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Uniform Convergence Analysis for Singularly Perturbed Elliptic Problems with Parabolic Layers

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Abstract. In this paper, using Lin's integral identity technique, we prove the optimal uniform convergence $\mathcal{O}(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y)$ in the L^2-norm for singularly perturbed problems with parabolic layers. The error estimate is achieved by bilinear finite elements on a Shishkin type mesh. Here N_x and N_y are the number of elements in the x- and y-directions, respectively. Numerical results are provided supporting our theoretical analysis.

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Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

It is well-known that singularly perturbed problems [14] appear in many application areas, and solving them efficiently is very challenging due to its inherent multiscales. Hence singularly perturbed problems often serve as challenging benchmark problems. In the past decade, many novel numerical methods for solving these problems have been proposed (e.g., [9, 11] and references cited therein).

In this paper, we consider the singularly perturbed elliptic problem:

$$-\varepsilon^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}+u=f(x,y) \quad \text{in } \Omega\equiv(0,1)^2, \tag{1.1}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $0<\varepsilon\ll 1$. This problem is different from most singularly perturbed problems in that it not only has the ordinary exponential boundary layer at $y=1$, but also has

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parabolic boundary layers at both $x = 0$ and $x = 1$. Such parabolic layer problems have been studied by many researchers (e.g., [1,3,10,12,15]). Li [3] has studied (1.1)-(1.2) using bilinear finite elements with Shishkin type meshes, but only first-order convergence in the L^2-norm was proved. Later, Roos *et al.* [10, 15] studied similar problems, and only first-order convergence in the energy norm was obtained for bilinear finite elements. However, we observed almost second-order convergence in the L^2-norm in our numerical tests [3]. In this paper, we fill the gap by providing a rigorous proof for this phenomenon. We like to remark that the proof is not trivial and needs to use the so-called Lin's integral identity technique developed in early 1990 by Lin and his group [5,6,8] for finite element superconvergence. More details can be found in a later book by Lin and Yan [7]. Li [2,4] was the first to use such technique to prove optimal uniform convergence for reaction-diffusion type singularly perturbed problems. Now it becomes an indispensable tool for proving optimal convergence for singularly perturbed problems [13,16].

The rest paper is organized as follows. In Section 2, we construct the Shishkin mesh based on the boundary layers of the analytic solution. Then we present our finite element method and the interpolation error in Section 3. In Section 4, we provide the optimal convergence error analysis. A numerical example is provided in Section 5 to support our theoretical analysis.

2. Solution decomposition and Shishkin mesh

To reduce technicality as Roos *et al.* [10, 15], we assume that the analytic solution of our problem (1.1)-(1.2) exists the following decomposition:

Theorem 2.1. *If* $f \in C^{3,\alpha}(\overline{\Omega})$*, for some* $\alpha \in (0,1)$*, satisfies the compatibility condition* $f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0$*, then the solution of* (1.1)-(1.2) *can be decomposed as*

$$u = S + E_1 + E_2 + E_3, \tag{2.1}$$

where for all $(x,y) \in \overline{\Omega}$ *and* $0 \le i + j \le 3$,

$$\begin{aligned}
&|\frac{\partial^{i+j}S}{\partial x^i \partial y^j}(x,y)| \le C, \quad |\frac{\partial^{i+j}E_1}{\partial x^i \partial y^j}(x,y)| \le C\epsilon^{-2j}e^{-(1-y)/\epsilon^2},\\
&|\frac{\partial^{i+j}E_2}{\partial x^i \partial y^j}(x,y)| \le C\epsilon^{-i}(e^{-x/\epsilon} + e^{-(1-x)/\epsilon}),\\
&|\frac{\partial^{i+j}E_3}{\partial x^i \partial y^j}(x,y)| \le C\epsilon^{-(i+2j)}e^{-(1-y)/\epsilon^2}(e^{-x/\epsilon} + e^{-(1-x)/\epsilon}).
\end{aligned}$$

To construct a Shishkin type mesh accordingly, we assume that the positive integers N_x and N_y are divisible by 4, where N_x and N_y denote the number of elements in the x- and y-directions, respectively. In the x-direction, we first divide the interval $[0,1]$ into the subintervals

$$[0,\sigma_x], \quad [\sigma_x, 1-\sigma_x], \quad [1-\sigma_x, 1] .$$

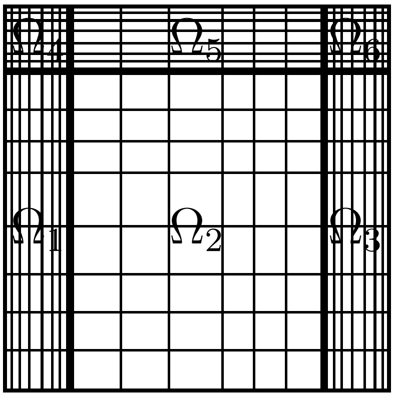

Figure 1: An exemplary Shishkin mesh for problem (1.1)-(1.2).

