Local Convergence for a Fifth Order Traub-Steffensen-Chebyshev-Like Composition Free of Derivatives in Banach Space

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Abstract. We present the local convergence analysis of a fifth order Traub-Steffensen-Chebyshev-like composition for solving nonlinear equations in Banach spaces. In earlier studies, hypotheses on the Fréchet derivative up to the fifth order of the operator under consideration is used to prove the convergence order of the method although only divided differences of order one appear in the method. That restricts the applicability of the method. In this paper, we extended the applicability of the fifth order Traub-Steffensen-Chebyshev-like composition without using hypotheses on the derivatives of the operator involved. Our convergence conditions are weaker than the conditions used in earlier studies. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

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1. Introduction

We are concerned with the problem of approximating a solution $x^*$ of the equation

$$F(x) = 0, \quad (1.1)$$

where $F : D \subseteq \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a Fréchet differentiable operator between Banach spaces $\mathcal{B}_1, \mathcal{B}_2$. Most of the solution methods for solving (1.1) are iterative and for iterative methods order of convergence is an important issue. Convergence analysis of higher order iterative methods require assumptions on the higher order Fréchet derivatives of the operator $F$. That restricts the applicability of these methods. In this study, we consider the local
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convergence for fifth order Traub-Steffensen-Chebyshev-like composition free of derivatives in Banach space studied by Sharma and Kumar in [16]. The method is defined by

\[ \begin{align*}
  y_n &= x_n - A_n^{-1} F(x_n), \\
  z_n &= y_n - A_n^{-1} F(y_n), \\
  x_{n+1} &= z_n - B_n A_n^{-1} F(z_n),
\end{align*} \]

(1.2a)

(1.2b)

(1.2c)

where

\[ \begin{align*}
  A_n &= [w_n, x_n; F], \\
  B_n &= 2I - A_n^{-1} [z_n, y_n; F], \\
  w_n &= x_n + \beta F(x_n),
\end{align*} \]

and \([, ; F]\) is a divided difference of order one on \(D^2\).

Throughout this paper \(L(\mathcal{B}_2, \mathcal{B}_1)\) denotes the set of bounded linear operators between \(\mathcal{B}_1\) and \(\mathcal{B}_2\) and \(B(z, \rho), \overline{B}(z, \rho)\) stand, respectively for the open and closed balls in \(\mathcal{B}_1\) with center \(z \in \mathcal{B}_1\) and of radius \(\rho > 0\).

The motivations for the construction of this method are that is derivative free, of convergence order five and efficient. Notice that in [16] favorable comparisons with other methods using similar information have been provided to show the advantages of the proposed method. The aim of this paper is not to present those comparisons but to extend the applicability of method (1.2) in the more general setting of a Banach space. We refer the reader to [16] for more detailed advantages, motivations and comparisons.

Convergence analysis in [16] is based on Taylor expansions and assumptions on the Fréchet derivative \(F\) up to the order five when \(\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^2\). That limits the applicability of this method. As a motivational example, let us define function \(F\) on \(I = [-\pi/2, \pi/2]\) by

\[ F(x) = \begin{cases} 
  x^5 \sin \frac{1}{x} + x^2 + x, & x \neq 0, \\
  0, & x = 0.
\end{cases} \]

Then, \(x^* = 0\), is a solution of \(F(x) = 0\). We have that

\[ \begin{align*}
  F'(x) &= 5x^4 \sin \frac{1}{x} - x^3 \sin \frac{1}{x} + 2x + 1, \\
  F''(x) &= 20x^3 \sin \frac{1}{x} - 5x^2 \cos \frac{1}{x} - 3x^2 \cos \frac{1}{x} - x \sin \frac{1}{x} + 2, \\
  F'''(x) &= 60x^2 \sin \frac{1}{x} - 36x \cos \frac{1}{x} + \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.
\end{align*} \]

Then, obviously function \(F\) does not have bounded third derivative in \(I\). In this study we use only assumptions on the first Fréchet derivative of the operator \(F\) in our convergence analysis, so that the method (1.2) can be applied to solve equations but the earlier results cannot be applied [1–18] (see Example 3.2). Moreover, we avoid the usage of high order derivatives, since we rely on the computational and approximate computational order of
convergence that do not require high order derivatives (see Remark 2.2). Notice that finding the radius of convergence is important because it shows the difficulty in obtaining close enough initial points, which otherwise constitute a “shot in the dark”. Our new tools of Lipschitz-type conditions allow us to compute a radius of convergence and computable error bounds not provided in [16]. This technique can be used on other iterative methods [9–11, 16–18].

