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Analytic and Experimental Studies of the Errors in Numerical Methods for the Valuation of Options

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Abstract. The value of a European option satisfies the Black-Scholes equation with appropriately specified final and boundary conditions. We transform the problem to an initial boundary value problem in dimensionless form. There are two parameters in the coefficients of the resulting linear parabolic partial differential equation. For a range of values of these parameters, the solution of the problem has a boundary or an initial layer. The initial function has a discontinuity in the first-order derivative, which leads to the appearance of an interior layer. We construct analytically the asymptotic solution of the equation in a finite domain. Based on the asymptotic solution we can determine the size of the artificial boundary such that the required solution in a finite domain in x and at the final time is not affected by the boundary. Also, we study computationally the behaviour in the maximum norm of the errors in numerical solutions in cases such that one of the parameters varies from finite (or pretty large) to small values, while the other parameter is fixed and takes either finite (or pretty large) or small values. Crank-Nicolson explicit and implicit schemes using centered or upwind approximations to the derivative are studied. We present numerical computations, which determine experimentally the parameter-uniform rates of convergence. We note that this rate is rather weak, due probably to mixed sources of error such as initial and boundary layers and the discontinuity in the derivative of the solution.

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Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

Initial and boundary layer phenomena give rise to an important sub–class of mathematical problems with non–smooth solutions. They arise when the underlying mathematical

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problem is a singular perturbation problem. In singular perturbation problems the coefficient of the highest derivative in the differential equation is multiplied by a small parameter, called the singular perturbation parameter. In what follows we consider a problem in which there is a large parameter, and its inverse plays the role of the singular perturbation parameter.

The value of a European option satisfies the Black-Scholes equation with appropriately specified final and boundary conditions, see, for example, [6, 8]. We denote its value by $C = C(S, t)$, where S is the current value of the underlying asset and t is the time. S and t are the independent variables. The value of the option also depends on σ, the volatility of the underlying asset; E, the exercise price; T, the expiry time and r, the interest rate.

The Black-Scholes equation governing $C(S, t)$ is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

The domain of the independent variables S, t is $(0, \infty) \times (0, T]$. The final condition at $t = T$ is

$$C(S, T) = \max(S - E, 0),$$

the boundary condition at $S = 0$ is

$$C(0, t) = 0,$$

and the boundary condition at $S = +\infty$ is

$$C(S, t) \sim S \text{ as } S \to \infty.$$

Typical ranges of values of T in years, r in percent per annum and σ in percent per annum arising in practice are

$$\frac{1}{12} \leq T \leq 30, \quad .01 \leq r \leq .2, \quad .01 \leq \sigma \leq .5.$$

2. Transformations of the equation

Standard approaches to the reformulation of the problem lead to new problems in which the free parameters of the problem appear in the coefficients of the equation, the initial and boundary conditions or the definition of the solution domain. Here we reformulate in such a way that the two independent parameters appear only in the coefficients of the equation. This enables us to study the range of problems of financial relevance in a systematic way.

First, to obtain a more familiar initial value problem, we transform the time variable t to $t' = T - t$, and we put $C'(S, t') = C(S, t)$. The domain is still $(0, \infty) \times (0, T]$, the equation becomes

$$\frac{\partial C'}{\partial t'} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C'}{\partial S^2} + rS\frac{\partial C'}{\partial S} - rC',$$

and the final condition in t is changed to an initial condition in t'. We further change the independent variables S, t' to the new independent variables x, τ by the transformations

$$S = Ee^x, \quad \tau = t'/T,$$

and the dependent variable C' to the new dependent variable v by the transformation

$$C'(S, t') = Ev(x, \tau).$$

In these variables the domain becomes $(-\infty, \infty) \times (0, 1]$ and the equation transforms to the dimensionless equation

$$k_1 \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k_2 - 1)\frac{\partial v}{\partial x} - k_2 v,$$

where

$$k_1 = \frac{1}{\frac{1}{2}\sigma^2 T}, \quad k_2 = \frac{r}{\frac{1}{2}\sigma^2}$$

are non-zero dimensionless parameters. Finally, defining $\varepsilon_1 = k_2/k_1$ and $\varepsilon_2 = 1/k_2$ we can rewrite the equation in the form

$$\frac{\partial v}{\partial \tau} = \varepsilon_1(\varepsilon_2 \frac{\partial^2 v}{\partial x^2} + (1 - \varepsilon_2)\frac{\partial v}{\partial x} - v),$$

where $\varepsilon_1 = rT$, $\varepsilon_2 = \sigma^2/2r$. The dimensionless initial condition is

$$v(x, 0) = \max(e^x - 1, 0),$$

and the dimensionless boundary conditions at $x = \pm\infty$ are

$$v(x, \tau) = 0 \text{ as } x \to -\infty,$$
$$v(x, \tau) \sim e^x \text{ as } x \to \infty.$$

3. Exact solution and its derivatives

In mathematical finance, in addition to the solution itself, some of the partial derivatives of the solution are of interest. These are known as the Greeks. There are exact analytic formulas for the solution and the Greeks for European call and put options.

