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A Parameter-Uniform Finite Difference Method for a Coupled System of Convection-Diffusion Two-Point Boundary Value Problems

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Abstract. A system of m (≥ 2) linear convection-diffusion two-point boundary value problems is examined, where the diffusion term in each equation is multiplied by a small parameter ε and the equations are coupled through their convective and reactive terms via matrices B and A respectively. This system is in general singularly perturbed. Unlike the case of a single equation, it does not satisfy a conventional maximum principle. Certain hypotheses are placed on the coupling matrices B and A that ensure existence and uniqueness of a solution to the system and also permit boundary layers in the components of this solution at only one endpoint of the domain; these hypotheses can be regarded as a strong form of diagonal dominance of B. This solution is decomposed into a sum of regular and layer components. Bounds are established on these components and their derivatives to show explicitly their dependence on the small parameter ε. Finally, numerical methods consisting of upwinding on piecewise-uniform Shishkin meshes are proved to yield numerical solutions that are essentially first-order convergent, uniformly in ε, to the true solution in the discrete maximum norm. Numerical results on Shishkin meshes are presented to support these theoretical bounds.

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Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

While the numerical analysis of singularly perturbed convection-diffusion problems has received much attention in recent years [6, 12, 14], the main focus has been on single equations of various types—systems of equations appear relatively rarely. Nevertheless

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coupled systems of convection-diffusion equations do appear in many applications, notably optimal control problems and in certain resistance-capacitor electrical circuits; see [7].

In this paper we consider a system of $m \geq 2$ convection-diffusion equations in the unknown vector function $\boldsymbol{u} = (u_1, u_2, \cdots, u_m)^T$. This system is coupled through its convective and reactive terms:

$$L\boldsymbol{u} := (-\varepsilon \boldsymbol{u}'' - B\boldsymbol{u}' + A\boldsymbol{u})(x) = \boldsymbol{f}(x), \quad x \in (0,1) \tag{1.1}$$

and it satisfies boundary conditions $\boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}$. Since the problem is linear there is no loss in generality in assuming homogeneous boundary conditions. Here $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times m$ matrices whose entries are assumed to lie in $C^3[0,1]$, and $\varepsilon > 0$ is a small diffusion parameter whose presence makes the problem singularly perturbed. We assume that $\boldsymbol{f} = (f_1, \cdots, f_m)^T \in (C^3[0,1])^m$.

Systems of this type from optimal control problems often have a different diffusion coefficient ε_i associated with the i^{th} equation for $i = 1, \cdots, m$, but with all ratios $\varepsilon_i/\varepsilon_j$ bounded by a fixed constant [7, p.503]; one can then rescale all equations to the form (1.1) with affecting the analysis and conclusions of this paper, so our assumption of a single value ε is not a restriction in this case.

Assumption 1.1. *In the matrices $B = (b_{ij})$ and $A = (a_{ij})$, for $i = 1, \cdots, m$ one has*

$$\beta_i := \min_{x \in [0,1]} b_{ii}(x) > 0 \tag{1.2a}$$

and

$$a_{ii}(x) \geq 0 \quad \textit{for } x \in [0,1]. \tag{1.2b}$$

Similar assumptions are often made in scalar convection-diffusion equations, where in particular any sign change or vanishing of the coefficient of the first-derivative term alters significantly the nature of the solution; see, e.g., [12]. Each component u_i of our solution $\boldsymbol{u}$ will exhibit a boundary layer and (1.2a) enables us to predict that the layer in $u_i(x)$ will be at $x = 0$.

Further hypotheses will be placed on B in Section 2, but our collective hypotheses are not strong enough to guarantee that the differential operator of (1.1) satisfies a standard maximum principle; see, e.g., [11, Example 2.1]. This excludes the most commonly-used tool in finite difference analysis of singularly perturbed differential equations and forces us to develop an alternative methodology.

Notation. Throughout the paper C denotes a generic constant that is independent of ε and any mesh, and can take on different values at different points in the argument. Write $\|\cdot\|_\infty$ for the norm on $L_\infty[0,1]$. Set

$$\|\mathbf{g}\|_\infty = \max\{\|g_1\|_\infty, \cdots, \|g_m\|_\infty\}$$

for any vector-valued function $\boldsymbol{g} = (g_1, \cdots, g_m)^T$ having $g_i \in L_\infty(0,1)$ for all i. For each $w \in W^{-1,\infty}$ define the norm

$$\|w\|_{-1,\infty} = \inf\{\|W\|_\infty : W' = w\}.$$

We shall also use the usual $L_1[0,1]$ norm $\|\cdot\|_{L_1}$.

1.1. Previous work on strongly coupled systems

When a system of singularly perturbed differential equations is coupled through their convective (first-order) terms, we describe it as *strongly coupled*. Singularly perturbed systems that are coupled only through their reactive (zero-order) terms are more easily analyzed and we do not consider them here.

In [1] an analysis of a strongly coupled convection-diffusion system (and of a numerical method that uses upwinding on an equidistant mesh) is carried out, but the matrix B there is assumed to be Hermitian, which is restrictive, and the nature of the mesh means that one cannot expect any accurate computation of the layers.

In [9] Linß considers the strongly coupled system (1.1) where (1.2a) is replaced by

$$\text{either} \quad \min_{x\in[0,1]} b_{ii}(x) > 0 \qquad \text{or} \qquad \max_{x\in[0,1]} b_{ii}(x) < 0 \quad \text{for } i = 1, \cdots, m. \tag{1.3}$$

Then layers can appear in solution components at both $x = 0$ and $x = 1$. He also assumes (1.2b) and

$$b'_{ii} + a_{ii} \geq 0 \quad \text{on } [0,1]. \tag{1.4}$$

Given (1.3), one can then ensure both (1.2b) and (1.4) by a simple change of dependent variable (as is pointed out in [9]) but this change of variable modifies many terms in (1.1), which will affect the results of Andreev [4] that are invoked in our later analysis and in [9]. In the current paper, (1.2b) and (1.4) cannot simply be assumed to follow from (1.2a) without affecting some of our subsequent work—in particular Assumption 2.1 would become more restrictive.

Linß permits the use of a diffusion coefficients ε_i for the i^{th} equation for $i = 1, \cdots, m$. A general numerical analysis of (1.1) for upwinding on an arbitrary mesh appears in [9] but no concrete convergence result is proved for any specific numerical method.

Finally, the special case $m = 2$ is considered in [11], where the continuous problem and a numerical method for its solution are both analyzed under the hypothesis that B is an M-matrix; in [5] this case is also examined but the problem is simplified by a weaker coupling hypothesis.

2. A priori bound on solution

Using our hypothesis (1.2a) we imitate the analysis of [9], but unlike [9] we do not assume (1.4).

Consider first the scalar convection-diffusion two-point boundary value problem

$$-\varepsilon v''(x) - r(x)v'(x) + q(x)v(x) = p(x) \quad \text{on } (0,1), \qquad v(0) = v(1) = 0, \tag{2.1}$$

where $0 < \underline{r} \leq r(x) \leq R$ and $0 \leq q(x) \leq Q$ on $[0,1]$. Set

$$R^* = \int_{x=0}^{1} \left| \left(\frac{1}{r(x)} \right)' \right| dx \quad \text{and} \quad \tilde{R} = \left(1 + \frac{Q}{\underline{r}} \right) \left(R^* + \frac{2}{\underline{r}} \right).$$

Then by the stability theory of Andreev [4, Theorem 3.1], which is based on a careful analysis of Green's functions, one has

$$\|v\|_\infty \le \frac{1}{\underline{r}}\,\|p\|_{L_1}, \tag{2.2a}$$

$$\|v\|_\infty \le \tilde{R}\,\|p\|_{-1,\infty}\,. \tag{2.2b}$$

Correspondingly, for $i = 1, \cdots, m$ set

$$\tilde{R}_i = \left(1 + \frac{\|a_{ii}\|_\infty}{\beta_i}\right)\left(R_i^* + \frac{2}{\beta_i}\right) \quad \text{where} \quad R_i^* := \int_{x=0}^{1} \left|\left(\frac{1}{b_{ii}(x)}\right)'\right| dx.$$

Define the $m \times m$ matrix $\Upsilon = (\gamma_{ij})$ by

$$\gamma_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\left[(\beta_i)^{-1}\|b'_{ij} + a_{ij}\|_{L_1} + \tilde{R}_i\|b_{ij}\|_\infty\right] & \text{if } i \neq j. \end{cases}$$

Assumption 2.1. *The matrix Υ is inverse monotone, i.e., $\Upsilon^{-1} = (y_{ij})$ exists with $y_{ij} \ge 0$ for all i and j.*

This assumption implies that B is strictly diagonally dominant and hence invertible. It also enables us to derive the following bound on $\|\boldsymbol{u}\|_\infty$.