Uniform meshes are then used on each subinterval, with $N_x/4$ points on each of $[0,\sigma_x]$ and $[1-\sigma_x,1]$, and $N_x/2$ points on $[\sigma_x,1-\sigma_x]$, where

$$\sigma_x = 3\varepsilon \ln N_x \le \frac{1}{4}.$$

In the y-direction, we follow the same way by dividing the interval $[0,1]$ into the subintervals

$$[0,1-\sigma_y], \quad [1-\sigma_y,1] \;.$$

Uniform meshes are then used on each subinterval, with $N_y/2$ points on each subinterval, here

$$\sigma_y = 3\varepsilon^2 \ln N_y \le \frac{1}{2}.$$

For easy reference, we introduce the following subdomains (see Fig. 1):

$$\begin{aligned}
&\Omega_1 \equiv [0,\sigma_x]\times[0,1-\sigma_y], \quad \Omega_2 \equiv [\sigma_x,1-\sigma_x]\times[0,1-\sigma_y],\\
&\Omega_3 \equiv [1-\sigma_x,1]\times[0,1-\sigma_y], \quad \Omega_4 \equiv [0,\sigma_x]\times[1-\sigma_y,1],\\
&\Omega_5 \equiv [\sigma_x,1-\sigma_x]\times[1-\sigma_y,1], \quad \Omega_6 \equiv [1-\sigma_x,1]\times[1-\sigma_y,1].
\end{aligned}$$

3. The finite element method and interpolation error

The weak formulation of (1.1)-(1.2) is: Find $u \in H_0^1(\Omega)$ such that

$$B(u,\xi) \equiv \varepsilon^2(\nabla u,\nabla \xi) + (u_y,\xi) + (u,\xi) = (f,\xi), \quad \forall \xi \in H_0^1(\Omega), \tag{3.1}$$

where $(\cdot,\cdot)$ denotes the usual $L^2(\Omega)$ inner product and $H_0^1(\Omega)$ is the usual Sobolev space.

Let us denote the weighted energy norm

$$|||v||| \equiv \{\varepsilon^2\|\nabla v\|^2 + \|v\|^2\}^{1/2} \;, \; \forall \; v \in H_0^1(\Omega).$$

Then it is easy to see that for any $\xi \in H_0^1(\Omega)$, we have

$$B(\xi,\xi) = \varepsilon^2 ||\nabla \xi||^2 + (\xi,\xi) = |||\xi|||^2. \tag{3.2}$$

Let $S_h(\Omega) \subset H_0^1(\Omega)$ be the conforming bilinear finite element space. Our finite element method can be defined as:

$$B(u_h, \xi_h) = (f, \xi_h), \quad \forall \xi_h \in S_h(\Omega). \tag{3.3}$$

Furthermore, let

$$w^I = \Pi_x \Pi_y w = \Pi_y \Pi_x w$$

be the bilinear interpolation of w, where Π_x and Π_y are the interpolations in the x- and y-direction, respectively.

Lemma 3.1. *For the solution u of* (1.1)-(1.2)*, we have*

$$\begin{aligned}
&||u-u^I||_{\infty,\Omega_1\cup\Omega_3} \le C(N_x^{-2}\ln^2 N_x + N_y^{-2}), \quad ||u-u^I||_{\infty,\Omega_2} \le C(N_x^{-2} + N_y^{-2}),\\
&||u-u^I||_{\infty,\Omega_4\cup\Omega_6} \le C(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y),\\
&||u-u^I||_{\infty,\Omega_5} \le C(N_x^{-2} + N_y^{-2}\ln^2 N_y).
\end{aligned}$$

Proof. From the interpolation property [3, p.55] and Theorem 2.1, we have

$$\begin{aligned}
&||u-u^I||_{\infty,\Omega_1} \le ||u-\Pi_x u||_{\infty,\Omega_1} + ||\Pi_x(u-\Pi_y u)||_{\infty,\Omega_1}\\
\le\ & ||u-\Pi_x u||_{\infty,\Omega_1} + ||u-\Pi_y u||_{\infty,\Omega_1}\\
\le\ & Ch_{x,\Omega_1}^2 ||u_{xx}||_{\infty,\Omega_1} + Ch_{y,\Omega_1}^2 ||(S+E_2)_{yy}||_{\infty,\Omega_1} + ||E_1+E_3||_{\infty,\Omega_1}\\
\le\ & C(\epsilon N_x^{-1}\ln N_x)^2 \epsilon^{-2} + CN_y^{-2} + C \max_{0\le y\le 1-\sigma_y} e^{-(1-y)/\epsilon^2}\\
\le\ & C(N_x^{-2}\ln^2 N_x + N_y^{-2}),
\end{aligned}$$

where in the last step we used the fact

$$\max_{0\le y\le 1-\sigma_y} e^{-(1-y)/\epsilon^2} \le e^{-\sigma_y/\epsilon^2} = N_y^{-3}.$$

