e_u\right)\right| \leqslant C_\varepsilon\int_0^\eta \left\|\overline{K}_\tau * \nabla\left(\overline{u}_\tau-u_\tau\right)\right\|^2+\varepsilon\int_0^\eta \|\nabla e_u\|^2\\ \leqslant &C_\varepsilon\int_0^\eta \left|\overline{K}_\tau\right|^2\int_0^\eta \|\nabla\left(\overline{u}_\tau-u_\tau\right)\|^2+\varepsilon\int_0^\eta \|\nabla e_u\|^2 \leqslant C_\varepsilon\tau^2+\varepsilon\int_0^\eta \|\nabla e_u\|^2,\end{aligned}$$

and

$$\begin{aligned}&\left|\int_0^\eta \left(\overline{K}_\tau * \nabla e_u, \nabla e_u\right)\right| \leqslant C_\varepsilon\int_0^\eta \left\|\overline{K}_\tau * \nabla e_u\right\|^2+\varepsilon\int_0^\eta \|\nabla e_u\|^2\\ \leqslant &C_\varepsilon\int_0^\eta\int_0^t \left|\overline{K}_\tau(t-s)\right|^2\|\nabla e_u(s)\|^2\,\mathrm{d}s\mathrm{d}t+\varepsilon\int_0^\eta \|\nabla e_u\|^2\\ \leqslant &C_\varepsilon\int_0^\eta\int_0^t \|\nabla e_u\|^2+\varepsilon\int_0^\eta \|\nabla e_u\|^2,\end{aligned}$$

where we combined the Cauchy and Young inequalities with (3.17) in the first estimate, with Lemma 3.4(ii), Lemma 2.2(i) and (3.17) in the second estimate and with Lemma 3.4(i) and Lemma 2.2(i) in the last estimate. Taking into account the results from

Lemma 3.4(ii) and Lemma 3.5 and the regularity of the solution, we deduce in a similar way that

$$\begin{aligned}&\left|\int_0^\eta (e_{\overline{K}} * \nabla u, \nabla e_u)\right|\\ \leqslant& C_\varepsilon \int_0^\eta |e_{\overline{K}}|^2 \int_0^\eta \|\nabla u\|^2 + \varepsilon \int_0^\eta \|\nabla e_u\|^2\\ \leqslant& C_\varepsilon \left(\tau^2 + \int_0^\eta \|e_u\|^2\right) + \varepsilon \int_0^\eta \|\nabla e_u\|^2 .\end{aligned}$$

For the last two terms in the LHS of (3.20), we deduce from the Cauchy inequality, Lemma 2.2(i), Lemma 3.1(i) and the Young inequality that

$$\begin{aligned}&\left|\int_0^\eta (K_0 \nabla \overline{u}_\tau(t)\tau, \nabla e_u) - \int_0^\eta \left(\int_{\tau\lfloor t\rfloor_\tau}^t \overline{K}_\tau(s)\nabla\overline{u}_\tau(t-s), \nabla e_u\right)\right|\\ \leqslant& C_\varepsilon \tau^2 + \varepsilon \int_0^\eta \|\nabla e_u\|^2 .\end{aligned}$$

Further, it is easy to see that

$$\begin{aligned}|T_1| &\leqslant C\left(\tau^2 + \int_0^\eta \|e_u\|^2\right),\\ |T_3| &\leqslant C\left(\int_0^\eta \|\overline{u}_\tau(t-\tau) - u_\tau(t)\|_\Gamma^2 + \int_0^\eta \|e_u\|_\Gamma^2\right)\\ &\leqslant C_\varepsilon\left(\tau^2 + \int_0^\eta \|e_u\|^2\right) + \varepsilon \int_0^\eta \|\nabla e_u\|^2 ,\end{aligned}$$

for which we used standard techniques, together with the Lipschitz continuity of f and (3.16) in the first estimate and together with the Lipschitz continuity of α, the triangle inequality, (3.18) and (3.6) in the second estimate. Moreover, from the Cauchy and Young inequalities, Lemma 3.4(ii), the regularity of g and Lemma 3.5, we successively deduce that

$$\begin{aligned}|T_4| &\leqslant C\left(\int_0^\eta \|e_{\overline{K}} * \overline{g}_\tau\|_\Gamma^2 + \int_0^\eta \|e_u\|_\Gamma^2\right)\\ &\overset{(3.6)}{\leqslant} C\int_0^\eta \|e_{\overline{K}} * \overline{g}_\tau\|_\Gamma^2 + \varepsilon \int_0^\eta \|\nabla e_u\|^2 + C_\varepsilon \int_0^\eta \|e_u\|^2\\ &\leqslant C\int_0^\eta |e_{\overline{K}}|^2 \int_0^\eta \|\overline{g}_\tau\|_\Gamma^2 + \varepsilon \int_0^\eta \|\nabla e_u\|^2 + C_\varepsilon \int_0^\eta \|e_u\|^2\\ &\leqslant C_\varepsilon\left(\tau^2 + \int_0^\eta \|e_u\|^2\right) + \varepsilon \int_0^\eta \|\nabla e_u\|^2 .\end{aligned}$$