Lemma 2.1. *Any solution $\boldsymbol{u} = (u_1, \cdots, u_m)^T$ of (1.1) must satisfy*

$$\|u_i\|_\infty \le \sum_{j=1}^{m} y_{ij}\tilde{R}_j\,\|f_j\|_{-1,\infty} \quad \textit{for } i = 1, \cdots, m. \tag{2.3}$$

Proof. We use a variant of the proof of [9, Theorem 1]. For $i = 1, \cdots, m$, the i^{th} equation in (1.1) can be rearranged as

$$-\varepsilon u_i'' - b_{ii}u_i' + a_{ii}u_i = f_i + \sum_{\substack{j=1\\ j\neq i}}^{m}\left[(b_{ij}u_j)' - (b'_{ij} + a_{ij})u_j\right], \qquad u_i(0) = u_i(1) = 0.$$

Write $u_i = u_{i1} + u_{i2}$, where

$$-\varepsilon u_{i1}'' - b_{ii}u_{i1}' + a_{ii}u_{i1} = -\sum_{\substack{j=1\\ j\neq i}}^{m}(b'_{ij} + a_{ij})u_j, \qquad u_{i1}(0) = u_{i1}(1) = 0, \tag{2.4a}$$

$$-\varepsilon u_{i2}'' - b_{ii}u_{i2}' + a_{ii}u_{i2} = f_i + \sum_{\substack{j=1\\ j\neq i}}^{m}(b_{ij}u_j)', \qquad u_{i2}(0) = u_{i2}(1) = 0. \tag{2.4b}$$

By Assumption 1.1 we can apply (2.2a) to (2.4a) and (2.2b) to (2.4b); this yields

$$\|u_i\|_\infty \le \|u_{i1}\|_\infty + \|u_{i2}\|_\infty \le \tilde{R}_i\|f_i\|_{-1,\infty} + \sum_{\substack{j=1\\ j\neq i}}^{m}\left(\frac{\|b_{ij}' + a_{ij}\|_{L_1}}{\beta_i} + \tilde{R}_i\|b_{ij}\|_\infty\right)\|u_j\|_\infty.$$

Taking the u_j terms to the left-hand side, we have

$$\|u_i\|_\infty + \sum_{\substack{j=1\\ j\neq i}}^{m}\gamma_{ij}\|u_j\|_\infty \le \tilde{R}_i\|f_i\|_{-1,\infty} \quad \text{for } i=1,\cdots,m.$$

Writing this system in matrix-vector form then multiplying by Υ^{-1} yields the desired result. □

Corollary 2.1. *The system* (1.1) *has a unique solution* $\boldsymbol{u}$.

Proof. When the data $\boldsymbol{f} \equiv \mathbf{0}$, inequality (2.3) implies that (1.1) has only the trivial solution $\boldsymbol{u} = \mathbf{0}$. The result follows. □

3. Decomposition of the solution

Most analysis of numerical methods for scalar convection-diffusion problems decompose the solution of the boundary value problem into a sum of a regular component (whose derivatives up to some order are bounded on [0,1] independently of ε) and a layer component (which has large derivatives in the layer region but dies off exponentially fast outside this region). We now perform the analogous construction for the solution $\boldsymbol{u}$ of our system (1.1), but it is a more delicate matter than in the scalar case to analyze the behaviour of the layer term in this decomposition, as we shall see.

Define the *homogeneous reduced problem* by

$$-B\hat{\boldsymbol{v}}' + A\hat{\boldsymbol{v}} = \mathbf{0} \quad \text{on } (0,1), \quad \hat{\boldsymbol{v}}(1) = \mathbf{0}, \tag{3.1}$$

where $\hat{\boldsymbol{v}} = (\hat{v}_1,\cdots,\hat{v}_m)^T$.

Assumption 3.1. *The problem* (3.1) *has only the trivial solution* $\hat{\boldsymbol{v}} \equiv \mathbf{0}$.

Abrahamsson et al. [1, (1.5)] make the same assumption. As B is invertible, this assumption is equivalent to the assumption that the operator $\hat{\boldsymbol{v}} \mapsto -\hat{\boldsymbol{v}}' + B^{-1}A\hat{\boldsymbol{v}}$ has a fundamental solution matrix $Y(x)$ on [0,1], i.e., that Y is a solution of the system $-Y' + B^{-1}AY = 0$ with $Y(t) = I_m$ (the $m \times m$ identity matrix) for some $t \in [0,1]$.

Now define the *reduced solution* $\boldsymbol{v_0} = (v_{01},\cdots,v_{0m})^T$ of (1.1) to be the solution in $\left(C^2[0,1]\right)^m$ of the problem

$$-B\boldsymbol{v_0}' + A\boldsymbol{v_0} = \boldsymbol{f} \quad \text{on } (0,1), \quad \boldsymbol{v_0}(1) = \mathbf{0}. \tag{3.2}$$

Our assumption that (3.1) has only the trivial solution implies that $\boldsymbol{v_0}$ is well defined.

Define the *regular component* of $\boldsymbol{u}$ to be

$$\boldsymbol{v} = \boldsymbol{v_0} + \varepsilon \boldsymbol{v_1} + \varepsilon^2 \boldsymbol{v_2} \tag{3.3a}$$

where $\boldsymbol{v}_0$ is the reduced solution, $\boldsymbol{v_1} = (v_{11}, v_{12}, \cdots, v_{1m})^T$ is the solution of

$$-B\boldsymbol{v_1}' + A\boldsymbol{v_1} = \boldsymbol{v_0}'' \quad \text{on } (0,1), \quad \boldsymbol{v_1}(1) = \boldsymbol{0} \tag{3.3b}$$

and $\boldsymbol{v_2} = (v_{21}, v_{22}, \cdots, v_{2m})^T$ is the solution of the boundary value problem

$$L\boldsymbol{v_2} = \boldsymbol{v_1}'', \quad \boldsymbol{v_2}(0) = \boldsymbol{v_2}(1) = \boldsymbol{0}. \tag{3.3c}$$

Then clearly

$$\|\boldsymbol{v_0}^{(k)}\|_\infty + \|\boldsymbol{v_1}^{(k)}\|_\infty \le C \quad \text{for } k = 0,1,2,3. \tag{3.4}$$

Applying Lemma 2.1 to (3.3c) and recalling (3.4), we get

$$\|\boldsymbol{v_2}\|_\infty \le C. \tag{3.5}$$

From (3.5) we deduce a bound on $\|\boldsymbol{v_2}'\|_\infty$. Fix $i \in \{1, \cdots, m\}$ and $x \in [0,1]$. Choose an interval $[x^-, x^+]$ of length ε with $x \in [x^-, x^+] \subset [0,1]$. By the mean value theorem there exists $x^* \in [x^-, x^+]$ such that

$$|v_{2i}'(x^*)| = \left| \frac{v_{2i}(x^+) - v_{2i}(x^-)}{\varepsilon} \right| \le C\varepsilon^{-1}. \tag{3.6}$$

On the other hand, the i^{th} equation of (3.3c) gives

$$\begin{aligned}|\varepsilon v_{2i}'(x) - \varepsilon v_{2i}'(x^*)| &= \left| \int_{x^*}^{x} \varepsilon v_{2i}''(s)\, ds \right| = \left| \int_{x^*}^{x} \left[-v_{1i}''(s) + \sum_{k=1}^{m} (-b_{ik} v_{2k}' + a_{ik} v_{2k})(s) \right] ds \right| \\ &= \left| \sum_{k=1}^{m} \left[-(b_{ik} v_{2k})(x) + (b_{ik} v_{2k})(x^*) + \int_{x^*}^{x} (-v_{1i}'' + b_{ik}' v_{2k} + a_{ik} v_{2k})(s)\, ds \right] \right| \le C,\end{aligned}$$

by (3.4) and (3.5). Combining this inequality with (3.6), we get

$$|v_{2i}'(x)| \le C\varepsilon^{-1} \quad \text{for } i = 1, \cdots, m \text{ and } x \in [0,1].$$

It then follows from $L\boldsymbol{v_2} = \boldsymbol{v_1}''$, (3.4) and (3.5) that

$$\|\boldsymbol{v_2}^{(k)}\|_\infty \le C\varepsilon^{-k} \quad \text{for } k = 1,2,3.$$