Exact estimate holds true on Ω_3. Similarly, we can obtain

$$\begin{aligned}
&||u-u^I||_{\infty,\Omega_2} \le ||u-\Pi_x u||_{\infty,\Omega_2} + ||u-\Pi_y u||_{\infty,\Omega_2}\\
\le\ & Ch_{x,\Omega_2}^2 ||(S+E_1)_{xx}||_{\infty,\Omega_2} + ||E_2+E_3||_{\infty,\Omega_2}\\
&+Ch_{y,\Omega_2}^2 ||(S+E_2)_{yy}||_{\infty,\Omega_2} + ||E_1+E_3||_{\infty,\Omega_2}\\
\le\ & CN_x^{-2} + C\max_{\sigma_x\le x\le 1-\sigma_x}[e^{-x/\epsilon} + e^{-(1-x)/\epsilon}] + CN_y^{-2} + C\max_{0\le y\le 1-\sigma_y} e^{-(1-y)/\epsilon^2}\\
\le\ & C(N_x^{-2} + N_y^{-2}),
\end{aligned}$$

where in the last step we used the fact

$$\max_{\sigma_x \le x \le 1-\sigma_x} [e^{-x/\epsilon} + e^{-(1-x)/\epsilon}] \le 2e^{-\sigma_x/\epsilon} = 2N_x^{-3}.$$

Also we have

$$\begin{aligned}
&\|u-u^I\|_{\infty,\Omega_4} \le \|u-\Pi_x u\|_{\infty,\Omega_4} + \|u-\Pi_y u\|_{\infty,\Omega_4}\\
\le\ & Ch^2_{x,\Omega_4}\|u_{xx}\|_{\infty,\Omega_4} + Ch^2_{y,\Omega_4}\|u_{yy}\|_{\infty,\Omega_4}\\
\le\ & C(\epsilon N_x^{-1}\ln N_x)^2(1+\epsilon^{-2}) + C(\epsilon^2 N_y^{-1}\ln N_y)^2(1+\epsilon^{-4})\\
\le\ & C(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y).
\end{aligned}$$

Exact estimate holds true on Ω_6. Finally, we have

$$\begin{aligned}
&\|u-u^I\|_{\infty,\Omega_5} \le Ch^2_{x,\Omega_5}\|(S+E_1)_{xx}\|_{\infty,\Omega_5} + \|E_2+E_3\|_{\infty,\Omega_5} + Ch^2_{y,\Omega_5}\|u_{yy}\|_{\infty,\Omega_5}\\
\le\ & CN_x^{-2} + C\max_{\sigma_x \le x \le 1-\sigma_x}[e^{-x/\epsilon} + e^{-(1-x)/\epsilon}] + C(\epsilon^2 N_y^{-1}\ln N_y)^2(1+\epsilon^{-4})\\
\le\ & C(N_x^{-2} + N_y^{-2}\ln^2 N_y).
\end{aligned}$$

This completes the proof of Lemma 3.1. □

4. Optimal uniform convergence

For any rectangular element τ, let (x_τ, y_τ) be the center of the element, and $2h_x, 2h_y$ be the length of the element in the x- and y-direction, respectively. Furthermore, we denote

$$E(x) = \frac{1}{2}[(x-x_\tau)^2 - h_x^2], \quad F(y) = \frac{1}{2}[(y-y_\tau)^2 - h_y^2].$$

First, we have a simple result:

Lemma 4.1. *On any rectangle element τ, we have*

$$\|(w-w^I)_x\|_{\infty,\tau} \le 3\|w_x\|_{\infty,\tau}, \quad \|(w-w^I)_y\|_{\infty,\tau} \le 3\|w_y\|_{\infty,\tau}, \quad \forall\ w \in C^1(\tau).$$

Proof. Let the four vertexes of τ: $P_1(x_1,y_1), P_2(x_2,y_1), P_3(x_2,y_2), P_4(x_1,y_2)$. The corresponding function values of w are $w_i, i=1,2,3,4$. Hence the bilinear interpolation of w on τ can be written as

$$\begin{aligned}
w^I\ =\ & w_1\cdot\frac{(x_2-x)(y_2-y)}{4h_xh_y} + w_2\cdot\frac{(x-x_1)(y_2-y)}{4h_xh_y}\\
&+w_3\cdot\frac{(x-x_1)(y-y_1)}{4h_xh_y} + w_4\cdot\frac{(x_2-x)(y-y_1)}{4h_xh_y},
\end{aligned}$$

from which we obtain

$$\|(w-w^I)_x\|_{\infty,\tau} = \|w_x - (w_2 - w_1)\frac{(y_2 - y)}{4h_x h_y} - (w_3 - w_4)\frac{(y-y_1)}{4h_x h_y}\|_{\infty,\tau}$$
$$= \|w_x - \frac{\partial w}{\partial x}(\eta_1, y_1)\frac{(y_2-y)}{2h_y} - \frac{\partial w}{\partial x}(\eta_2, y_2)\frac{(y-y_1)}{2h_y}\|_{\infty,\tau} \le 3\|w_x\|_{\infty,\tau}.$$

In the same way, we can prove the inequality for derivative with respect to y. □

To prove the optimal uniform convergence, we need the following integral identities:

Lemma 4.2. [5] *For any rectangle element τ and $v \in Q_1(\tau)$, we have*

$$(a)\ \int_\tau (w-w^I)_x v_x dxdy = \int_\tau F(y) w_{xy^2}[v_x - \frac{2}{3}(y-y_\tau)v_{xy}]dxdy, \tag{4.1}$$

$$(b)\ \int_\tau (w-w^I)_y v_y dxdy = \int_\tau E(x) w_{x^2y}[v_y - \frac{2}{3}(x-x_\tau)v_{xy}]dxdy, \tag{4.2}$$

$$(c)\ \int_\tau (w-w^I)_y v dxdy = \int_\tau \{E(x)w_{x^2y}[v-(y-y_\tau)v_y - \frac{2}{3}(x-x_\tau)v_x$$
$$+\frac{2}{3}(x-x_\tau)(y-y_\tau)v_{xy}] + \frac{1}{3}F(y)(y-y_\tau)w_{y^3}v_y - \frac{1}{3}h_y^2 w_{y^3} v\}dxdy$$
$$+\frac{1}{3}h_y^2(\int_{y=y_\tau+h_y} - \int_{y=y_\tau-h_y})w_{y^2}v dx. \tag{4.3}$$

Lemma 4.3. *For any $\xi \in S_h(\Omega)$ and the solution u of* (1.1)-(1.2)*, we have*

$$(a)\quad \epsilon^2((u-u^I)_x, \xi_x) \le CN_y^{-2}\ln^2 N_y |||\xi|||, \tag{4.4}$$

$$(b)\quad \epsilon^2((u-u^I)_y, \xi_y) \le CN_x^{-2}\ln^2 N_x |||\xi|||. \tag{4.5}$$

Proof. (a) Denote $\tilde{E} = E_1 + E_3$. Hence we have

$$\epsilon^2((u-u^I)_x, \xi_x)$$
$$= \epsilon^2((S-S^I)_x, \xi_x) + \epsilon^2((E_2 - E_2^I)_x, \xi_x) + \epsilon^2((\tilde{E} - \tilde{E}^I)_x, \xi_x). \tag{4.6}$$

By the identity (4.1) and Theorem 2.1, we obtain

$$\epsilon^2((S+E_2)-(S+E_2)^I)_x, \xi_x) \le \epsilon^2 h_y^2 \|(S+E_2)_{xy^2}\|_{\infty,\Omega}|\Omega|^{1/2}\|\xi_x\|_{0,\Omega}$$
$$\le C\epsilon^2 N_y^{-2}\epsilon^{-1}\|\xi_x\| \le CN_y^{-2}|||\xi|||. \tag{4.7}$$

By Lemma 4.1, we have

$$\epsilon^2((\tilde{E}-\tilde{E}^I)_x, \xi_x)_{0,\Omega_1\cup\Omega_2\cup\Omega_3} \le C\epsilon^2\|\tilde{E}_x\|_{\infty,\Omega_1\cup\Omega_2\cup\Omega_3}\|\xi_x\|_{0,\Omega_1\cup\Omega_2\cup\Omega_3}$$
$$\le \epsilon^2 \cdot C\epsilon^{-1} \max_{0\le y\le 1-\sigma_y} e^{-(1-y)/\epsilon^2}\|\xi_x\|_{0,\Omega}$$
$$\le C\epsilon N_y^{-2}\|\xi_x\|_{0,\Omega} \le CN_y^{-2}|||\xi|||. \tag{4.8}$$

By the identity (4.1) and Theorem 2.1, we obtain

$$\begin{aligned}&\epsilon^2((\tilde{E}-\tilde{E}^I)_x,\xi_x)_{0,\Omega_4\cup\Omega_5\cup\Omega_6}\le C\epsilon^2h_y^2||\tilde{E}_{xy^2}||_{\infty,\Omega_4\cup\Omega_5\cup\Omega_6}||\xi_x||_{0,\Omega_4\cup\Omega_5\cup\Omega_6}\\ \le\ &\epsilon^2\cdot C(\frac{\epsilon^2\ln N_y}{N_y})^2\cdot\epsilon^{-1}\cdot\epsilon^{-4}||\xi_x||_{0,\Omega}\le CN_y^{-2}\ln^2N_y|||\xi|||. \end{aligned}\tag{4.9}$$

Substitution of the above estimates into (4.6) completes the proof of part (a).