Likewise, using also Theorem 2.1(iii), the boundedness and Lipschitz continuity of α, the MVT, the triangle inequality, (3.16) and (3.17), we find that

$$\left|\sum_{k=5}^{7} T_k\right| \leqslant C_\varepsilon \left(\tau^2 + \int_0^\eta \|e_u\|^2\right) + \varepsilon \int_0^\eta \|\nabla e_u\|^2 .$$

For the estimation of T_8 and T_{10}, we use the Cauchy and triangle inequalities, the regularity of g, the boundedness of α, Lemma 2.2(i), the Young inequality and (3.6) to obtain

$$|T_8 + T_{10}| \leqslant C_\varepsilon \left(\tau^2 + \int_0^\eta \|e_u\|^2\right) + \varepsilon \int_0^\eta \|\nabla e_u\|^2 .$$

Finally,

$$\begin{aligned}|T_9| &\leqslant C \int_0^\eta \int_{t-\tau\lfloor t\rfloor_\tau}^{t} \|\overline{u}_\tau(s-\tau) - \overline{u}_\tau(s)\|_\Gamma^2 + C \int_0^\eta \int_{t-\tau\lfloor t\rfloor_\tau}^{t} \|e_u\|_\Gamma^2 \\ &\leqslant C_\varepsilon \left(\tau^2 + \int_0^\eta \|e_u\|^2\right) + \varepsilon \int_0^\eta \|\nabla e_u\|^2 ,\end{aligned}$$

which follows from the Cauchy inequality, Lemma 2.2(i), the Young inequality, (3.18) and (3.6). Collecting all estimates, we arrive at

$$\int_0^\eta \|\nabla e_u\|^2 + \frac{1}{2}\|e_u\|^2 \leqslant C_\varepsilon \left(\tau^2 + \int_0^\eta \|e_u\|^2 + \int_0^\eta \int_0^t \|\nabla e_u\|^2\right) + \varepsilon \int_0^\eta \|\nabla e_u\|^2 .$$

Fixing ε small enough and using the Grönwall lemma, we eventually obtain that

$$\int_0^\eta \|\nabla e_u\|^2 + \|e_u\|^2 \leqslant C\tau^2,$$

which is valid for every $\eta \in [0,T]$. This, combined with Lemma 3.5, concludes the proof. □

The following corollary follows from Lemma 2.2, Lemma 3.3(ii) and the Arzelà-Ascoli theorem [24, Theorem 1.5.3].

Corollary 3.1. *Let the assumptions of Theorem* 3.1 *be fulfilled. Then* $\overline{K}_\tau \to K$ *in* $L^2(0,T)$. *Moreover,* $K_\tau \to K$ *in* $C([0,T])$ *with* $\partial_t K \in L^2(0,T)$ *and there exists a positive constant* C *such that*

$$\int_0^T |K_\tau(t) - K(t)|^2 \, dt \leqslant C\tau^2.$$

4. Numerical experiment

The aim of the following simulations is to demonstrate the established error estimate in Corollary 3.1. For the implementaion, the finite element library DOLFIN [25, 26] from the FEniCS project [27] is used.

In every experiment, the domain Ω equals the unit interval. The number of time discretization intervals is chosen to be $n = 2^j, j = 3, \ldots, 8$, such that the time step τ for the equidistant time partitioning equals $2^{-j}T, j = 3, \ldots, 8$ respectively. At every discrete time step, the resulting elliptic problems (see step 6 in the algorithm) are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. For this space discretization, a fixed uniform mesh of 50 intervals is used. The L^2-error between the numerical and exact kernel is approximated by the Simpson's rule for the several values of the timestep τ:

$$
\begin{aligned}
E_K(\tau) &= \int_0^T |K_\tau(t) - K(t)|^2 \,\mathrm{d}t \\
&\approx \sum_{i=1}^{n} \frac{\tau}{6}\left[(K_{i-1} - K(t_{i-1}))^2 + 4\left(\frac{K_{i-1}+K_i}{2} - K\left(\frac{t_{i-1}+t_i}{2}\right)\right)^2 + (K_i - K(t_i))^2\right].
\end{aligned}
$$