The exact solution of the Black-Scholes equation satisfying the given initial and boundary conditions is

$$v(x, \tau) = e^x N(d^+(x, \tau)) - e^{-\varepsilon_1 \tau} N(d^-(x, \tau)).$$

Here N is the cumulative distribution function for the normal distribution with mean 0 and standard deviation 1

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}s^2} ds,$$

and its arguments d^+, d^- are given by

$$d^+(x,\tau) = \frac{x + \varepsilon_1(1+\varepsilon_2)\tau}{\sqrt{2\tau\varepsilon_1\varepsilon_2}},$$
$$d^-(x,\tau) = \frac{x + \varepsilon_1(1-\varepsilon_2)\tau}{\sqrt{2\tau\varepsilon_1\varepsilon_2}}.$$

The x-derivative of the solution v is

$$\frac{\partial}{\partial x} v(x,\tau) = e^x N(d^+(x,\tau)), \tag{3.1}$$

and the τ-derivative is

$$\frac{\partial}{\partial \tau} v(x,\tau) = \exp(-\varepsilon_1\tau)[\varepsilon_1 N(d^-) + \frac{1}{2}\sqrt{\frac{\varepsilon_1\varepsilon_2}{\pi\tau}} \exp(-\frac{1}{2}(d^-)^2)]. \tag{3.2}$$

4. Classification of singular perturbation problems

The ranges of values of ε_1 and ε_2 corresponding to the above ranges of the parameters T, r, and σ are

$$.00083 \le \varepsilon_1 \le 6,$$
$$.00025 \le \varepsilon_2 \le 12.5.$$

We see that both parameters have comparable ranges and that they lie approximately in the common interval $[2^{-12}, 2^4]$. Since these are the ranges that arise in practical financial applications, we can classify the problems of financial interest into the four classes of problems given in Table 1.

Table 1: Taxonomy of problems.

	$\varepsilon_2 \in (0,1)$	$\varepsilon_2 = \mathcal{O}(1)$
$\varepsilon_1 \in (0,1)$	$C_{0,0}$	$C_{0,1}$
$\varepsilon_1 = \mathcal{O}(1)$	$C_{1,0}$	$C_{1,1}$

We see that singular perturbation problems arise in all classes other than the class $C_{1,1}$. This means that it is necessary to determine whether or not boundary and/or interior layers arise in the solutions of these classes of problems.

5. Singularities in the continuous problem

In order to obtain accurate numerical approximations of the solution and its derivatives it is necessary to take account of the singularities of the problem. Each of the following singularities is a potential source of numerical difficulties:

- The domain of the exact solution is infinite in the space variable, so artificial boundaries and boundary conditions may be required to define the numerical solutions on a finite domain, depending on whether the method is explicit or implicit in the time-like variable τ.
- The discontinuity in the x-derivative of the initial condition can cause numerical errors, which may propagate into the solution domain.
- The presence of large and/or small parameters multiplying the coefficients of the differential equation may give rise to boundary and/or interior layers in the solution and its derivatives, which, if not treated appropriately, will cause errors in the numerical solution.

In order to study the effect of these singularities on the errors in the numerical approximations, it is advisable to isolate them from each other and to deal with them one at a time.

6. Asymptotic analysis of the singularity in the initial condition for (2.3)

We now examine, analytically, whether or not the discontinuity in the x-derivative of the initial condition gives rise to numerical errors that propagate into the solution domain. For the problem considered here we may discuss this using the exact expression for the solution given in Section 3. However, for problems for which an exact solution in closed form is unavailable, we are forced to compute numerical approximations. This gives rise to a further difficulty, in that we cannot compute over an unbounded domain $(-\infty, \infty)$. The standard way of overcoming this is to compute on a bounded domain $[-L, L]$, where L is a sufficiently large number. We use the following approximate boundary conditions on the artificial boundaries $v(-L,\tau) \approx 0$ and $v(L,\tau) \approx e^L$. Note that the exact expression of the solution is not valid in the finite interval and/or in the case of variable coefficients, so we do an asymptotic analysis of (2.3) in this section in order to provide a meaningful choice of L. The technique can be applied also for finite domains. We refer to [3, 7] for general asymptotic techniques of singularly perturbed parabolic equations.

6.1. Case 1: $\varepsilon_1 \ll 1$

Since the initial condition $v(x,0) = \max(e^x - 1, 0)$ does not have a continuous derivative at $x = 0$, we first construct an approximation $\phi(x)$ of it, where $\phi(x)$ has a continuous second order derivative. That is, we construct a polynomial $p(x)$ in the interval $(-\delta, \delta)$

satisfying $p(-\delta)=0, p'(-\delta)=0, p''(-\delta)=0, p(\delta)=e^{\delta}-1, p'(\delta)=e^{\delta}, p''(\delta)=e^{\delta}$ and define

$$\phi(x)=\begin{cases} e^x-1, & x\geq\delta,\\ p(x), & -\delta\leq x\leq\delta,\\ 0, & x<-\delta.\end{cases}$$

It is easy to see that

$$p(x)=(x+\delta)^3(a+b(x-\delta)+c(x-\delta)^2),$$

where

$$a=\frac{e^{\delta}-1}{8\delta^3},\ b=\frac{e^{\delta}-3(e^{\delta}-1)/2\delta}{8\delta^3},$$
$$c=\frac{e^{\delta}-3e^{\delta}/\delta+3(e^{\delta}-1)/\delta^2}{16\delta^3}.$$

It is not difficult to verify that $p(x)$ is monotonically increasing in $(-\delta,\delta)$, $\phi(x)\in C^2(-\infty,\infty)$, and $|\phi(x)-v(x,0)|\leq M\delta^2$, where M is a generic constant independent of δ. Also we have that in $(-\delta,\delta)$:

$$p(x)=\mathcal{O}(\delta),\ \ p\prime(x)=\mathcal{O}(1),\ \ p\prime\prime(x)=\mathcal{O}(1/\delta).$$