As $\boldsymbol{v} = \boldsymbol{v_0} + \varepsilon \boldsymbol{v_1} + \varepsilon^2 \boldsymbol{v_2}$, this inequality, (3.4) and (3.5) imply that

$$\|\boldsymbol{v}^{(k)}\|_\infty \le C(1 + \varepsilon^{2-k}) \quad \text{for } k = 0,1,2,3. \tag{3.7}$$

3.1. Layer components

We now decompose the solution $\boldsymbol{u}$ of (1.1) as the sum of the regular component $\boldsymbol{v} = (v_1, \cdots, v_m)^T$ and m layer components $\boldsymbol{w}_i$ for $i = 1, \cdots, m$. First, for each i define the constant $m \times 1$ vector $\boldsymbol{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T$, where the 1 appears in the i^{th} position. Then set

$$\boldsymbol{u} = \boldsymbol{v} + \sum_{i=1}^{m} [(u_i - v_i)(0)] \boldsymbol{w}_i$$

where for $i = 1, \cdots, m$, the vector function $\boldsymbol{w}_i$ satisfies the system

$$L\boldsymbol{w}_i = \boldsymbol{0}, \quad \boldsymbol{w}_i(0) = \boldsymbol{e}_i, \quad \boldsymbol{w}_i(1) = \boldsymbol{0}. \tag{3.8}$$

To analyze $\boldsymbol{w}_i$, fix i and write $\boldsymbol{w}_i(x) = \hat{\boldsymbol{w}}_i(x) + (1 - x)\boldsymbol{e}_i$; then the vector function $\hat{\boldsymbol{w}}_i$ satisfies $\hat{\boldsymbol{w}}_i(0) = \hat{\boldsymbol{w}}_i(1) = \boldsymbol{0}$ with $L\hat{\boldsymbol{w}}_i(x) = -L\big((1 - x)\boldsymbol{e}_i(x)\big)$ on $(0, 1)$, so one can apply Lemma 2.1 to $\hat{\boldsymbol{w}}_i$, which gives existence and uniqueness of $\hat{\boldsymbol{w}}_i$ (and hence also of $\boldsymbol{w}_i$) and leads to the bound

$$\|\boldsymbol{w}_i\|_\infty \le C \quad \text{for } i = 1, \cdots, m, \tag{3.9}$$

where C is some constant.

We show that the $\boldsymbol{w}_i$ decay exponentially away from $x = 0$. Let $j \in \{1, \cdots, m\}$ be arbitrary but fixed. Write $\boldsymbol{w}_j = (w_{j1}, \cdots, w_{jm})^T$. Introduce the vector function $\mathbf{z} = (z_1, \cdots, z_m)^T$ defined by

$$z_i(x) = e^{\alpha x/\varepsilon} w_{ji}(x) \quad \text{on } [0, 1] \quad \text{for } i = 1, \cdots, m, \tag{3.10}$$

where the positive constant α is yet to be specified. Then $z_i(0) = \delta_{ij}$, $z_i(1) = 0$, where δ_{ij} is the Kronecker delta. Our aim is to show that $\|z_i\|_\infty \le C$ for all i, which would give $|w_{ji}(x)| \le Ce^{-\alpha x/\varepsilon}$ for all x; we shall derive this bound by imitating the proof of Lemma 2.1. Now

$$\boldsymbol{0} = L\boldsymbol{w}_j = -\varepsilon \boldsymbol{w}_j'' - B\boldsymbol{w}_j' + A\boldsymbol{w}_j.$$

The i^{th} equation of this system is, for $i = 1, \cdots, m$,

$$\begin{aligned}
0 &= -\varepsilon w_{ji}'' + \sum_{k=1}^{m} (-b_{ik} w_{jk}' + a_{ik} w_{jk}) \\
&= e^{-\alpha x/\varepsilon} \left[-\varepsilon \left(z_i'' - \frac{2\alpha}{\varepsilon} z_i' + \frac{\alpha^2}{\varepsilon^2} z_i \right) + \sum_{k=1}^{m} b_{ik} \left(-z_k' + \frac{\alpha}{\varepsilon} z_k + a_{ik} z_k \right) \right],
\end{aligned} \tag{3.11}$$

where we substituted from (3.10). That is, for $i = 1, \cdots, m$ one has

$$\begin{aligned}
&- \varepsilon^2 z_i'' + \varepsilon(2\alpha - b_{ii}) z_i' + [\alpha(b_{ii} - \alpha) + \varepsilon b_{ii} a_{ii}] z_i = \sum_{k \ne i} b_{ik} (-\alpha z_k - \varepsilon a_{ik} z_k + \varepsilon z_k') \\
&= \varepsilon \sum_{k \ne i} (b_{ik} z_k)' - \sum_{k \ne i} b_{ik} (\alpha + \varepsilon a_{ik} + \varepsilon b_{ik}') z_k.
\end{aligned} \tag{3.12}$$

Set

$$\beta = \min_i \beta_i.$$

To ensure that the zero-order terms in the differential operator of the system (3.12) engender stability, we choose α to satisfy $0 < \alpha < \beta$ (a more precise choice will be made later). Write $z_i = \phi_i + \psi_i$ for $i = 1, \cdots, m$, where these functions are defined by

$$\begin{cases} -\varepsilon^2 \phi_i'' + \varepsilon(2\alpha - b_{ii})\phi_i' + [\alpha(b_{ii} - \alpha) + \varepsilon b_{ii} a_{ii}]\phi_i = -\sum_{k\neq i} b_{ik}(\alpha + \varepsilon a_{ik} + \varepsilon b_{ik}')z_k, \\ \phi_i(0) = \delta_{ij}, \ \phi(1) = 0, \end{cases} \tag{3.13}$$

and

$$\begin{cases} -\varepsilon^2 \psi_i'' + \varepsilon(2\alpha - b_{ii})\psi_i' + [\alpha(b_{ii} - \alpha) + \varepsilon b_{ii} a_{ii}]\psi_i = \varepsilon \sum_{k\neq i} (b_{ik} z_k)', \\ \quad \psi_i(0) = \psi(1) = 0. \end{cases} \tag{3.14}$$

Observe that

$$\alpha(b_{ii} - \alpha) + \varepsilon b_{ii} a_{ii} \geq \alpha(\beta - \alpha)$$

by Assumption 1.1 and $0 < \alpha < \beta$. Applying a maximum principle to (3.13) with a constant barrier function, one gets

$$\begin{aligned} \|\phi_i\|_\infty &\leq \max\left\{ \frac{\|\sum_{k\neq i}(\alpha + \varepsilon a_{ik} + \varepsilon b_{ik}')b_{ik} z_k\|_\infty}{\alpha(\beta - \alpha)}, 1 \right\} \\ &\leq 1 + \frac{\sum_{k\neq i}(\alpha + \varepsilon \|a_{ik}\|_\infty + \varepsilon \|b_{ik}'\|_\infty)\, \|b_{ik}\|_\infty \, \|z_k\|_\infty}{\alpha(\beta - \alpha)}. \end{aligned} \tag{3.15}$$

Next, consider (3.14). Set $\kappa = \max_i \max_{x\in[0,1]} |2\alpha - b_{ii}(x)|$. By Lemma A.1, whose proof is deferred to the Appendix,

$$\|\psi_i\|_\infty \leq \frac{1}{\sqrt{\alpha(\beta - \alpha)}} \left[1 + \frac{2\kappa}{\sqrt{\alpha(\beta - \alpha)}} \right] \Big\| \sum_{k\neq i} b_{ik} z_k \Big\|_\infty. \tag{3.16}$$

Now $\|z_i\|_\infty \leq \|\phi_i\|_\infty + \|\psi_i\|_\infty$; invoking (3.15) and (3.16) then rearranging, we get

$$\|z_i\|_\infty - \frac{1}{\alpha(\beta - \alpha)} \sum_{k\neq i} \left[\alpha + \sqrt{\alpha(\beta - \alpha)} + 2\kappa + \varepsilon \|a_{ik}\|_\infty + \varepsilon \|b_{ik}'\|_\infty \right] \|b_{ik}\|_\infty \, \|z_k\|_\infty \leq 1 \tag{3.17}$$

for $i = 1, \cdots, m$.