(b) By the identity (4.2) and Theorem 2.1, we obtain

$$\begin{aligned}&\epsilon^2(S-S^I)_y,\xi_y)\le C\epsilon^2h_x^2||S_{x^2y}||_{\infty,\Omega}|\Omega|^{1/2}||\xi_y||_{0,\Omega}\\ \le\ &C\epsilon^2N_x^{-2}||\xi_y||\le CN_x^{-2}|||\xi|||.\end{aligned}\tag{4.10}$$

By the same argument, we can easily obtain

$$\begin{aligned}&\epsilon^2(E_2-E_2^I)_y,\xi_y)_{\Omega_2}\le\epsilon^2Ch_x^2||E_{2,x^2y}||_{\infty,\Omega}|\Omega_2|^{1/2}||\xi_y||_{0,\Omega_2}\\ \le\ &C\epsilon^2N_x^{-2}\max_{\sigma_x\le x\le 1-\sigma_x}\epsilon^{-2}(e^{-x/\epsilon}+e^{-(1-x)/\epsilon})||\xi_y||_{0,\Omega_2}\le CN_x^{-4}N_y||\xi||_{0,\Omega_2},\end{aligned}\tag{4.11}$$

$$\begin{aligned}&\epsilon^2(E_2-E_2^I)_y,\xi_y)_{\Omega_1\cup\Omega_4}\le\epsilon^2Ch_x^2||E_{2,x^2y}||_{\infty,\Omega}|\Omega_1\cup\Omega_4|^{1/2}||\xi_y||_{0,\Omega_1\cup\Omega_4}\\ \le\ &C\epsilon^2(\frac{\epsilon\ln N_x}{N_x})^2\cdot\epsilon^{-2}||\xi_y||_{0,\Omega_1\cup\Omega_4}\le CN_x^{-2}\ln^2N_x|||\xi|||,\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}&\epsilon^2(E_2-E_2^I)_y,\xi_y)_{\Omega_5}\le\epsilon^2Ch_x^2||E_{2,x^2y}||_{\infty,\Omega_5}|\Omega_5|^{1/2}||\xi_y||_{0,\Omega_5}\\ \le\ &C\epsilon^2N_x^{-2}\cdot\epsilon^{-2}(\epsilon^2)^{1/2}||\xi_y||_{0,\Omega_5}\le CN_x^{-2}|||\xi|||.\end{aligned}\tag{4.13}$$

Using Lemma 4.1, Theorem 2.1 and the inverse estimate, we obtain

$$\begin{aligned}&\epsilon^2(\tilde{E}-\tilde{E}^I)_y,\xi_y)_{\Omega_1\cup\Omega_2\cup\Omega_3}\le C\epsilon^2||\tilde{E}_y||_{\infty,\Omega_1\cup\Omega_2\cup\Omega_3}||\xi_y||_{0,\Omega_1\cup\Omega_2\cup\Omega_3}\\ \le\ &C\epsilon^2\max_{0\le y\le 1-\sigma_y}\epsilon^{-2}e^{-(1-y)/\epsilon^2}||\xi_y||_{0,\Omega_1\cup\Omega_2\cup\Omega_3}\le CN_y^{-3}N_y||\xi||_{0,\Omega}\le CN_y^{-2}|||\xi|||.\end{aligned}$$

Similarly, by the identity (4.2) and Theorem 2.1, we have

$$\begin{aligned}&\epsilon^2(E_1-E_1^I)_y,\xi_y)_{\Omega_5}\le C\epsilon^2h_x^2||E_{1,x^2y}||_{\infty,\Omega_5}|\Omega_5|^{1/2}||\xi_y||_{0,\Omega_5}\\ \le\ &C\epsilon^2N_x^{-2}\cdot\epsilon^{-2}(\epsilon^2)^{1/2}||\xi_y||_{0,\Omega_5}\le CN_x^{-2}|||\xi|||.\end{aligned}\tag{4.14}$$

Using Lemma 4.1 and Theorem 2.1, we obtain

$$\begin{aligned}&\epsilon^2(E_3-E_3^I)_y,\xi_y)_{\Omega_5}\le\epsilon^2||E_{3,y}||_{\infty,\Omega_5}|\Omega_5|^{1/2}||\xi_y||_{0,\Omega_5}\\ \le\ &C\epsilon^2\max_{\sigma_x\le x\le 1-\sigma_x}\epsilon^{-2}(e^{-x/\epsilon}+e^{-(1-x)/\epsilon})\cdot(\epsilon^2)^{1/2}||\xi_y||_{0,\Omega_5}\le CN_x^{-2}|||\xi|||.\end{aligned}\tag{4.15}$$

Finally, by the identity (4.2) and Theorem 2.1, we have

$$\begin{aligned}&\epsilon^2(\tilde{E}-\tilde{E}^I)_y,\xi_y)_{\Omega_4}\le\epsilon^2h_x^2||\tilde{E}_{x^2y}||_{\infty,\Omega_4}|\Omega_4|^{1/2}||\xi_y||_{0,\Omega_4}\\ \le\ &C\epsilon^2(\frac{\epsilon\ln N_x}{N_x})^2\cdot\epsilon^{-4}\cdot\epsilon^{3/2}||\xi_y||_{0,\Omega_4}\le CN_x^{-2}\ln^2N_x|||\xi|||.\end{aligned}\tag{4.16}$$