The reduced problem corresponding to (2.3) is

$$\frac{\partial v_0}{\partial\tau}=0,\ \ v_0(x,0)=\max(e^x-1,0),$$

and so the reduced solution is

$$v_0(x,\tau)=\max(e^x-1,0).$$

Based on the construction of $\phi(x)$, we have

$$v_0(x,\tau)-\phi(x)=\begin{cases} 0, & x\notin(-\delta,\delta),\\ \mathcal{O}(\delta), & x\in(-\delta,\delta).\end{cases}$$

We can also estimate $v(x,\tau)-\phi(x)$, where $v(x,\tau)$ is the solution of (2.3). It is easy to see that

$$v(x,0)-\phi(x)=v_0(x,0)-\phi(x)=\mathcal{O}(\delta)$$

and

$$\begin{aligned} P(v-\phi)\ &\equiv\ \frac{\partial(v-\phi)}{\partial\tau}-\varepsilon_1(\varepsilon_2\frac{\partial^2(v-\phi)}{\partial x^2}+(1-\varepsilon_2)\frac{\partial(v-\phi)}{\partial x}-(v-\phi))\\ &=\ \varepsilon_1(\varepsilon_2\frac{\partial^2\phi}{\partial x^2}+(1-\varepsilon_2)\frac{\partial\phi}{\partial x}-\phi)\\ &\equiv\ \varepsilon_1F(\varepsilon_2,x),\end{aligned}$$

with

$$F(\varepsilon_2, x) = \begin{cases} \mathcal{O}(1+\varepsilon_2/\delta), & x \in (-\delta, \delta), \\ \mathcal{O}(1), & x \notin (-\delta, \delta). \end{cases}$$

The parabolic operator P satisfies the maximum principle

$$P\Phi \geq 0, \Phi|_{\tau=0} \geq 0 \Rightarrow \Phi \geq 0$$

for all $\tau, x \in (-\infty, \infty)$. We construct the barrier function

$$\Phi(x,\tau) = \pm(v(x,\tau) - \phi) + M[\varepsilon_1(1+\varepsilon_2/\delta)\tau + \delta].$$

Then it is easy to verify that if M is sufficiently large $L\Phi \geq 0$ and $\Phi|_{\tau=0} \geq 0$. Hence $\Phi \geq 0$ for all x, τ. That is

$$|v(x,\tau) - \phi(x)| \leq M[\varepsilon_1(1+\varepsilon_2/\delta)\tau + \delta], \quad \tau \leq 1,$$

or

$$\begin{aligned} v(x,\tau) &= v_0(x,\tau) + \phi(x) - v_0(x,\tau) + \mathcal{O}(\varepsilon_1(1+\varepsilon_2/\delta)\tau + \delta) \\ &= \max(e^x - 1, 0) + \mathcal{O}(\varepsilon_1(1+\varepsilon_2/\delta)\tau + \delta) \\ &= \max(e^x - 1, 0) + \mathcal{O}(\sqrt{\varepsilon_1\varepsilon_2}), \end{aligned}$$

taking $\delta = \sqrt{\varepsilon_1\varepsilon_2}$. So $v_0(x,\tau) = \max(e^x - 1, 0)$ is a good approximation to $v(x,\tau)$ when ε_1 or ε_2 is small. On examination of $v_0(x,\tau)$, we see that the discontinuity in the derivative of the initial condition $v(x,0) = \max(e^x - 1, 0)$ at $x = 0$ propagates in the vertical τ-direction in the $x-\tau$ plane. If our finite difference scheme satisfies a discrete maximum principle, then we expect similar behaviour of the numerical solution.

Financially we are interested in the option price in only a relatively small region of x at the final time $\tau = 1$, say, a region $x \in (-2, 2)$ at $\tau = 1$ (often just one value of x at $\tau = 1$). Based on our asymptotic analysis we see that the inaccuracies in the boundary conditions on the artificial boundaries $x = -L$ and $x = L$ propagate in the vertical τ-direction in the $x - \tau$ plane. So it is probably safe to choose $L = 2 + \mathcal{O}(\sqrt{\varepsilon_1\varepsilon_2})$ to calculate the option price in the interval $x \in (-2, 2)$ at $\tau = 1$.

6.2. Case 2: $\varepsilon_1 = \mathcal{O}(1), \varepsilon_2 \ll 1$

Without loss of generality we let $\varepsilon_1 = 1$. The reduced problem corresponding to (2.3) is now

$$\frac{\partial v_0}{\partial \tau} = \frac{\partial v_0}{\partial x} - v_0\,, \quad v_0(x,0) = \max(e^x - 1, 0).$$