Define the $m \times m$ matrix $\Theta = (\theta_{ik})$ by

$$\theta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ -[\alpha(\beta - \alpha)]^{-1} \left[\alpha + \sqrt{\alpha(\beta - \alpha)} + 2\kappa + \varepsilon \|a_{ik}\|_\infty + \varepsilon \|b_{ik}'\|_\infty \right] \|b_{ik}\|_\infty & \text{if } i \neq k. \end{cases}$$

Assumption 3.2. *One can choose $\alpha \in (0, \beta)$ such that Θ is inverse monotone.*

Remark 3.1. To show the sharpest possible decay rate for $\boldsymbol{w}_i$ one would like to choose α very close to β, but Assumption 3.2 constrains us in this regard.

Remark 3.2. If one chooses $\alpha = \beta/2$, then it is straightforward to verify that Θ will be a strictly diagonally dominant M-matrix (and therefore inverse-monotone) for all sufficiently small ε, if for all i one has

$$\sum_{j\neq i} \frac{4}{\beta^2}(\beta + 2\kappa)\|b_{ij}\|_\infty \leq C_1 < 1 \tag{3.18}$$

for some constant C_1. This inequality combines a diagonal dominance requirement on the matrix B with an equilibration condition on the rows of B because of the presence of κ.

Writing the system (3.17) in matrix-vector form then multiplying by Θ^{-1} yields $\|z_i\|_\infty \leq C$ for all i. Returning to (3.10), it follows that

$$|w_{ji}(x)| \leq Ce^{-\alpha x/\varepsilon} \quad \text{for } i, j = 1, \cdots, m \text{ and } x \in [0, 1]. \tag{3.19}$$

We also need to show that $w'_{ji}(x)$ decays exponentially. Fix $j, i \in \{1, \cdots, m\}$ and $x \in [0, 1]$. Choose an interval $[x^-, x^+]$ of length ε with $x \in [x^-, x^+] \subset [0, 1]$. By the mean value theorem there exists $x^* \in [x^-, x^+]$ such that

$$|w'_{ji}(x^*)| = \left|\frac{w_{ji}(x^+) - w_{ji}(x^-)}{\varepsilon}\right| \leq C\varepsilon^{-1}[e^{-\alpha x_+/\varepsilon} - e^{-\alpha x_-/\varepsilon}] \leq C\varepsilon^{-1}e^{-\alpha x/\varepsilon}, \tag{3.20}$$

where we used (3.19) and $|x - x^\pm| \leq \varepsilon$. On the other hand, (3.11) gives

$$\begin{aligned}
|\varepsilon w'_{ji}(x) - \varepsilon w'_{ji}(x^*)| &= \left|\int_{x^*}^{x} \varepsilon w''_{ji}(s)\,ds\right| \\
&= \left|\int_{x^*}^{x} \sum_{k=1}^{m}(-b_{ik}w'_{jk} + a_{ik}w_{jk})(s)\,ds\right| \\
&= \left|\sum_{k=1}^{m}\left[-(b_{ik}w_{jk})(x) + (b_{ik}w_{jk})(x^*) + \int_{x^*}^{x}(b'_{ik} + a_{ik})(s)w_{jk}(s)\,ds\right]\right| \\
&\leq C\left[e^{-\alpha x/\varepsilon} + e^{-\alpha x^*/\varepsilon} + \int_{x^*}^{x} e^{-\alpha s/\varepsilon}\,ds\right] \\
&\leq Ce^{-\alpha x/\varepsilon},
\end{aligned}$$

where we used (3.19) and $|x - x^*| \leq \varepsilon$. Combining this inequality with (3.20), we get

$$|w'_{ji}(x)| \leq C\varepsilon^{-1}e^{-\alpha x/\varepsilon} \quad \text{for } i, j = 1, \cdots, m \text{ and } x \in [0, 1]. \tag{3.21}$$

Recalling that $L\boldsymbol{w}_i = \mathbf{0}$, we deduce from (3.19) and (3.21) that for $x \in [0,1]$ one has

$$|w_{ji}^{(k)}(x)| \le C\varepsilon^{-k} e^{-\alpha x/\varepsilon} \quad \text{for } k = 2,3 \ \text{ and } \ i,j = 1,\cdots,m.$$

The analysis of Section 3 is summarized in the following result.

Theorem 3.1. *In addition to the hypotheses of Section* 1*, let Assumptions* 3.1 *and* 3.2 *be satisfied. Then there exists a constant C such that the solution* $\boldsymbol{u}$ *of* (1.1) *can be decomposed as*

$$\boldsymbol{u} = \boldsymbol{v} + \sum_{i=1}^{m} [(u_i - v_i)(0)] \boldsymbol{w}_i$$

where

$$\|\boldsymbol{v}^{(j)}\|_\infty \le C(1 + \varepsilon^{2-j}) \quad \text{for } j = 0,1,2,3,$$

and for each $\boldsymbol{w}_i = (w_{i1},\cdots,w_{im})^T$ *and* $x \in [0,1]$ *one has*

$$|w_{ik}^{(j)}(x)| \le C\varepsilon^{-j} e^{-\alpha x/\varepsilon} \quad \text{for } j = 0,1,2,3 \ \text{ and } \ k = 1,\cdots,m.$$

4. Numerical method and analysis

We use a Shishkin mesh, which is constructed as follows. Let N be an even positive integer. Choose $\alpha \in (0,\beta)$ to satisfy Assumption 3.2 then choose $k \ge 1/\alpha$. Partition the domain $[0,1]$ into two subintervals $[0,\sigma_k]$ and $[\sigma_k,1]$, where the *transition point* is

$$\sigma_k := \min\left\{\frac{1}{2}, k\varepsilon \ln N\right\}. \tag{4.1}$$

Subdivide the subinterval $[0,\sigma_k]$ by the equidistant mesh $\{x_i\}_{i=0}^{N/2}$ and subdivide $[\sigma_k,1]$ by the equidistant mesh $\{x_i\}_{i=N/2}^{N}$. Typically for small ε the mesh is fine on $[0,\sigma_k]$ and coarse on $[\sigma_k,1]$. We write h and H for the mesh widths on $[0,\sigma_k]$ and $[\sigma_k,1]$ respectively. Set $\Omega_\sigma^N = \{x_k\}_{k=1}^{N-1}$.

We introduce the difference operators

$$D^+ v_i = \frac{v_{i+1} - v_i}{\bar{h}_i}, \quad D^- v_i = \frac{v_i - v_{i-1}}{h_i} \quad \text{and} \quad \delta^2 v_i = D^+(D^- v_i),$$

where $h_i = x_i - x_{i-1}$ and $\bar{h}_i = (h_i + h_{i+1})/2$ for each i. The operator $D^+ v_i$ is an upwinded approximation of $v'(x_i)$ and $\delta^2 v_i$ is the standard central difference approximation of $v''(x_i)$. Note that the operator $D^+ v_i$ coincides with standard upwinding when $h_i = h_{i+1}$. When we apply these operators to a vector-valued mesh function $\mathbf{V}$, this means that they are applied separately to each component of $\mathbf{V}$.

Our discretization of problem (1.1) is

$$L^N \mathbf{U}(x_k) \equiv \left(-\varepsilon\delta^2 \mathbf{U} - BD^+\mathbf{U} + A\mathbf{U}\right)(x_k) = \mathbf{f}(x_k) \quad \text{for } \ x_k \in \Omega_\sigma^N, \tag{4.2a}$$

$$\mathbf{U}(0) = \boldsymbol{u}(0), \quad \mathbf{U}(1) = \boldsymbol{u}(1). \tag{4.2b}$$

To analyze (4.2), first consider the scalar convection-diffusion two-point boundary value problem (2.1) and the associated finite difference scheme

$$\begin{aligned} &-\varepsilon\delta^2 V(x_k) - r(x_k)D^+V(x_k) + q(x_k)V(x_k) = p(x_k) \quad \text{for } x_k \in \Omega_\sigma^N, \\ &V(0) = V(1) = 0, \end{aligned} \tag{4.3}$$

which was investigated by Andreev [2,3]. Recall that $0 < \underline{r} \le r(x) \le R$ and $0 \le q(x) \le Q$ on $[0,1]$. Set

$$R'' = \sum_{i=1}^{N} \left| \frac{1}{r(x_i)} - \frac{1}{r(x_{i-1})} \right| \quad \text{and} \quad \hat{R} = \left(1 + \frac{Q}{\underline{r}}\right)\left(R'' + \frac{2}{\underline{r}}\right).$$