Substitution of the above estimates into

$$\epsilon^2((u-u^I)_x,\xi_x)=\epsilon^2((S-S^I)_x,\xi_x)+\epsilon^2((E_2-E_2^I)_x,\xi_x)+\epsilon^2((\tilde{E}-\tilde{E}^I)_x,\xi_x)$$

concludes the proof of part (b). □

Lemma 4.4. *For any $\xi\in S_h(\Omega)$ and the solution u of* (1.1)-(1.2)*, we have*

$$((u-u^I)_y,\xi)\le C(N_x^{-2}\ln^2 N_x+N_y^{-2}\ln^2 N_y)|||\xi|||. \tag{4.17}$$

Proof. Denote $\tilde{E}=E_1+E_3$. Hence we have

$$((u-u^I)_y,\xi)=((S-S^I)_y,\xi)+((E_2-E_2^I)_y,\xi)+((\tilde{E}-\tilde{E}^I)_y,\xi). \tag{4.18}$$

Using (4.3) with $v=\xi=0$ on $\partial\Omega$ and the inverse estimate, we have

$$\begin{aligned}
&((S-S^I)_y,\xi)_{\Omega_1\cup\Omega_4}\\
\le\;& C\left(h_x^2||S_{x^2y}||_{0,\Omega_1\cup\Omega_4}+h_y^2||S_{y^3}||_{0,\Omega_1\cup\Omega_4}\right)||\xi||_{0,\Omega_1\cup\Omega_4}\\
&+C(h^2_{y,\Omega_1}-h^2_{y,\Omega_4})\int_0^{\sigma_x}(S_{y^2}\xi)(x,1-\sigma_y)dx\\
\le\;& C(\epsilon^2N_x^{-2}\ln^2 N_x+N_y^{-2})||\xi||_{0,\Omega}+CN_y^{-2}\int_{1-\sigma_y}^1\int_0^{\sigma_x}(S_{y^3}\xi+S_{y^2}\xi_y)dydx\\
\le\;& C(N_x^{-2}\ln^2 N_x+N_y^{-2})||\xi||_{0,\Omega}\\
&+CN_y^{-2}\left(||S_{y^3}||_{0,\Omega_4}||\xi||_{0,\Omega_4}+||S_{y^2}||_{\infty,\Omega_4}|\Omega_4|^{1/2}||\xi_y||_{0,\Omega_4}\right)\\
\le\;& C(N_x^{-2}\ln^2 N_x+N_y^{-2})|||\xi|||.
\end{aligned} \tag{4.19}$$

By symmetry, we have

$$((S-S^I)_y,\xi)_{\Omega_3\cup\Omega_6}\le C(N_x^{-2}\ln^2 N_x+N_y^{-2})|||\xi|||.$$

Similarly, we have

$$\begin{aligned}
&((S-S^I)_y,\xi)_{\Omega_2\cup\Omega_5}\\
\le\;& C(h_x^2||S_{x^2y}||_{0,\Omega_2\cup\Omega_5}+h_y^2||S_{y^3}||_{0,\Omega_2\cup\Omega_5})||\xi||_{0,\Omega_2\cup\Omega_5}\\
&+C(h^2_{y,\Omega_2}-h^2_{y,\Omega_5})\int_0^{\sigma_x}(S_{y^2}\xi)(x,1-\sigma_y)dx\\
\le\;& C(N_x^{-2}+N_y^{-2})||\xi||_{0,\Omega}+CN_y^{-2}\int_{1-\sigma_y}^1\int_0^{\sigma_x}(S_{y^3}\xi+S_{y^2}\xi_y)dydx\\
\le\;& C(N_x^{-2}+N_y^{-2})||\xi||_{0,\Omega}+CN_y^{-2}(||S_{y^3}||_{0,\Omega_5}||\xi||_{0,\Omega_5}+||S_{y^2}||_{\infty,\Omega_5}|\Omega_5|^{1/2}||\xi_y||_{0,\Omega_5})\\
\le\;& C(N_x^{-2}+N_y^{-2})|||\xi|||.
\end{aligned} \tag{4.20}$$

By Lemma 4.1, Theorem 2.1 and the inverse estimate, we obtain

$$
\begin{aligned}
&(\tilde{E}-\tilde{E}^I,\xi_y)_{\Omega_1\cup\Omega_2\cup\Omega_3}\\
\leq\ & C||\tilde{E}||_{\infty,\Omega_1\cup\Omega_2\cup\Omega_3}||\xi_y||_{0,\Omega_1\cup\Omega_2\cup\Omega_3}\leq C\max_{0\leq y\leq 1-\sigma_y}e^{-(1-y)/\epsilon^2}||\xi_y||_{0,\Omega_1\cup\Omega_2\cup\Omega_3}\\
\leq\ & Ce^{-\sigma_y/\epsilon^2}||\xi_y||_{0,\Omega_1\cup\Omega_2\cup\Omega_3}=CN_y^{-3}||\xi_y||_{0,\Omega_1\cup\Omega_2\cup\Omega_3}\leq CN_y^{-2}|||\xi|||.
\end{aligned}
$$