This can be written in the form

$$\frac{\partial (e^{-x} v_0)}{\partial \tau} = \frac{\partial (e^{-x} v_0)}{\partial x},$$

so

$$e^{-x}v_0(x,\tau) = e^{-(\tau+x)}\max(e^{\tau+x}-1,0)$$

or

$$v_0(x,\tau) = e^{-\tau}\max(e^{\tau+x}-1,0).$$

Now we want to estimate $v(x,\tau)-v_0(x,\tau)$. Due to the discontinuity in the derivative of $v_0(x,\tau)$ along the straight line $x+\tau=0$ we need to construct a smooth function $\phi(x)$ to approximate the initial condition $v(x,0)$ in order to estimate $v_(x,\tau)-v_0(x,\tau)$. Then the reduced problem corresponding to (2.3) satisfying the smooth initial condition ϕ has the form

$$\frac{\partial \bar{v}_0}{\partial \tau} = \frac{\partial \bar{v}_0}{\partial x} - \bar{v}_0\,, \quad \bar{v}_0(x,0) = \phi(x),$$

where ϕ is the same function as in Case 1. We then get

$$\bar{v}_0(x,\tau) = e^{-\tau}\phi(\tau+x).$$

From the results obtained in Case 1 for ϕ, we have $\bar{v}_0(x,\tau)\in C^2((-\infty,\infty)\times[0,1])$ and $\bar{v}_0(x,\tau)-v_0(x,\tau)=\mathcal{O}(\delta)e^{-\tau}$. Then

$$\begin{aligned} P(v-\bar{v}_0) \equiv\ & \frac{\partial(v-\bar{v}_0)}{\partial\tau} - \varepsilon_2\frac{\partial^2(v-\bar{v}_0)}{\partial x^2} - (1-\varepsilon_2)\frac{\partial(v-\bar{v}_0)}{\partial x} + (v-\bar{v}_0) \\ =\ & \varepsilon_2(\frac{\partial^2\bar{v}_0}{\partial x^2} + \frac{\partial\bar{v}_0}{\partial x}) \\ \equiv\ & \varepsilon_2\bar{F}(x). \end{aligned}$$

From the properties of ϕ we see that

$$F(x) = \begin{cases} \mathcal{O}(1+1/\delta), & x+\tau\in(-\delta,\delta), \\ \mathcal{O}(1), & \text{otherwise.} \end{cases}$$

Similarly to Case 1 we construct the barrier function

$$\Phi(x,\tau) = \pm(v(x,\tau)-\bar{v}_0(x,\tau)) + M[\varepsilon_2(1+1/\delta)\tau+\delta],$$

and we verify that, for M sufficiently large, $L\Phi\ge 0$ and $\Phi|_{\tau=0}\ge 0$, and so $\Phi\ge 0$ for all x,τ. Hence

$$|v(x,\tau)-\bar{v}_0(x,\tau)| \le M[\varepsilon_2(1+1/\delta)\tau+\delta],\ \tau\le 1,$$

or

$$|v(x,\tau)-v_0(x,\tau)| \le M[\varepsilon_2(1+1/\delta)\tau+\delta],\ \tau\le 1.$$

So $v(x,\tau)=v_0(x,\tau)+\mathcal{O}(\sqrt{\varepsilon_2})$ if we take $\delta=\sqrt{\varepsilon_2}$. That is $v_0(x,\tau)$ is a good approximation to $v(x,\tau)$ if ε_2 is small. From the expression for $v_0(x,\tau)$ we see that the non-smoothness of the initial value $v(x,0)$ at $x=0$ propagates along the straight line $x+\tau=0$ in the $x-\tau$ plane (i.e., 135^o direction). Furthermore, any error at $\tau=0$ decays along $x+\tau=0$ at a rate $e^{-\tau}$.

From the above analysis we can get a restriction on choosing the position of the artificial boundaries. If we are interested in the option price in the interval $x \in (-2, 2)$ at the terminal time $\tau = 1$, we should choose L at least as large as $2 + 1 + \mathcal{O}(\sqrt{\varepsilon_2}) = 3 + \mathcal{O}(\sqrt{\varepsilon_2})$. This ensures that the inaccuracy at the artificial boundary $x = L$ does not propagate into the interval $(-2, 2)$ at $\tau = 1$ (along the line $x + \tau = 0$).

7. Numerical methods

Since there may be thin layers and other singular behaviour due to the singularities of the problem, we need to measure the error in the numerical approximations in the maximum norm. Due to the lack of theoretical error estimates in the maximum norm for finite element methods, we prefer to use finite difference methods to obtain controllable numerical approximations.

In our computations we use centered and upwind finite difference operators in the x-variable and explicit, implicit and Crank-Nicolson (midpoint) finite difference operators in the τ-variable. We use appropriately constructed uniform and fitted meshes. The uniform x-mesh is $\{X(i)\}_1^{M+2}$, where $X(1), X(M+2)$ are respectively the left,right boundary points. The uniform τ-mesh is $\{T(j)\}_1^{N+1}$, where $T(1) = 0$ and $T(N+1) = 1$. We use a tensor product of these one-dimensional uniform meshes to obtain a uniform mesh on the two -dimensional domain. For our fitted meshes we use the techniques described in [2–5] to construct one-dimensional fitted meshes and again take the tensor product to obtain fitted meshes on the two-dimensional domain.

We use the following notation for the finite difference operators

$$D_x^+\phi_i = \frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i},\ D_x^-\phi_i = \frac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}},\ D_x^0\phi_i = \frac{\phi_{i+1} - \phi_{i-1}}{x_{i+1} - x_{i-1}},\ D_t^+\psi_j = \frac{\psi_{j+1} - \psi_j}{t_{j+1} - t_j}.$$

The finite difference equations, centered in x, are then

$$\begin{aligned} D_\tau^+ V_{i,j} = {} & \theta\varepsilon_1(\varepsilon_2 D_x^+ D_x^- V_{i,j+1} + (1-\varepsilon_2) D_x^0 V_{i,j+1} - V_{i,j+1}) \\ & + (1-\theta)\varepsilon_1(\varepsilon_2 D_x^+ D_x^- V_{i,j} + (1-\varepsilon_2) D_x^0 V_{i,j} - V_{i,j}) \end{aligned}$$

for $2 \le i \le M+1,\ \ 1 \le j \le N$, where $V_{i,j}$ is the numerical approximation to the exact solution $v(X_i, T_j)$. The explicit, implicit and Crank-Nicolson operators in t correspond respectively to $\theta = 0$, $\theta = 1$ and $\theta = 1/2$.