Clearly $R'' \le \|(1/r)'\|_\infty$, so R'' and $\hat{R}$ are bounded independently of the mesh. The stability bounds of [2, Theorem 2.1] give

$$\|V\|_\infty \le \frac{1}{\underline{r}} \|p\|_{1,d}, \tag{4.4a}$$

$$\|V\|_\infty \le \hat{R}\, \|p\|_{-1,\infty,d} \tag{4.4b}$$

where the discrete norms here are defined by

$$\|p\|_{1,d} = \sum_{i=0}^{N} |p(x_i)|\bar{h}_i \quad \text{and} \quad \|p\|_{-1,\infty,d} = \min_C \max_{0<i<N} \left| \sum_{j=i}^{N-1} p(x_j)\bar{h}_j - C \right|.$$

Returning to the system (4.2), for $i = 1, \cdots, m$ set

$$R_i'' = \sum_{j=1}^{N} \left| \frac{1}{b_{ii}(x_j)} - \frac{1}{b_{ii}(x_{j-1})} \right| \quad \text{and} \quad \hat{R}_i = \left(1 + \frac{\|a_{ii}\|_\infty}{\beta_i}\right)\left(R_i'' + \frac{2}{\beta_i}\right).$$

Define the $m \times m$ matrix $\Upsilon_d = (\zeta_{ij})$ by

$$\zeta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\left[(\beta_i)^{-1}\|D^+b_{ij} + a_{ij}\|_{1,d} + \hat{R}_i\|b_{ij}\|_\infty\right] & \text{if } i \neq j. \end{cases}$$

When analyzing our discretization, this matrix is the analogue of the matrix Υ that was used in Section 2 to investigate the original system (1.1).

Assumption 4.1. *(i) The matrix Υ_d is inverse monotone, i.e., $\Upsilon_d^{-1} = (z_{ij})$ exists with $z_{ij} \ge 0$ for all i and j; (ii) there exists a constant C such that $z_{ij} \le C$ for all i and j.*

This assumption will be satisfied if, e.g., there exists a constant C_2 such that

$$\sum_{\substack{j=1 \\ j\neq i}}^{m} |\zeta_{ij}| \le C_2 < 1 \quad \text{for } i = 1, \cdots, m.$$

Since

$$|\zeta_{ij}| \le \frac{1}{\beta_i}\left(\|b'_{ij}\|_\infty + \|a_{ij}\|_\infty\right) + \left(1+\frac{\|a_{ii}\|_\infty}{\beta_i}\right)\left[\left\|\left(\frac{1}{b_{ii}}\right)'\right\|_\infty + \frac{2}{\beta_i}\right]\|b_{ij}\|_\infty =: \phi_{ij},$$

we see that Assumption 4.1 will be satisfied if for each i one has

$$\sum_{j\neq i}\phi_{ij} \le C_2 < 1, \tag{4.5}$$

i.e., the diagonal entries of B are sufficiently dominant relative to the off-diagonal entries of B and the entries of A. Furthermore, it is easy to verify that (4.5) is also a sufficient condition for Assumption 2.1 to hold.

Remark 4.1. Consider a simple case of the class of problems defined in (1.1): B a constant matrix and $A \equiv 0$. Suppose that we choose $\alpha = \beta/2$. Then $\Upsilon = \Upsilon_d$ and the off-diagonal elements in the matrices Υ, Θ and Υ_d are

$$\gamma_{ij} = \zeta_{ij} = -\frac{2}{\beta_i}|b_{ij}|, \quad \theta_{ij} = -\frac{4}{\beta^2}(\beta+2\kappa)|b_{ij}|.$$

In this case, a sufficient condition for all the matrices Υ, Θ and Υ_d to be inverse monotone and for Assumption 4.1 to be satisfied is that

$$M\sum_{\substack{j=1\\ j\neq i}}^{m}|b_{ij}| < \beta \quad \text{for } i=1,\cdots,m, \quad \text{where } M = 4+8\max_i\left(\frac{b_{ii}}{\beta}-1\right). \tag{4.6}$$

With Assumption 4.1 one can prove the following result.

Lemma 4.1. *Any solution* $(U_1,\cdots,U_m)^T$ *of* (4.2) *must satisfy*

$$\|U_i\|_\infty \le \sum_{j=1}^{m} z_{ij}\hat{R}_j\,\|f_j\|_{-1,\infty,d} \quad \textit{for } i=1,\cdots,m. \tag{4.7}$$

Proof. For $i=1,\cdots,m$, the i^{th} equation in (4.2) can be rewritten as

$$\begin{aligned}&(-\varepsilon\delta^2 U_i - b_{ii}D^+U_i + a_{ii}U_i)(x_k)\\ &= f_i(x_k) + \sum_{\substack{j=1\\ j\neq i}}^{m}\left[D^+(b_{ij}(x_{k-1})U_j(x_k)) - [D^+b_{ij}(x_{k-1}) + a_{ij}(x_k)]U_j(x_k)\right]\end{aligned}$$

for $k=1,\cdots,N-1$, since

$$b(x_k)D^+V(x_k) = D^+(b(x_{k-1})V(x_k)) - V(x_k)D^+b(x_{k-1}).$$

Note that $\|D^+(pq)\|_{-1,d} \le \|p\|_\infty \|q\|_\infty$. Analogously to the proof of Lemma 2.1, we can use (4.4a) and (4.4b) to get

$$\|U_i\|_\infty \le \hat{R}_i \|f_i\|_{-1,\infty,d} + \sum_{\substack{j=1\\ j\neq i}}^{m} \left(\frac{\|D^+ b_{ij} + a_{ij}\|_{1,d}}{\beta_i} + \hat{R}_i \|b_{ij}\|_\infty \right) \|U_j\|_\infty.$$

Taking the U_j terms to the left-hand side, we have

$$\|U_i\|_\infty + \sum_{\substack{j=1\\ j\neq i}}^{m} \zeta_{ij} \|U_j\|_\infty \le \hat{R}_i \|f_i\|_{-1,\infty,d} \quad \text{for } i = 1, \cdots, m.$$

Write this system in matrix-vector form then multiply by Υ_d^{-1} to obtain the desired result. □

Corollary 4.1. *The system* (4.2) *has a unique solution.*

4.1. Truncation error analysis

The truncation error analysis that we present below for the system (4.2) is akin to the truncation error analysis given in [3] for the scalar case. The solution of (4.2) can be written as the sum

$$\mathbf{U} = \mathbf{V} + \sum_{i=1}^{m} [(u_i - v_i)(0)] \mathbf{W}_i$$

where, analogously to the construction of Section 3, we define $\mathbf{V}$ and $\mathbf{W}_i$ by

$$L^N \mathbf{V} = \mathbf{f}, \quad \mathbf{V}(0) = \boldsymbol{v}(0),\ \mathbf{V}(1) = \boldsymbol{v}(1),$$
$$L^N \mathbf{W}_i = \mathbf{0}, \quad \mathbf{W}_i(0) = \boldsymbol{w}_i(0),\ \mathbf{W}_i(1) = \mathbf{0}.$$

Now the truncation error is

$$L^N(\mathbf{U} - \boldsymbol{u}) = \varepsilon(\delta^2 \boldsymbol{u} - \boldsymbol{u}'') + B(D^+ \boldsymbol{u} - \boldsymbol{u}')$$

and one also has

$$L^N(\mathbf{U} - \boldsymbol{u}) = L^N(\mathbf{V} - \boldsymbol{v}) + \sum_{i=1}^{m} [(u_i - v_i)(0)] L^N(\mathbf{W}_i - \boldsymbol{w}_i).$$

In the special case when the mesh is uniform ($\sigma_k = 0.5$), one can deduce from Theorem 3.1 that

$$\|L^N(\mathbf{U} - \boldsymbol{u})\|_\infty \le CN^{-1}(\ln N)^2.$$

Thus assume that $\sigma_k = k\varepsilon \ln N$. From the bounds in Theorem 3.1 on the derivatives of the regular component $\boldsymbol{v}$, one can see that

$$\max_i |(L^N(\mathbf{V} - \boldsymbol{v})(x_j))_i| \le CN^{-1} \quad \text{for } x_j \neq \sigma_k \quad \text{and } \max_i |(L^N(\mathbf{V} - \boldsymbol{v})(\sigma_k))_i| \le C.$$

Hence

$$\|L^N(\mathbf{V}-\boldsymbol{v})\|_{1,d} \le CN^{-1}.$$

By using an integral representation for the truncation error and the bounds on the derivatives of the layer components $\boldsymbol{w}_i$ in Theorem 3.1, we get

$$\begin{aligned}&\bar{h}_j|L^N(\mathbf{W}_i-\boldsymbol{w}_i)(x_j)|\\ &\le C\varepsilon^{-1}h_j e^{-\alpha x_{j-1}/\varepsilon}(1-e^{-\alpha h_j/\varepsilon})+C\varepsilon^{-1}h_{j+1}e^{-\alpha x_j/\varepsilon}(1-e^{-\alpha h_{j+1}/\varepsilon}).\end{aligned} \tag{4.8}$$