The rest of the paper is structured as follows. In Section 2 we present the local convergence analysis of the method (1.2). We also provide a radius of convergence, computable error bounds and a uniqueness result. Numerical examples are given in the concluding Section 3.

2. Local convergence

It is convenient for the local convergence analysis of method (1.2) to define some parameter and scalar functions. Let \( \beta \in \mathbb{R} \) be a parameter and let \( \omega_0 : [0, +\infty)^2 \to [0, +\infty) \) be a function continuous nondecreasing with \( \omega_0(0, 0) = 0 \). Define parameter \( r_0 \) by

\[
    r_0 = \sup \{ t \geq 0 : \omega_0(\delta t, t) < 1 \}.
\]  

(2.1)

Let \( v : [0, r_0) \to [0, +\infty), \omega_1 : [0, r_0)^2 \to [0, +\infty) \) be continuous and nondecreasing functions. Define functions \( g_1 \) and \( h_1 \) on the interval \([0, r_0)\) by

\[
    g_1(t) = \frac{\omega_1(|\beta|v_0(t, t))}{1 - \omega_0(\delta t, t)}, \quad h_1(t) = g_1(t) - 1.
\]

Suppose that

\[
    \omega_1(0, 0) < 1.
\]  

(2.2)

Using (2.1) and (2.2), we get that \( h_1(0) = \frac{\omega_1(0, 0)}{1 - \omega_0(0, 0)} - 1 < 0 \) and \( h_1(t) \to +\infty \) as \( t \to r_0^- \). Then, by intermediate value theorem equation \( h_1(t) = 0 \) has solutions in \((0, r_0)\). Denote by \( r_1 \) the smallest such solution. Let \( v : [0, r_0) \to [0, +\infty) \) be a continuous and nondecreasing function. Define functions \( g_2 \) and \( h_2 \) on the interval \([0, r_0)\) by

\[
    g_2(t) = \left( 1 + \frac{v(g_1(t))}{1 - \omega(\delta t, t)} \right) g_1(t), \quad h_2(t) = g_2(t) - 1.
\]  

(2.3)

Suppose that

\[
    (1 + v(0))\omega_1(0, 0) < 1.
\]  

(2.4)

We get that \( h_2(0) < 0 \) and \( h_2(t) \to +\infty \) as \( t \to r_0^- \). Denote by \( r_2 \) the smallest solution of equation \( h_2(t) = 0 \) in the interval \((0, r_0)\). Let \( \omega : [0, r_0)^2 \to [0, +\infty) \) be a continuous and
nondecreasing function. Define functions $b, g_3$ and $h_3$ on the interval $[0, r_0)$ by

$$b(t) = \frac{1 + \omega((\delta + g_2(t))t, (1 + g_1(t))t)}{1 - \omega_0(\delta t, t)},$$

$$g_3(t) = \left(1 + \frac{b(t)v(g_2(t)t)}{1 - \omega_0(\delta t, t)}\right)g_2(t),$$

$$h_3(t) = g_3(t) - 1.$$

Suppose that

$$(1 + (1 + \omega(0, 0)v(0))(1 + v(0))\omega_1(0, 0) < 1. \quad (2.5)$$

We obtain by (2.5) that $h_3(0) < 0$ and $h_3(t) \to +\infty$ as $t \to r_0^-$. Denote by $r_3$ the smallest solution of equation $h_3(t) = 0$ in the interval $(0, r_0)$. Define the radius of convergence $r$ by

$$r = \min\{r_i\}, \quad i = 1, 2, 3. \quad (2.6)$$

Then, for each $t \in [0, r)$ we have:

$$0 \leq g_i(t) < 1, \quad (2.7a)$$

$$0 \leq \omega_0(\delta t, t) < 1. \quad (2.7b)$$

Finally define parameter $R^*$ by

$$R^* = \max\{r, \delta r\}. \quad (2.8)$$

Next, the local convergence analysis of method (1.2) is presented.

**Theorem 2.1.** Let $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a continuous operator and let $[., .; F] : \Omega^2 \to L(\mathcal{B}_1, \mathcal{B}_2)$ be a divided difference of order one on $\Omega^2$ for $F$. Suppose: there exists $x^* \in \Omega$ and function $\omega_0 : [0, +\infty)^2 \to [0, +\infty)$ continuous and nondecreasing such that for each $x, y, \in \Omega$,

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \quad (2.9a)$$

$$\|F'(x^*)^{-1}([x, y; F] - F(x^*))\| \leq \omega_0(\|x - x^*\|, \|y - x^*\|). \quad (2.9b)$$