Furthermore, the standard finite difference equations, upwind in x, are

$$\begin{aligned} D_\tau^+ V_{i,j} = {} & \theta\varepsilon_1(\varepsilon_2 D_x^+ D_x^- V_{i,j+1} + (1-\varepsilon_2)^+ D_x^+ V_{i,j+1} + (1-\varepsilon_2)^- D_x^- V_{i,j+1} - V_{i,j+1}) \\ & + (1-\theta)\varepsilon_1(\varepsilon_2 D_x^+ D_x^- V_{i,j} + (1-\varepsilon_2)^+ D_x^+ V_{i,j} + (1-\varepsilon_2)^- D_x^- V_{i,j} - V_{i,j}) \end{aligned}$$

for $2 \le i \le M+1,\ \ 1 \le j \le N$, where for any quantity ϕ we use the notation

$$\phi^+ = \frac{1}{2}(\phi + |\phi|),\ \ \phi^- = \frac{1}{2}(\phi - |\phi|).$$

The explicit, implicit and Crank-Nicolson operators in t correspond respectively to $\theta = 0$, $\theta = 1$ and $\theta = 1/2$. The initial condition is

$$V_{i,1} = \max\{e^{X_i} - 1, 0\}$$

for $1 \leq i \leq M + 2$.

8. Numerical treatment of the infinite domain

As discussed in the previous section, since the domain of the exact solution is infinite in the space variable x and since the numerical domain must be finite, it is necessary to construct artificial boundaries at finite values of x and, in the case of implicit in τ methods, to impose boundary conditions on these artificial boundaries.

For vanilla European options these boundary conditions pose no problem, because we know the exact solution, and so the values of the exact solution on the artificial boundaries give us the exact boundary conditions. This means that we have isolated the errors due to the infinite x-domain from the other singularities. Therefore we can study the *exact* errors in the numerical solutions of the dimensionless Black-Scholes equation for vanilla European options arising from the discontinuity in the derivative of the initial condition and from boundary and interior layers arising from large or small values of the coefficients. However, in most cases when we are using numerical methods to compute the value of an option, we do not know the exact solution in advance, and so we must use techniques other than the above. We do not pursue this matter here.

9. Numerical solutions

We compute solutions for the extreme problems of each of the four classes $C_{0,0}, C_{0,1}, C_{1,0}, C_{1,1}$, which correspond to values of $(\varepsilon_1, \varepsilon_2) = (2^{q_1}, 2^{q_2})$ where (q_1, q_2) take successively the values $(4, 4)$, $(4, -12)$, $(-12, 4)$ and $(-12, -12)$. We make these computations using a variety of different finite difference operators. But due to the page limitation we only show a few results. All other computational results may be found in our extensive research report [1].

We use a variety of different meshes with $N = N_x = N_\tau$ in all cases. For the results we show below the temporal discretization is either Crank-Nicolson or implicit Euler, and the finite difference operators in x are all centered. The meshes are uniform in all cases with N ranging from 16 to 512. Some incomplete results for piecewise-uniform (fitted) meshes may be found in [1], which show a little bit better uniform convergence rate. But improvement is not significant due to other sources of error, such as discontinuity in the derivative of the solution. Further numerical experiments need be carried out and reported in near future.

We present computational results in tables which come in pairs. The first of the pair is a table of the error in the maximum norm for the particular value of the free parameter (row) and a particular value of the discretization parameter N (column). The corresponding entry is the maximum norm over the whole solution domain for that particular problem.

Table 2: Errors for $\varepsilon_2 = 2^4$ with implicit Euler, centered operator, uniform mesh.

$\varepsilon_1 \backslash$ N	16	32	64	128	256	512
2^4	2.453393e-02	4.090640e-02	5.558417e-02	6.274202e-02	6.084209e-02	5.062901e-02
2^3	2.632904e-02	4.197406e-02	5.625624e-02	5.799328e-02	4.948874e-02	3.603977e-02
2^2	3.000451e-02	5.020275e-02	5.522310e-02	4.836619e-02	3.562738e-02	2.406533e-02
2^1	3.907600e-02	4.994743e-02	4.617159e-02	3.481138e-02	2.378890e-02	1.630502e-02
2^0	4.039252e-02	4.192629e-02	3.322032e-02	2.324136e-02	1.612738e-02	1.136702e-02
2^{-1}	3.431060e-02	3.017361e-02	2.217039e-02	1.577734e-02	1.125498e-02	7.997312e-03
2^{-2}	2.432899e-02	2.003486e-02	1.506957e-02	1.102729e-02	7.923309e-03	5.641428e-03
2^{-3}	2.275167e-02	1.352622e-02	1.052052e-02	7.756225e-03	5.585447e-03	3.982248e-03
2^{-4}	2.072235e-02	1.121119e-02	7.349421e-03	5.446558e-03	3.934450e-03	2.810166e-03
2^{-5}	1.832281e-02	9.877670e-03	5.132265e-03	3.801068e-03	2.763318e-03	1.980233e-03
2^{-6}	1.513077e-02	8.107008e-03	4.189694e-03	2.617093e-03	1.928074e-03	1.390899e-03
2^{-7}	1.147099e-02	5.997526e-03	3.066488e-03	1.752015e-03	1.327155e-03	9.704550e-04
2^{-8}	7.455886e-03	3.780394e-03	1.898913e-03	1.105050e-03	8.883369e-04	6.679661e-04
2^{-9}	4.409734e-03	2.135460e-03	1.039922e-03	6.077083e-04	5.604101e-04	4.471075e-04
2^{-10}	2.967134e-03	1.340273e-03	6.195048e-04	3.178791e-04	3.085679e-04	2.820994e-04
2^{-11}	2.442735e-03	1.101596e-03	4.709120e-04	2.622652e-04	1.640630e-04	1.554208e-04
2^{-12}	2.478695e-03	1.150821e-03	5.397733e-04	2.295145e-04	1.286998e-04	8.369763e-05