By Cauchy-Schwarz inequality and Lemma 3.1, we easily obtain

$$
\begin{aligned}
&(\tilde{E}-\tilde{E}^I,\xi_y)_{\Omega_4\cup\Omega_5\cup\Omega_6}\leq||\tilde{E}-\tilde{E}^I||_{0,\Omega_4\cup\Omega_5\cup\Omega_6}||\xi_y||_{0,\Omega_4\cup\Omega_5\cup\Omega_6}\\
\leq\ & C(N_x^{-2}\ln^2N_x+N_y^{-2}\ln^2N_y)|\Omega_4\cup\Omega_5\cup\Omega_6|^{1/2}||\xi_y||_{0,\Omega_4\cup\Omega_5\cup\Omega_6}\\
\leq\ & C(N_x^{-2}\ln^2N_x+N_y^{-2}\ln^2N_y)|||\xi|||. \qquad (4.21)
\end{aligned}
$$

By Lemma 4.1 and Theorem 2.1, we have

$$
\begin{aligned}
&(E_2-E_2^I,\xi_y)_{\Omega_2}\leq C||E_2||_{\infty,\Omega_2}|\Omega_2|^{1/2}||\xi_y||_{0,\Omega_2}\\
\leq\ & C\max_{\sigma_x\leq x\leq 1-\sigma_x}(e^{-x/\epsilon}+e^{-(1-x)/\epsilon})||\xi_y||_{0,\Omega_2}\\
\leq\ & Ce^{-\sigma_x/\epsilon}||\xi_y||_{0,\Omega_2}\leq CN_x^{-3}||\xi_y||_{0,\Omega_2}\leq CN_x^{-3}N_y||\xi||_{0,\Omega_2}.
\end{aligned}
$$

Similarly, by Lemma 4.1 and Theorem 2.1, we obtain

$$
\begin{aligned}
(E_2-E_2^I,\xi_y)_{\Omega_5}\ &\leq\ C||E_2||_{\infty,\Omega_5}|\Omega_5|^{1/2}||\xi_y||_{0,\Omega_5}\\
&\leq\ CN_x^{-2}\epsilon||\xi_y||_{0,\Omega_5}\leq CN_x^{-2}|||\xi|||. \qquad (4.22)
\end{aligned}
$$

Using (4.3) with $v=\xi=0$ on $\partial\Omega$, we have

$$
\begin{aligned}
&((E_2-E_2^I)_y,\xi)_{\Omega_1\cup\Omega_4}\\
\leq\ & C(h_x^2||E_{2,x^2y}||_{0,\Omega_1\cup\Omega_4}+h_y^2||E_{2,y^3}||_{0,\Omega_1\cup\Omega_4})||\xi||_{0,\Omega_1\cup\Omega_4}\\
&+\frac{1}{3}(h_{y,\Omega_1}^2-h_{y,\Omega_4}^2)\int_0^{\sigma_x}(E_{2,y^2}\xi)(x,1-\sigma_y)dx\\
\leq\ & C((\frac{\epsilon\ln N_x}{N_x})^2\cdot\epsilon^{-2}+N_y^{-2})||\xi||_{0,\Omega}+CN_y^{-2}\int_{1-\sigma_y}^1\int_0^{\sigma_x}(E_{2,y^3}\xi+E_{2,y^2}\xi_y)dydx\\
\leq\ & C(N_x^{-2}\ln^2N_x+N_y^{-2})||\xi||_{0,\Omega}\\
&+CN_y^{-2}(||E_{2,y^3}||_{0,\Omega_4}||\xi||_{0,\Omega_4}+||E_{2,y^2}||_{\infty,\Omega_4}|\Omega_4|^{1/2}||\xi_y||_{0,\Omega_4})\\
\leq\ & C(N_x^{-2}\ln^2N_x+N_y^{-2})|||\xi|||. \qquad (4.23)
\end{aligned}
$$

By symmetry, we have

$$((E_2-E_2^I)_y,\xi)_{\Omega_3\cup\Omega_6}\leq C(N_x^{-2}\ln^2N_x+N_y^{-2})|||\xi|||.$$

Substitution of the above estimates into (4.18) concludes the proof. □

Using the above Lemmas, we obtain our main result:

Theorem 4.1. *Let u be the solution of* (1.1)-(1.2)*, u_h be the bilinear finite element solution of* (3.3)*. Then we have*

$$||u-u_h||_{0,\Omega} \leq C(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y).$$

Proof. Denote $\xi = u^I - u_h$. We have the error equation

$$\begin{aligned}&\epsilon^2(\nabla\xi,\nabla\xi)+(\xi_y,\xi)+(\xi,\xi)\\ =&\epsilon^2(\nabla(u^I-u),\nabla\xi)+((u^I-u)_y,\xi)+(u^I-u,\xi).\end{aligned} \tag{4.24}$$

By the coercivity (3.2) and Lemmas 3.1, 4.3 and 4.4, we obtain

$$||u^I-u_h||_{0,\Omega} \leq |||u^I-u_h||| \leq C(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y),$$

which along with the triangle inequality and Lemma 3.1 completes our proof. □

5. Numerical results

Here we solve the problem (1.1)-(1.2) with a properly chosen f such that we have the exact solution

$$u(x,y) = y(1-e^{-(1-y)/\epsilon^2})(1-e^{-x/\epsilon})(1-e^{-(1-x)/\epsilon}).$$

We solved the problem with various ϵ from 10^{-2} to 10^{-7} and different rectangular meshes with $N_x = N_y = 12, 24$, and 48. The errors in the L^2-norm and the corresponding convergence rate $r = (\ln e_N - \ln e_{2N})/\ln 2$ are provided in Table 1. Here e_N and e_{2N} denote the L^2 errors obtained with N and $2N$ piecewise uniform elements in both directions. From Table 1, we can see that the L^2 errors are indeed second-order (more obviously when ϵ is small), as we predicted in our theoretical analysis.