Let $\Omega_0 = \Omega \cap B(x^*, r_0)$. There exist $\delta \geq 0$ functions $v, v_0 : [0, r_0) \to [0, +\infty)$, $\omega, \omega_1 : [0, r_0)^2 \to [0, +\infty)$, such that for each $x, y, z, u \in \Omega_0$

$$\|I + [x, x^*; F]\| \leq \delta, \quad (2.10a)$$

$$\|[x, x^*; F]\| \leq v_0(\|x - x^*\|), \quad (2.10b)$$

$$\|F'(x^*)^{-1}[x, x^*; F]\| \leq v(\|x - x^*\|), \quad (2.10c)$$

$$\|F'(x^*)^{-1}([x, y; F] - [y, x^*; F])\| \leq \omega_1(\|x - y\|, \|y - x^*\|), \quad (2.10d)$$

$$\|F'(x^*)^{-1}([x, y; F] - [z, u; F])\| \leq \omega(\|x - z\|, \|y - u\|), \quad (2.10e)$$

$$B(x^*, R^*) \subseteq \Omega, \quad (2.10f)$$
By the Banach perturbation Lemma

\[ \|y_n - x^*\| \leq g_1(\|x_n - x^*\|) \|x_n - x^*\| \leq \|x_n - x^*\| < r, \]  

(2.11a)

\[ \|z_n - x^*\| \leq g_2(\|x_n - x^*\|) \|x_n - x^*\| \leq \|x_n - x^*\|, \]  

(2.11b)

\[ \|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|) \|x_n - x^*\| \leq \|x_n - x^*\|, \]  

(2.11c)

where the functions \( g_i, i = 1, 2, 3 \) are defined previously. Furthermore, if there exists for \( R_1 \geq r \) such that

\[ \omega_0(R_1, 0) < 1 \text{ or } \omega_0(0, R_1) < 1, \]  

(2.12)

then the limit point \( x^* \) is the only solution of equation \( F(x) = 0 \) in \( \Omega_1 := \Omega \cap B(x^*, R_1) \).

Proof. Estimates (2.11a)- (2.11c) are shown using induction on \( k \). We shall show \( A_0^{-1} \in L(\mathcal{B}_2, \mathcal{R}_1) \) which implies \( y_0, z_0 \) and \( x_1 \) are well defined by method (1.2) for \( n = 0 \). Using (2.1), (2.6), (2.7b), (2.8), (2.9b) and (2.10a), we have in turn that

\[ \|F'(x^*)^{-1}([w_0, x_0; F] - F'(x^*))\| \leq \omega_0(\delta \|x_0 - x^*\|, \|x_0 - x^*\|) \]

\[ \leq \omega_0(\delta \|x_0 - x^*\|, \|x_0 - x^*\|) \]

\[ \leq \omega_0(\delta r, r) \leq \omega_0(\delta r_0, r_0) < 1, \]  

(2.13)

where we also used

\[ \|w_0 - x^*\| = \|(I + \beta [x_0, x^*; F])(x_0 - x^*)\| \leq \delta \|x_0 - x^*\|. \]

By the Banach perturbation Lemma [2,8] and (2.13), we get that \( A_0^{-1} \in L(\mathcal{B}_2, \mathcal{R}_1) \) and

\[ \|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - \omega_0(\delta \|x_0 - x^*\|, \|x_0 - x^*\|)}. \]  

(2.14)

Using the first substep of method (1.2) for \( n = 0 \), (2.6), (2.7a) (for \( i = 1 \)), (2.10b), (2.10d) and (2.14), we get in turn that

\[ \|y_0 - x^*\| = \|(x_0 - x^*) - A_0^{-1}F'(x_0)\| \]

\[ = \|(A_0^{-1}F'(x^*))F'(x^*)^{-1}([w_0, x_0; F] - [x_0, x^*; F])(x_0 - x^*)\| \]

\[ \leq \omega_1(\|w_0 - x_0\|, \|x_0 - x^*\|) \|x_0 - x^*\| \]

\[ \leq \frac{\omega_1(\|v_0(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|)}{1 - \omega_0(\delta \|x_0 - x^*\|, \|x_0 - x^*\|)} \]

\[ \leq \frac{g_1(\|x_0 - x^*\|) \|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \]  