Table 3: ε_1-uniform rates for $\varepsilon_2 = 2^4$ with implicit Euler, centered operator, uniform mesh.

N	16	32	64	128	256	*
D^N	0.023079	0.030264	0.034479	0.032673	0.027309	–
p^N	-0.390971	-0.188137	0.077602	0.258760	–	-0.390971
$C^N_{p^*}$	-0.025079	-0.025079	-0.021789	-0.015747	-0.010037	-0.010037
$C^*_{p^*}N^{-p^*}$	-0.029674	-0.038911	-0.051023	-0.066905	-0.087731	–

Thus, the first table of a pair represents the solution of 15×6=90 distinct problems. The second table of the pair displays experimentally determined values of the error parameters for the particular method under consideration. It uses the experimental technique introduced in [2] (Chapter 8). The entries are derived from the same numerical solutions that are used in the first table of the pair. The first row gives the maximum of the two successive mesh differences, the second row gives a local rate of convergence, the last entry of the row being $p*$ the minimum of the other entries and the computed parameter-uniform rate of convergence. Negative values of the entries in this row indicate that there is no parameter-uniform convergence for the current numerical method and parameter values; positive values indicate that there is parameter-uniform convergence. The third row gives local error constants, the last entry of the row being the maximum of the other entries and the computed parameter uniform error constant C^*_{p*}. The final row is the experimentally determined parameter-uniform error bound. The three dashes in each of these tables indicates that an entry there is not applicable. The final column heading * indicates that the two entries in that column are the value of $p*$ and the value of C^*_{p*}. The captions of the tables include the information about the finite difference operator and the mesh used to generate the displayed results.

In the following two tables we see no ϵ_1-uniform convergence (note the anomalous negative values in Table 2). They indicate however that the centered finite difference scheme in x is stable for these practical values of ε_1 and ε_2.

Table 4: Errors for $\varepsilon_1 = 2^{-12}$ with implicit Euler, centered operator, uniform mesh.

$\varepsilon_2 \setminus$ N	16	32	64	128	256	512
2^4	2.150369e-03	1.071562e-03	5.262806e-04	2.235133e-04	1.266895e-04	8.595719e-05
2^3	2.162615e-03	1.098546e-03	5.406129e-04	2.638044e-04	1.125697e-04	6.326434e-05
2^2	2.135008e-03	1.101413e-03	5.541875e-04	2.721031e-04	1.320528e-04	5.650668e-05
2^1	1.681422e-03	1.109473e-03	5.592291e-04	2.792237e-04	1.368251e-04	6.612159e-05
2^0	1.033723e-03	1.113409e-03	5.678942e-04	2.828175e-04	1.404178e-04	6.865276e-05
2^{-1}	5.636745e-04	8.789767e-04	5.817293e-04	2.872926e-04	1.422653e-04	7.041010e-05
2^{-2}	3.140630e-04	5.438439e-04	5.897123e-04	2.942714e-04	1.444812e-04	7.135061e-05
2^{-3}	1.886587e-04	3.093299e-04	4.702735e-04	3.074571e-04	1.479914e-04	7.244912e-05
2^{-4}	1.258463e-04	1.860592e-04	3.017493e-04	3.200211e-04	1.546133e-04	7.420946e-05
2^{-5}	9.441244e-05	1.241724e-04	1.848022e-04	2.644570e-04	1.677014e-04	7.752933e-05
2^{-6}	7.868859e-05	9.317927e-05	1.234855e-04	1.817468e-04	1.831393e-04	8.408618e-05
2^{-7}	7.082493e-05	7.767024e-05	9.270917e-05	1.234617e-04	1.613718e-04	9.710378e-05
2^{-8}	6.689267e-05	6.991260e-05	7.729731e-05	9.280297e-05	1.224198e-04	1.138982e-04
2^{-9}	6.492643e-05	6.603300e-05	6.958545e-05	7.741369e-05	9.349331e-05	1.099901e-04
2^{-10}	6.394328e-05	6.409300e-05	6.572804e-05	6.970743e-05	7.808548e-05	9.367729e-05
2^{-11}	6.345170e-05	6.312295e-05	6.379896e-05	6.585139e-05	7.034750e-05	7.962140e-05
2^{-12}	6.320591e-05	6.263792e-05	6.283432e-05	6.392264e-05	6.647269e-05	7.183095e-05

Table 5: ε_2-uniform rates for $\varepsilon_1 = 2^{-12}$ with implicit Euler, centered operator, uniform mesh.