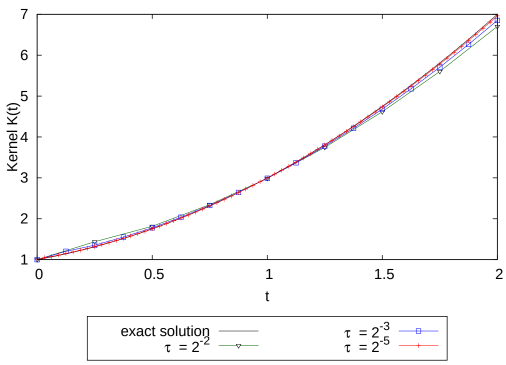
4.1. Experiment 1

For the first experiment, we prescribe the exact solution as

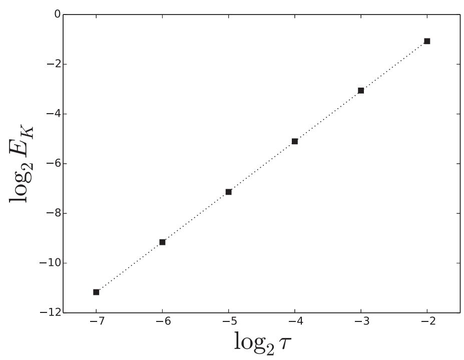
$$u(x,t) = \left(1 + t + t^2\right)\left(\cos(\pi x) + 1\right), \tag{4.1a}$$

$$K(t) = 1 + t + t^2, \qquad x \in [0,1], \quad t \in [0,2]. \tag{4.1b}$$

The functions h, f and α are given by respectively $h(x,t) = 2x + 2t + 1$, $f(s) = s + 5$ and $\alpha(s) = s - 2$. In Figure 1(a) the exact kernel K is compared with the numerical



(a) The exact solution (4.1b) is approximated better as τ becomes smaller



(b) The error $E_K(\tau)$ decreases with decreasing time step

Figure 1: Kernel reconstruction in Experiment 1.

solution for $\tau = 2^{-2}$, 2^{-3} and $\tau = 2^{-5}$. It can be seen that the approximations become better as the time step τ decreases. We draw the same conclusion from Figure 1(b), where $\log_2 E_K$ is plotted as a function of $\log_2 \tau$. The linear regression line through the data points is given by $\log_2 E_K = 2.0234 \log_2 \tau + 2.9896$. This is in accordance with the predicted convergence rate in Corollary 3.1.

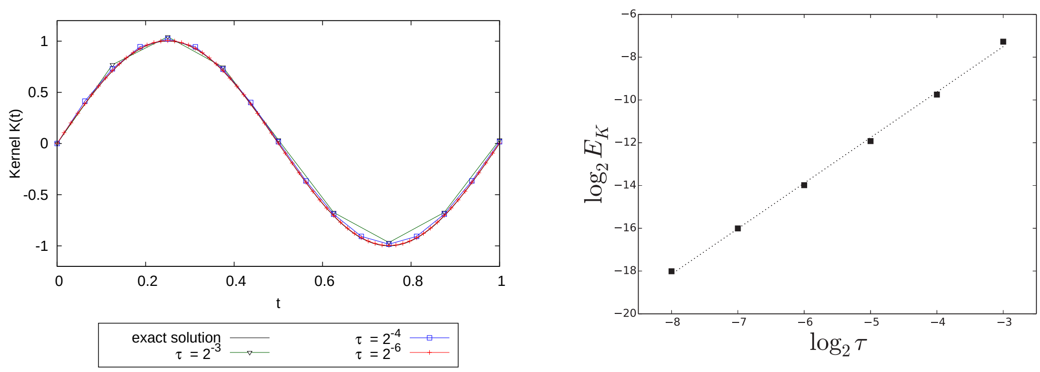
4.2. Experiment 2

In the second experiment, the unknown kernel is sinusoidal, i.e.

$$u(x,t) = \left(1 + t + t^2\right)\left(\cos(\pi x) + 1\right), \tag{4.2a}$$

$$K(t) = \sin(2\pi t), \qquad x \in [0,1], \quad t \in [0,1]. \tag{4.2b}$$

The functions h, f and α are given by respectively $h(x,t) = 2x + 2t + 1$, $f(s) = s - 5$ and $\alpha(s) = s + 2$. In Figures 2(a)–2(b) the results of the numerical experiment are depicted. They can be interpreted analogously as in the first experiment. The linear regression line through the data points in Figure 2(b) is given by $\log_2 E_K = 2.1290 \log_2 \tau - 1.1151$, which also supports the theoretically obtained convergence rate in Corollary 3.1.



(a) The exact solution (4.2b) is approximated better as τ becomes smaller

(b) The error $E_K(\tau)$ decreases with decreasing time step

Figure 2: Kernel reconstruction in Experiment 2.

5. Conclusion

The semilinear parabolic problem (1.1) of second order with an unknown solely time-dependent convolution kernel K has been considered. The numerical scheme from [18] for reconstructing the unknown convolution kernel from the additional integral measurement (1.2) has been adapted such that higher stability results are valid. Using these estimates, it has been proved that, under appropriate conditions of the data, the convergence of the numerical approximations K_τ and $\overline{K}_\tau$ to K is of first order in time:

$$\|K_\tau(t) - K(t)\|_{L^2(0,T)} + \|\overline{K}_\tau(t) - K(t)\|_{L^2(0,T)} \approx \mathcal{O}(\tau).$$

This means that strong convergence of the approximations to the kernel K has been obtained, whereas in [18] this convergence had only been proved in weak sense. Two numerical experiments have supported the theoretically obtained result. Finally, it is worth pointing out that the techniques used in this article might be applicable to other problems in other settings.

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