(2.15)
which shows (2.11a) for $n = 0$ and $y_0 \in B(x^*, r)$. Then, by the second substep of method (1.2) for $n = 0$, (2.6), (2.7a) (for $i = 2$), (2.10c), (2.14) and (2.15), we obtain in turn that

\[
\|z_0 - x^*\| \leq \|y_0 - x^*\| + \|A_0^{-1}F'(x^*)\| \|F'(x^*)^{-1}F(y_0)\|
\]

\[
\leq \left(1 + \frac{v(\|y_0 - x^*\|)}{1 - \omega_0(\|x_0 - x^*\|, \|x_0 - x^*\|)}\right) \|y_0 - x^*\|
\]

\[
\leq \left(1 + \frac{v(g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - \omega_0(\|x_0 - x^*\|, \|x_0 - x^*\|)}\right) g_1(\|x_0 - x^*\|) \|x_0 - x^*\|
\]

\[
g_2(\|x_0 - x^*\|) \|x_0 - x^*\| \leq \|x_0 - x^*\| < r,
\]

(2.16)

which shows (2.11b) holds for $n = 0$ and $z_0 \in B(x^*, r)$. Next, we need an estimate on $\|B_0\|$. Using (2.10e), (2.14)–(2.16) we get in turn that

\[
\|B_0\| \leq \|A_0^{-1}F'(x^*)\| \left[ \|F'(x^*)^{-1}F'(x^*)\| + \|F'(x^*)^{-1}([w_0, x_0; F] - [z_0, y_0; F])\| \right]
\]

\[
\leq \|A_0^{-1}F'(x^*)\| \left[ 1 + \omega(\|w_0 - z_0\|, \|y_0 - x_0\|) \right]
\]

\[
\leq \|A_0^{-1}F'(x^*)\| \left[ 1 + \omega((\delta + g_2(\|x_0 - x^*\|))\|x_0 - x^*\|, (1 + g_1(\|x_0 - x^*\|)\|x_0 - x^*\|) \right]
\]

\[
\leq b(\|x_0 - x^*\|).
\]

(2.17)

Then, from the last substep of method (1.2), (2.7a) (for $i = 3$), (2.14), (2.16) and (2.17), we have in turn that

\[
\|x_1 - x^*\| \leq \|z_0 - x^*\| + \|B_0\| \|A_0^{-1}F'(x^*)\| \|F'(x^*)^{-1}F(z_0)\|
\]

\[
\leq \left(1 + \frac{b(\|x_0 - x^*\|)v(\|z_0 - x^*\|)}{1 - \omega_0(\|x_0 - x^*\|, \|x_0 - x^*\|)}\right) \|z_0 - x^*\|
\]

\[
\leq g_3(\|x_0 - x^*\|) \|x_0 - x^*\| \leq \|x_0 - x^*\| < r,
\]

(2.18)

which shows (2.11c) for $n = 0$ and $x_1 \in B(x^*, r)$. The induction for (2.11a)-(2.11c) is completed in an analogous way, if we replace $x_0, y_0, z_0, x_1$ by $x_k, y_k, z_k, x_{k+1}$, respectively in the previous estimates. Then, from the estimate

\[
\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r,
\]

(2.19)

where $c = g_3(\|x_0 - x^*\|) \in [0, 1)$, we deduce that

\[
\lim_{k \to \infty} x_k = x^*, \quad x_{k+1} \in B(x^*, r).
\]

The uniqueness part is shown by assuming $y^* \in \Omega_1$ with $F(y^*) = 0$. Define linear operator $T$ by $T = [y^*, x^*; F]$. Using (2.9b) and (2.12), we have in turn that

\[
\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \omega_0(0, \|y^* - x^*\|) \leq \omega_0(0, R_1) < 1,
\]

(2.20)

so $T$ is invertible. If then follows from the identity $0 = F(y^*) - F(x^*) = T(y^* - x^*)$ that $x^* = y^*$.  \(\square\)
Remark 2.1. Method (1.2) is not changing if we use the new instead of the old conditions [16]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [18]

$$\xi = \frac{\ln \|x_{n+2} - x^*\|}{\ln \|x_{n+1} - x\|}, \quad \text{for each } n = 1, 2, \cdots,$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \|x_{n+2} - x_{n+1}\|}{\ln \|x_{n+1} - x_n\|}, \quad \text{for each } n = 0, 1, \cdots,$$

instead of the error bounds obtained in Theorem 2.1. Notice that the computation of $\xi^*$ does not require knowledge of $x^*$. In some cases verifying Lipschitz-type conditions also does not require knowledge of solution $x^*$. As an example consider $F$ being Fréchet differentiable and satisfying the autonomous differentiable equation [2,8]

$$F'(x) = P(F(x)), \quad (2.21)$$

where $P : \mathbb{R} \rightarrow \mathbb{R}$ is a known continuous function and say $\mathcal{R}_1 = \mathcal{R}_2 = \mathbb{R}$. We have $F'(x^*) = P(F(x^*)) = P(0)$, which is known. By choosing for simplicity $[x, y; F] = \frac{1}{2}(F'(x) + F'(y))$, we can then verify conditions (2.10a)–(2.10e) without knowledge of the solution $x^*$. As an example, let $F(x) = e^x - 1$, then choose $P(x) = x + 1$, so that (2.21) is satisfied.