Table 1: L^2 errors and convergence rates.

	L^2 error	L^2 error	rate	L^2 error	rate
ϵ	$N=12$	$N=24$		$N=48$	
1.0E-02	1.29448E-02	5.07218E-03	1.35	1.71577E-03	1.56
1.0E-03	4.68345E-03	1.70260E-03	1.46	5.60574E-04	1.60
1.0E-04	2.71983E-03	7.86218E-04	1.79	2.26156E-04	1.80
1.0E-05	2.43805E-03	6.24585E-04	1.96	1.57606E-04	1.98
1.0E-06	2.40807E-03	6.06082E-04	1.99	1.49021E-04	2.02
1.0E-07	2.40386E-03	6.02303E-04	2.00	1.47396E-04	2.03

The numerical solution and corresponding pointwise error are plotted for a special case with $\epsilon = 10^{-5}$ and $N = 24$ and $N = 25$ in Fig. 2.

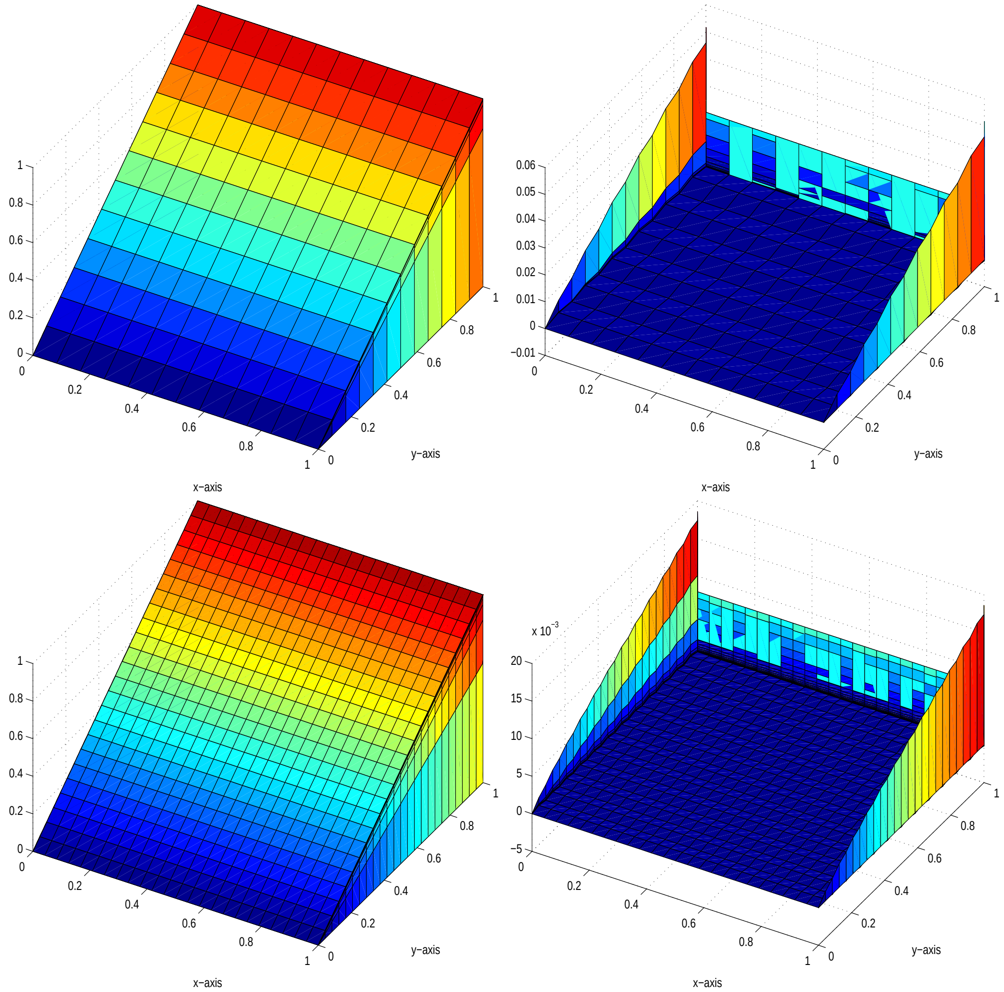

Figure 2: Results on $N = 24$ mesh: approximate solution (Top Left); pointwise error (Top Right); Results on $N = 48$ mesh: approximate solution (Bottom Left); pointwise error (Bottom Right).

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