For the points $x_j=\sigma_k$ and $x_j=\sigma_k+H$ we derive an alternative bound on the truncation error:

$$\begin{aligned}\bar{h}_jL^N(\mathbf{W}_i-\boldsymbol{w}_i) &= \bar{h}_j(\varepsilon(\delta^2\boldsymbol{w}_i-\boldsymbol{w}_i'')+B(D^+\boldsymbol{w}_i-\boldsymbol{w}_i'))\\ &= (\varepsilon I+\bar{h}_jB)D^+\boldsymbol{w}_i-\varepsilon D^-\boldsymbol{w}_i-\bar{h}_j\varepsilon\boldsymbol{w}_i''+B\boldsymbol{w}_i')\\ &= (\varepsilon I+\bar{h}_jB)D^+\boldsymbol{w}_i-\varepsilon D^-\boldsymbol{w}_i-\bar{h}_jA\boldsymbol{w}_i.\end{aligned}$$

By Theorem 3.1, for each layer function $\boldsymbol{w}_i$ one has

$$|\varepsilon D^+\boldsymbol{w}_i(x_j)|\le Ce^{-\alpha x_j/\varepsilon}\mathbf{1} \quad \text{and} \quad |\bar{h}_jBD^+\boldsymbol{w}_i(x_j)|\le Ce^{-\alpha x_j/\varepsilon}\mathbf{1}.$$

Using these bounds we deduce that

$$\bar{h}_j|L^N(\mathbf{W}_i-\boldsymbol{w}_i)(x_j)|\le Ce^{-\alpha x_{j-1}/\varepsilon}\mathbf{1} \quad \text{for } x_j=\sigma_k,\sigma_k+H. \tag{4.9}$$

By (4.8) and (4.9), we get

$$\begin{aligned}\|L^N(\mathbf{W}_i-\boldsymbol{w}_i)\|_{1,d} &\le C\frac{h}{\varepsilon}+\sum_{j=N/2}^{N/2+1}\bar{h}_j|L^N(\mathbf{W}_i-\boldsymbol{w}_i)(x_j)|+C\frac{H}{\varepsilon}e^{-\alpha(\sigma_k+H)/\varepsilon}+CH\\ &\le CN^{-1}\ln N.\end{aligned}$$

The above bounds and Lemma 4.1 (with $\mathbf{U}$ replaced by $\mathbf{U}-\boldsymbol{u}$) imply the following convergence result for our method (4.2).

Theorem 4.1. *Let Assumptions* 1.1, 3.1, 3.2 *and* 4.1 *all be satisfied. Then*

$$\|\boldsymbol{u}-\boldsymbol{U}\|_\infty \le \begin{cases} CN^{-1}\ln N & \text{if } \sigma_k<0.5,\\ CN^{-1}\ln^2 N & \text{if } \sigma_k=0.5.\end{cases}$$

where $\boldsymbol{u}$ *is the solution of* (1.1) *and* $\boldsymbol{U}$ *is the solution of* (4.2).

Remark 4.2. Theorem 4.1 remains valid when the upwinded operator D^+V_i used to approximate the convective terms in (4.2) is replaced by the standard upwind operator $\bar{h}_iD^+V_i/h_{i+1}$, but the off-diagonal elements in the matrix Υ_d then increase in magnitude, which restricts the applicability of the convergence result.

Table 1: Two mesh differences D_ε^N, ε-uniform two-mesh difference D^N, ε-uniform orders p^N as defined in (5.2) and computed error constants C_1^N for Example 5.1 with $k=0.275$.

	N							
ε	8	16	32	64	128	256	512	1024
10^0	8.655e-2	5.538e-2	3.154e-2	1.681e-2	8.693e-3	4.422e-3	2.230e-3	1.120e-3
10^{-1}	1.549e-1	1.068e-1	6.989e-2	4.359e-2	2.609e-2	1.509e-2	8.530e-3	4.736e-3
10^{-2}	1.630e-1	1.114e-1	7.284e-2	4.549e-2	2.720e-2	1.575e-2	8.900e-3	4.941e-3
10^{-3}	1.638e-1	1.118e-1	7.312e-2	4.567e-2	2.731e-2	1.581e-2	8.935e-3	4.962e-3
10^{-4}	1.639e-1	1.119e-1	7.315e-2	4.569e-2	2.732e-2	1.582e-2	8.939e-3	4.964e-3
10^{-5}	1.639e-1	1.119e-1	7.316e-2	4.569e-2	2.732e-2	1.582e-2	8.939e-3	4.964e-3
10^{-6}	1.639e-1	1.119e-1	7.316e-2	4.569e-2	2.732e-2	1.582e-2	8.939e-3	4.964e-3
10^{-7}	1.639e-1	1.119e-1	7.316e-2	4.569e-2	2.732e-2	1.582e-2	8.939e-3	4.964e-3
D^N	1.639e-1	1.119e-1	7.316e-2	4.569e-2	2.732e-2	1.582e-2	8.939e-3	4.964e-3
p^N	0.942	0.903	0.921	0.954	0.977	0.992	1.001	
C_1^N	0.630	0.645	0.675	0.703	0.721	0.730	0.734	0.733

5. Numerical results

In this section we use (4.2) to compute solutions for some specific cases of (1.1) in order to support Theorem 4.1.

Example 5.1. *Let*

$$B = \begin{pmatrix} 5+2x & 1+3x^2 & 3-x \\ 1+2e^{-4x} & 5-x^2 & x^3 \\ 1 & 2(2+x)/(1+x) & 6 \end{pmatrix}, \ A=0 \quad \textit{and} \quad \boldsymbol{f} = \begin{pmatrix} 1 \\ -4-4x \\ -12+2x^2 \end{pmatrix}.$$

The boundary conditions are $\boldsymbol{u}(0)=\boldsymbol{u}(1)=0$.

In Example 5.1 one sees that $\beta=4$. The analysis in Sections 3 and 4 requires that $0<\alpha<\beta$ and $k \geq 1/\alpha$, where the transition point on the Shishkin mesh is

$$\sigma_k = \min\left\{\frac{1}{2}, k\varepsilon \ln N\right\}. \tag{5.1}$$

Thus we should choose $k>0.25$.

In Table 1 we show computational results for Example 5.1 with $k=0.275$, $\varepsilon = 1, 10^{-1}, \cdots, 10^{-7}$ and $N=16, 32, \cdots, 1024$. The definition of σ_k implies that an equidistant mesh is used when computing the first row of the table while a piecewise uniform mesh is used in all other rows. As the exact solution of the example is unknown, we follow the standard approach [6] by computing, for each N and ε, the two-mesh difference D_ε^N defined by

$$D_\varepsilon^N = \|\tilde{\mathbf{U}}^{2N} - \mathbf{U}^N\|$$

where $\tilde{\mathbf{U}}^{2N}$ is computed on the mesh obtained by bisecting $\bar{\Omega}^N$. Then the ε-uniform two-mesh difference is defined to be

$$D^N = \max_{\varepsilon=1,10^{-1},\cdots,10^{-7}} D_\varepsilon^N.$$

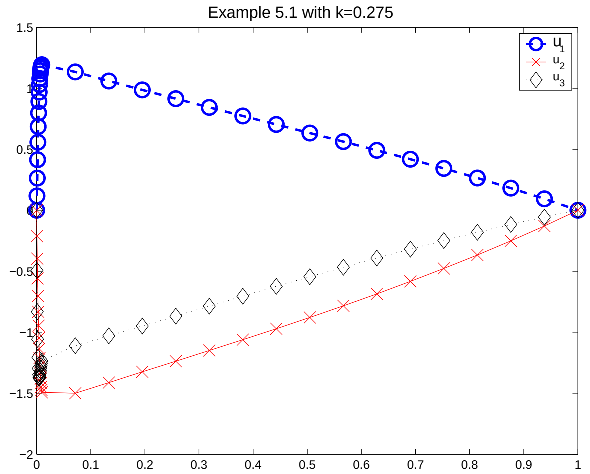

Figure 1: Computed solution for Example 5.1 when $\varepsilon = 0.01$, $N = 32$ and $k = 0.275$.