3. Numerical examples

The numerical examples are presented in this section. We choose

$$[x, y; F] = \int_0^1 F'(y + \theta(x - y)) d\theta.$$

Example 3.1. Looking back at the motivational example, we see that conditions (2.9b)-(2.10f) are satisfied, if we choose $x^* = 0$, $F'(0) = 1$, $\omega_0(s, t) = 9.0482(s + t)$,

$$\delta = \max_{-\frac{\pi}{2} \leq s, y \leq \frac{\pi}{2}} |1 + [x, y; F]| = 9.0482,$$

$v(t) = v_0(t) = 1 + 9.0482t$, $\omega_1(s, t) = 9.0482t$ and $\omega(s, t) = 16.5263(s + t)$. The parameters are:

$$r_1 = 0.0121, \quad r_2 = 0.0080 = r_3 = 0.0079 = r.$$

We cannot compare this example with earlier ones such as [16], since no computable radius of convergence was given there. But if it was given under the usual Rheinbold and
Then, the Fréchet-derivative is given by
\[
\frac{\partial}{\partial x} \omega_0(s, t) = \frac{\partial}{\partial x} \omega_1(s, t) = \frac{\partial}{\partial x} \omega(s, t) = 16.5263(s + t),
\]
\[\tilde{v}_0(t) = \tilde{v}(t) = 1 + 16.5263t,
\]
\[
\tilde{\delta} = \max_{\frac{1}{2} \leq x \leq \frac{3}{2}} |1 + [x, x^3; F]| = 7.12427
\]
leading to
\[
\tilde{r}_1 = 0.0070 = \tilde{r}, \quad \tilde{r}_2 = 0.0091, \quad \tilde{r}_3 = 0.0078.
\]
Notice that \(\omega_0(s, t)<\tilde{\omega}_0(s, t), \omega(s, t)=\tilde{\omega}(s, t), \omega_1(s, t)<\tilde{\omega}_1(s, t), \nu_0(t)<\nu(t)<\tilde{\nu}_0=\tilde{v}(t), \delta<\tilde{\delta}, \tilde{r}<r,\) justifying our approach. Notice also that the results in [16] cannot guarantee the convergence of method (1.2), since \(F''\) does not exist.

**Example 3.2.** Let \(\mathcal{B}_1 = \mathcal{B}_2 = C[0, 1], D = \tilde{U}(0, 1).\) Define function \(F\) on \(D\) by
\[
F(\varphi)(x) = \varphi(x) - \int_0^1 x \varphi(\theta)^3 d\theta.
\]
Then, the Fréchet-derivative is given by
\[
F'(\varphi(x))(x) = \varphi(x) - 3 \int_0^1 x \varphi(\theta)^2 \varphi(\theta) d\theta, \quad \text{for each } \zeta \in D.
\]
We get that \(x^* = 0, F'(x^*) = I, \omega_0(s, t) = \frac{3}{4}(s + t), \delta = \frac{3}{2}, \omega(s, t) = \frac{3}{4}(s + t), \nu_0(t) = \nu(t) = 1 + \frac{3}{2}t, \omega_1(s, t) = \frac{3}{2}t, \omega(s, t) = \frac{3}{2}(s + t), \delta = \frac{3}{2}.\) Then, the radius of convergence \(r\) is given by
\[
r_1 = 0.2963, \quad r_2 = 0.1823, \quad r_3 = 0.1135 = r.
\]
We cannot compare this example with earlier ones such as [16], since no computable radius of convergence was given there. But if it was given under the usual Rheinbold and Traub-type Lipschitz conditions [2–6, 8, 14, 17], then, the functions and parameters would have been instead \(\omega_0(s, t) = \omega_1(s, t) = \omega(s, t) = \frac{3}{4}(s + t), \nu_0(t) = \nu(t) = 1 + \frac{3}{2}t, \delta = 3\)
leading to
\[
r_1 = 0.1481, \quad r_2 = 0.2607, \quad r_3 = 0.0730 = \tilde{r}.
\]
Notice that \(\omega_0(s, t)<\omega_0(s, t), \omega(s, t)=\omega(s, t), \omega_1(s, t)<\omega_1(s, t), \nu_0(t)<\nu(t)<\tilde{\nu}_0=\tilde{v}(t), \delta<\tilde{\delta}, \tilde{r}<r,\) justifying our approach.

**References**