N	16	32	64	128	256	*
D^N	0.001100	0.000588	0.000293	0.000163	0.000101	–
p^N	0.902751	1.005400	0.843414	0.689216	–	0.689216
$C^N_{p^*}$	0.019574	0.016881	0.013559	0.012184	0.012184	0.019574
$C^*_{p^*} N^{-p^*}$	0.002896	0.001796	0.001114	0.000691	0.000428	–

Table 6: Errors for $\varepsilon_2 = 2^{-12}$ with implicit Euler, centered operator, uniform mesh.

$\varepsilon_1 \setminus$ N	16	32	64	128	256	512
2^4	1.239506e-01	9.901791e-02	8.700653e-02	6.936237e-02	5.123501e-02	3.522006e-02
2^3	7.135972e-02	8.420085e-02	6.645256e-02	4.984439e-02	3.656676e-02	2.591016e-02
2^2	9.406462e-02	7.395970e-02	5.531170e-02	3.991516e-02	2.802125e-02	1.933944e-02
2^1	8.556035e-02	6.358104e-02	4.572286e-02	3.204279e-02	2.197950e-02	1.474418e-02
2^0	7.875750e-02	5.564344e-02	3.879386e-02	2.659635e-02	1.790269e-02	1.178850e-02
2^{-1}	7.110934e-02	4.996162e-02	3.436220e-02	2.321418e-02	1.540262e-02	9.991684e-03
2^{-2}	5.562872e-02	4.030937e-02	2.797497e-02	1.892606e-02	1.252193e-02	8.073728e-03
2^{-3}	3.525937e-02	2.842637e-02	2.042722e-02	1.401375e-02	9.316075e-03	6.006740e-03
2^{-4}	3.486517e-02	1.742305e-02	1.387640e-02	9.860201e-03	6.653432e-03	4.314009e-03
2^{-5}	1.809540e-02	1.729796e-02	8.515393e-03	6.635837e-03	4.623495e-03	3.040960e-03
2^{-6}	8.591612e-03	8.905157e-03	8.402215e-03	4.082610e-03	3.101076e-03	2.109705e-03
2^{-7}	4.168283e-03	4.230312e-03	4.426837e-03	4.009483e-03	1.917741e-03	1.432678e-03
2^{-8}	2.050417e-03	2.054632e-03	2.104108e-03	2.216849e-03	1.872614e-03	9.410980e-04
2^{-9}	1.016536e-03	1.011520e-03	1.022621e-03	1.054917e-03	1.112533e-03	8.497149e-04
2^{-10}	5.060689e-04	5.017247e-04	5.036726e-04	5.129675e-04	5.338278e-04	5.498088e-04
2^{-11}	2.524808e-04	2.498426e-04	2.498906e-04	2.527312e-04	2.597458e-04	2.739578e-04
2^{-12}	1.261015e-04	1.246650e-04	1.244543e-04	1.254107e-04	1.280163e-04	1.335339e-04

In Tables 3 and 4 we consider cases where either ε_1 or ε_2 is small. We see that there is good ε_2-uniform convergence when ε_1 is small. However, we see in Tables 5 and 6 that, when ε_2 is small, the ε_1-uniform convergence is greatly reduced, even though the same

Table 7: ε_1-uniform rates for $\varepsilon_2 = 2^{-12}$ with implicit Euler, centered operator, uniform mesh.

N	16	32	64	128	256	*
D^N	0.098006	0.095139	0.081150	0.062331	0.043069	–
p^N	0.042839	0.229432	0.380643	0.533310	–	0.042839
$C^N_{p^*}$	3.772294	3.772294	3.314631	2.622684	1.866807	3.772294
$C^*_{p^*}N^{-p^*}$	3.349828	3.251822	3.156683	3.064328	2.974675	–

Table 8: Errors for $\varepsilon_2 = 2^4$ with Crank Nicolson, centered operator, uniform mesh.

$\varepsilon_1 \backslash$ N	16	32	64	128	256	512
2^4	3.894781e-01	3.488495e-01	2.888428e-01	2.206858e-01	1.559965e-01	1.054759e-01
2^3	2.632904e-02	4.197406e-02	5.625624e-02	5.799328e-02	4.948874e-02	3.603977e-02
2^2	3.000451e-02	5.020275e-02	5.522310e-02	4.836619e-02	3.562738e-02	2.406533e-02
2^1	3.907600e-02	4.994743e-02	4.617159e-02	3.481138e-02	2.378890e-02	1.630502e-02
2^0	4.039252e-02	4.192629e-02	3.322032e-02	2.324136e-02	1.612738e-02	1.136702e-02
2^{-1}	3.431060e-02	3.017361e-02	2.217039e-02	1.577734e-02	1.125498e-02	7.997312e-03
2^{-2}	2.432899e-02	2.003486e-02	1.506957e-02	1.102729e-02	7.923309e-03	5.641428e-03
2^{-3}	2.275167e-02	1.352622e-02	1.052052e-02	7.756225e-03	5.585447e-03	3.982248e-03
2^{-4}	2.072235e-02	1.121119e-02	7.349421e-03	5.446558e-03	3.934450e-03	2.810166e-03
2^{-5}	1.832281e-02	9.877670e-03	5.132265e-03	3.801068e-03	2.763318e-03	1.980233e-03
2^{-6}	1.513077e-02	8.107008e-03	4.189694e-03	2.617093e-03	1.928074e-03	1.390899e-03
2^{-7}	1.147099e-02	5.997526e-03	3.066488e-03	1.752015e-03	1.327155e-03	9.704550e-04
2^{-8}	7.455886e-03	3.780394e-03	1.898913e-03	1.105050e-03	8.883369e-04	6.679661e-04
2^{-9}	4.409734e-03	2.135460e-03	1.039922e-03	6.077083e-04	5.604101e-04	4.471075e-04
2^{-10}	2.967134e-03	1.340273e-03	6.195048e-04	3.178791e-04	3.085679e-04	2.820994e-04
2^{-11}	2.442735e-03	1.101596e-03	4.709120e-04	2.622652e-04	1.640630e-04	1.554208e-04
2^{-12}	2.478695e-03	1.150821e-03	5.397733e-04	2.295145e-04	1.286998e-04	8.369763e-05