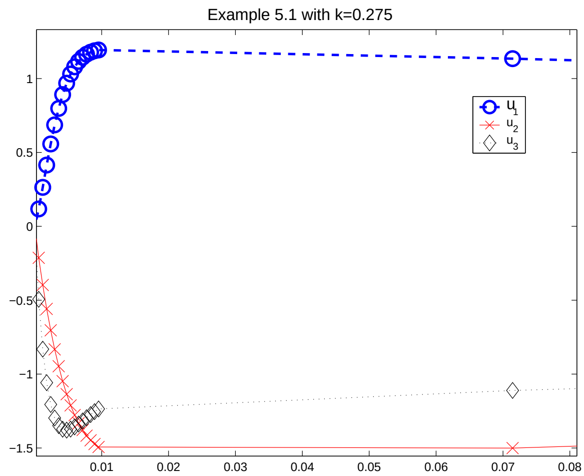

Figure 2: Blow-up of layer region in Fig. 1.

Assuming that one has a theoretical rate of convergence of the form $\mathcal{O}((N^{-1}\ln N)^p)$, an estimate of the computed rate of convergence is given by

$$p^N := \frac{\ln D^N - \ln D^{2N}}{\ln(2\ln N) - \ln(\ln(2N))}. \tag{5.2}$$

In the numerical experiments that we report, when $\varepsilon \le 0.1$ one has $\sigma_k = k\varepsilon \ln N$. To investigate whether the theoretical rate of convergence $\mathcal{O}(N^{-1}\ln N)$ predicted by Theorem 4.1 is observed, for each N we compute

$$C_1^N := D^N(N/\ln N).$$

In Table 1 the values of p^N are approaching the value 1 as N increases, which is what our theory predicts. This is also borne out by the behaviour of C_1^N, which appears to be converging to a positive value.

Table 2: Computed values of p^N for different values of k, for Example 5.1.

	N						
k	8	16	32	64	128	256	512
0.1	0.606	0.535	1.046	0.394	0.156	0.643	0.647
0.2	1.064	1.001	0.975	0.994	1.003	1.009	1.013
0.25	1.034	0.918	0.934	0.959	0.985	0.997	1.005
0.275	0.942	0.903	0.921	0.954	0.977	0.992	1.001
0.3	0.855	0.885	0.916	0.938	0.966	0.987	0.998
0.4	0.879	0.737	0.855	0.889	0.940	0.968	0.986
0.5	0.806	0.692	0.794	0.870	0.911	0.952	0.970
0.6	0.652	0.756	0.701	0.814	0.896	0.937	0.916
0.8	0.408	0.598	0.658	0.768	0.857	0.914	0.915
1.0	0.297	0.392	0.718	0.654	0.801	0.894	0.934
2.0	0.657	0.318	0.099	0.574	0.629	0.807	0.859

Table 3: Computed values of p^N with $k = 0.4$ and various values of η, for Example 5.2.

	N						
η	8	16	32	64	128	256	512
0	0.911	0.788	0.856	0.895	0.935	0.961	0.869
1	0.886	0.845	0.871	0.904	0.943	0.966	0.983
2	0.989	0.931	0.895	0.916	0.951	0.972	0.986
3	1.270	1.030	0.953	0.953	0.962	0.980	0.984
3.5	1.208	1.190	1.114	1.080	1.018	0.982	0.972
4	1.009	1.201	0.472	0.755	1.220	0.261	0.550
4.5	1.157	-0.001	0.362	0.919	0.241	0.168	0.688
5	- 0.417	0.299	0.809	-0.402	0.086	0.656	-0.130

A representative computed solution for Example 5.1 is shown in Figs. 1 and 2.

To investigate the dependence of the method on the value of k, Table 2 presents the computed rates of convergence p_N for various k. We observe a degradation in the order of convergence as k is increased above 0.6 and when $k < 0.2$.

Example 5.2. *Let*

$$B = \begin{pmatrix} 5 & 3 & \eta \\ \eta & 5 & 3 \\ \eta & 3 & 6 \end{pmatrix}, \qquad A = 0 \qquad \textit{and} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ -4-4x \\ -12+2x^2 \end{pmatrix}.$$

Here η is a parameter that we shall vary in our numerical experiments. The boundary conditions are $\boldsymbol{u}(0) = \boldsymbol{u}(1) = 0$.

We use Example 5.2 to test numerically whether strict diagonal dominance is a necessary condition for convergence of our numerical method. Here $\beta = 5$ and strict diagonal dominance requires $|\eta| < 2$. The orders of convergence of the numerical method when applied to Example 5.2 for a range of η are given in Table 3. The method fails to be convergent apparently only when $\eta > 3.5$, which suggests that the numerical method (4.2)

Table 4: Maximum pointwise errors E_ε^N, ε-uniform errors E^N, ε-uniform orders q^N (as defined in (5.3)) and computed error constants $\tilde{C}_1^N, \tilde{C}_2^N$ for Example 5.3 with $k=0.4$.

	N							
ε	8	16	32	64	128	256	512	1024
10^0	2.631e-1	1.519e-1	8.113e-2	4.201e-2	2.141e-2	1.081e-2	5.430e-3	2.721e-3
10^{-1}	3.306e-1	2.560e-1	1.919e-1	1.263e-1	7.894e-2	4.701e-2	2.720e-2	1.539e-2
10^{-2}	3.114e-1	2.454e-1	1.835e-1	1.205e-1	7.567e-2	4.512e-2	2.612e-2	1.479e-2
10^{-3}	3.095e-1	2.443e-1	1.827e-1	1.199e-1	7.535e-2	4.493e-2	2.601e-2	1.473e-2
10^{-4}	3.093e-1	2.442e-1	1.826e-1	1.198e-1	7.532e-2	4.491e-2	2.600e-2	1.472e-2
10^{-5}	3.092e-1	2.442e-1	1.826e-1	1.198e-1	7.531e-2	4.491e-2	2.600e-2	1.472e-2
10^{-6}	3.092e-1	2.442e-1	1.826e-1	1.198e-1	7.531e-2	4.491e-2	2.600e-2	1.472e-2
10^{-7}	3.092e-1	2.442e-1	1.826e-1	1.198e-1	7.531e-2	4.491e-2	2.600e-2	1.472e-2
E^N	3.306e-1	2.560e-1	1.919e-1	1.263e-1	7.894e-2	4.701e-2	2.720e-2	1.539e-2
q^N	0.631	0.614	0.818	0.873	0.926	0.951	0.969	
$\tilde{C}_1^N$	1.272	1.477	1.772	1.944	2.083	2.170	2.232	2.274
$\tilde{C}_2^N$	0.612	0.533	0.511	0.467	0.429	0.391	0.358	0.328

may yield uniformly convergent numerical approximations to the solutions of a wider class of problems (1.1) than is covered by our current theory.

Finally, in Example 5.3 we consider a problem with a known analytical solution. Thus the numerical results can be assessed by means of exact pointwise errors in our computed solutions instead of the two-mesh differences used in our earlier examples.

Example 5.3. *Let*

$$B = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 4 & -2 \\ -1 & -2 & 4 \end{pmatrix}, \; A=0 \quad and \quad \boldsymbol{f} = \begin{pmatrix} -4 \\ 11 \\ -7 \end{pmatrix}.$$

The boundary conditions are $\boldsymbol{u}(0)=(-1,4,-1)^T, \boldsymbol{u}(1)=(e^{-1/\varepsilon}-2e^{-4/\varepsilon}+1, e^{-1/\varepsilon}+e^{-4/\varepsilon}+2e^{-6/\varepsilon}-2, e^{-1/\varepsilon}+e^{-4/\varepsilon}-2e^{-6/\varepsilon})^T$. *The true solution is*

$$\boldsymbol{u}=(1,1,1)^T e^{-x/\varepsilon}+(-2,1,1)^T e^{-4x/\varepsilon}+(0,2,-2)^T e^{-6x/\varepsilon}+(x,-2x,x-1)^T.$$

In Table 4 we show computational results for Example 5.3 with $k=0.4$, $\varepsilon = 1, 10^{-1}, \cdots, 10^{-7}$ and $N = 8,16,32,\cdots,1024$. As the exact solution of the example is known, we compute the maximum pointwise error E_ε^N

$$E_\varepsilon^N = ||\mathbf{U}^N - \mathbf{u}||$$

and the ε-uniform maximum pointwise error E^N

$$E^N = \max_{\varepsilon=1,10^{-1},\cdots,10^{-7}} E_\varepsilon^N.$$

Assuming that one has a theoretical rate of convergence of the form $\mathcal{O}((N^{-1}\ln N)^p)$, the computed rate of convergence q^N is given by

$$q^N := \frac{\ln E^N - \ln E^{2N}}{\ln(2\ln N) - \ln(\ln(2N))} \tag{5.3}$$

and estimates of the associated error constants by

$$\tilde{C}_1^N := E^N N(\ln N)^{-1}, \quad \tilde{C}_2^N := E^N N(\ln N)^{-2}.$$

The results in Table 4 again indicate that the numerical approximations $\boldsymbol{U}$ generated by (4.2) converge uniformly to the exact solution $\boldsymbol{u}$ of Example 5.3.