Table 9: ε_1-uniform rates for $\varepsilon_2 = 2^4$ with Crank Nicolson, centered operator, uniform mesh.

N	16	32	64	128	256	*
D^N	0.680949	0.564019	0.427296	0.304001	0.212088	–
p^N	0.271803	0.400508	0.491159	0.519415	–	0.271803
$C^N_{p^*}$	8.425321	8.425321	7.706237	6.619265	5.575336	8.425321
$C^*_{p^*}N^{-p^*}$	3.965547	3.284598	2.720580	2.253412	1.866465	–

numerical method is used. Note that there is slow ε_1 convergence although ε_2 is very small. The following four tables show two cases using the Crank Nicolson scheme and the centered spatial operator (in comparison with Tables 2-3 and 6-7). We see that, for these cases, if we replace the implicit Euler by the Crank Nicolson scheme both ε_1-uniform and ε_2-uniform rates are improved (probably due to higher accuracy of the Crank Nicolson scheme). Note, however, that this does not necessarily mean that such behaviour is always to be expected.

We have slow ε_2-uniform convergence.

10. Conclusion

In this short paper we have presented some results, but there are still interesting problems left for further studies. We have seen that the problem is complicated, since it has

Table 10: Errors for $\varepsilon_2 = 2^{-12}$ with Crank Nicolson, centered operator, uniform mesh.

$\varepsilon_1 \setminus$ N	16	32	64	128	256	512
2^4	1.027500e-01	6.461273e-02	3.719955e-02	2.860507e-02	1.647026e-02	8.560908e-03
2^3	7.135972e-02	8.420085e-02	6.645256e-02	4.984439e-02	3.656676e-02	2.591016e-02
2^2	9.406462e-02	7.395970e-02	5.531170e-02	3.991516e-02	2.802125e-02	1.933944e-02
2^1	8.556035e-02	6.358104e-02	4.572286e-02	3.204279e-02	2.197950e-02	1.474418e-02
2^0	7.875750e-02	5.564344e-02	3.879386e-02	2.659635e-02	1.790269e-02	1.178850e-02
2^{-1}	7.110934e-02	4.996162e-02	3.436220e-02	2.321418e-02	1.540262e-02	9.991684e-03
2^{-2}	5.562872e-02	4.030937e-02	2.797497e-02	1.892606e-02	1.252193e-02	8.073728e-03
2^{-3}	3.525937e-02	2.842637e-02	2.042722e-02	1.401375e-02	9.316075e-03	6.006740e-03
2^{-4}	3.486517e-02	1.742305e-02	1.387640e-02	9.860201e-03	6.653432e-03	4.314009e-03
2^{-5}	1.809540e-02	1.729796e-02	8.515393e-03	6.635837e-03	4.623495e-03	3.040960e-03
2^{-6}	8.591612e-03	8.905157e-03	8.402215e-03	4.082610e-03	3.101076e-03	2.109705e-03
2^{-7}	4.168283e-03	4.230312e-03	4.426837e-03	4.009483e-03	1.917741e-03	1.432678e-03
2^{-8}	2.050417e-03	2.054632e-03	2.104108e-03	2.216849e-03	1.872614e-03	9.410980e-04
2^{-9}	1.016536e-03	1.011520e-03	1.022621e-03	1.054917e-03	1.112533e-03	8.497149e-04
2^{-10}	5.060689e-04	5.017247e-04	5.036726e-04	5.129675e-04	5.338278e-04	5.498088e-04
2^{-11}	2.524808e-04	2.498426e-04	2.498906e-04	2.527312e-04	2.597458e-04	2.739578e-04
2^{-12}	1.261015e-04	1.246650e-04	1.244543e-04	1.254107e-04	1.280163e-04	1.335339e-04

Table 11: ε_1-uniform rates for $\varepsilon_2 = 2^{-12}$ with Crank Nicolson, centered operator, uniform mesh.

N	16	32	64	128	256	*
D^N	0.112010	0.074310	0.038860	0.030945	0.016014	–
p^N	0.592008	0.935279	0.328586	0.950330	–	0.328586
$C^N_{p^*}$	1.367599	1.139359	0.748217	0.748217	0.486255	1.367599
$C^*_{p^*}N^{-p^*}$	0.549923	0.437912	0.348717	0.277689	0.221128	–

several singularities simultaneously. This influences the error behavior of the numerical solution and reduces the convergence rate of known numerical methods. Further investigations, not undertaken here, are required to study separately the influence of singularities such as the boundary, interior and initial layers on the error behavior. Furthermore,the numerical investigations in this paper show no decisive evidence of advantages or disadvantages in using Crank-Nicolson or Euler (explicit or implicit) temporal discretizations in conjunction with centered or upwind discretizations of the first derivative.

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