Appendix: $(\|\cdot\|_\infty, \|\cdot\|_{-1,\infty})$ stability for a reaction-convection-diffusion equation

Consider the problem

$$Lu(x) := -\varepsilon^2 v''(x) - \varepsilon r(x) v'(x) + q^2(x) v(x) = h(x) \quad \text{on } (0,1), \tag{A.1a}$$
$$v(0) = v(1) = 0. \tag{A.1b}$$

Assume that $|r(x)| \le R$ and $q(x) \ge \underline{q} > 0$ on $[0,1]$. The smallness of the convective coefficient means that from the singular perturbation point of view, this convection-reaction-diffusion problem is similar in nature to a reaction-diffusion problem; see, e.g., [10, 13].

We wish to bound $\|v\|_\infty$ in terms of $\|h\|_{-1,\infty}$. The results of Andreev [4] are inapplicable here since the convection coefficient tends to zero as $\varepsilon \to 0$. Nevertheless we shall use the technique from [4] of treating L as a perturbation of a simpler operator.

Lemma A.1. *The solution v of* (A.1a) *satisfies*

$$\|v\|_\infty \le \frac{1}{\varepsilon \underline{q}} \left(1 + \frac{2R}{\underline{q}}\right) \|h\|_{-1,\infty}. \tag{A.2}$$

Proof. Define $M : C^2[0,1] \to C[0,1]$ by

$$Mw(x) := -\varepsilon^2 w''(x) + q^2(x) w(x).$$

Then for each $\xi \in (0,1)$, the Green's function $\hat{G}(x,\xi)$ associated with M and ξ is defined by

$$M\hat{G}(x,\xi) = \delta(x-\xi) \quad \text{for } x \in (0,1), \ \ \hat{G}(0,\xi) = \hat{G}(1,\xi) = 1.$$

In fact

$$\hat{G}(x,\xi) = \frac{1}{\varepsilon^2 g_1'(0)} \begin{cases} g_0(\xi) g_1(x) & \text{if } x \le \xi, \\ g_0(x) g_1(\xi) & \text{if } x \ge \xi, \end{cases} \tag{A.3}$$

where the functions g_0 and g_1 are defined by

$$Mg_0(x) = 0 \quad \text{on } (0,1), \quad g_0(0) = 1, \ g_0(1) = 0, \tag{A.4a}$$
$$Mg_1(x) = 0 \quad \text{on } (0,1), \quad g_1(0) = 0, \ g_1(1) = 1. \tag{A.4b}$$

The operator M satisfies a maximum principle. Hence $0 \le g_0 \le 1$, and (A.4a) then gives $g_0'' \ge 0$, whence $g_0' \le 0$. Similarly $g_1' \ge 0$. It is shown in the proof of [8, Lemma 2.2] that

$$\int_{\xi=0}^{1} |\hat{G}_\xi(x,\xi)|\, d\xi \le \frac{1}{\varepsilon \underline{q}} \quad \text{for } 0 \le x \le 1. \tag{A.5}$$

Fix $x \in (0,1)$. Let $G(x,\xi)$ be the Green's function associated with (A.1). Then

$$L^* G(x,\xi) := -\varepsilon^2 G_{\xi\xi}(x,\xi) + \varepsilon\big(r(\xi)G(x,\xi)\big)_\xi + q(\xi)G(x,\xi) = \delta(x-\xi) \quad \text{for } 0 < \xi < 1.$$

Rearranging, this is

$$MG(x,\xi) = \delta(x-\xi) - \varepsilon\big(r(\xi)G(x,\xi)\big)_\xi .$$

Hence

$$\begin{aligned} G(x,\xi) &= \int_{y=0}^{1} \Big[\delta(x-y) - \varepsilon\big(r(y)G(x,y)\big)_y\Big]\, \hat{G}(y,\xi)\, dy \\ &= \hat{G}(x,\xi) + \varepsilon \int_{y=0}^{1} \hat{G}_y(y,\xi) r(y) G(x,y)\, dy \\ &= \hat{G}(x,\xi) + \varepsilon \left(\int_{y=0}^{\xi} + \int_{y=\xi}^{1} \right) \hat{G}_y(y,\xi) r(y) G(x,y)\, dy. \end{aligned}$$

Differentiating, we get

$$\begin{aligned} G_\xi(x,\xi) =& \hat{G}_\xi(x,\xi) + \varepsilon \Bigg[\hat{G}_y(\xi^-,\xi) r(\xi) G(x,\xi) + \int_{y=0}^{\xi} \hat{G}_{y\xi}(y,\xi) r(y) G(x,y)\, dy \\ & - \hat{G}_y(\xi^+,\xi) r(\xi) G(x,\xi) + \int_{y=\xi}^{1} \hat{G}_{y\xi}(y,\xi) r(y) G(x,y)\, dy \Bigg]. \end{aligned}$$

But $M\hat{G}(x,\xi) = \delta(x-\xi)$ means that

$$G_y(\xi^-,\xi) - G_y(\xi^+,\xi) = \varepsilon^{-2}.$$

Thus

$$G_\xi(x,\xi) = \hat{G}_\xi(x,\xi) + \varepsilon^{-1} r(\xi) G(x,\xi) + \varepsilon \left(\int_{y=0}^{\xi} + \int_{y=\xi}^{1} \right) \hat{G}_{y\xi}(y,\xi) r(y) G(x,y)\, dy.$$

Via a maximum principle argument one can see that $G(x,y) \ge 0$, so

$$\begin{aligned} \int_{\xi=0}^{1} |G_\xi(x,\xi)|\, d\xi \le& \int_{\xi=0}^{1} |\hat{G}_\xi(x,\xi)|\, d\xi + \frac{R}{\varepsilon} \int_{\xi=0}^{1} G(x,\xi)\, d\xi \\ & + \varepsilon \left(\int_{\xi=0}^{1} \int_{y=0}^{\xi} + \int_{\xi=0}^{1} \int_{y=\xi}^{1} \right) |\hat{G}_{y\xi}(y,\xi)|.|r(y)|\, G(x,y)\, dy\, d\xi. \end{aligned} \tag{A.6}$$

Now (A.3) and the monotonicity properties of g_0' and g_1' imply that $\hat{G}_{y\xi} \le 0$. Thus

$$\begin{aligned}
&\varepsilon\left(\int_{\xi=0}^{1}\int_{y=0}^{\xi}+\int_{\xi=0}^{1}\int_{y=\xi}^{1}\right)|\hat{G}_{y\xi}(y,\xi)|.|r(y)|\,G(x,y)\,dy\,d\xi\\
&=-\varepsilon\left(\int_{\xi=0}^{1}\int_{y=0}^{\xi}+\int_{\xi=0}^{1}\int_{y=\xi}^{1}\right)\hat{G}_{y\xi}(y,\xi)\,|r(y)|\,G(x,y)\,dy\,d\xi\\
&=-\varepsilon\left(\int_{y=0}^{1}\int_{\xi=y}^{1}+\int_{y=0}^{1}\int_{\xi=0}^{y}\right)\hat{G}_{y\xi}(y,\xi)\,|r(y)|\,G(x,y)\,dy\,d\xi\\
&=-\varepsilon\int_{y=0}^{1}\left[-\hat{G}_y(y,y^+)+\hat{G}_y(y,y^-)\right]\,|r(y)|\,G(x,y)\,dy,
\end{aligned}$$

on carrying out the integrations and using $\hat{G}_y(y,0)=\hat{G}_y(y,1)=0$. But

$$\hat{G}_y(y,y^-)-\hat{G}_y(y,y^+)=-\varepsilon^{-2},$$

so we get

$$\begin{aligned}
&\varepsilon\left(\int_{\xi=0}^{1}\int_{y=0}^{\xi}+\int_{\xi=0}^{1}\int_{y=\xi}^{1}\right)|\hat{G}_{y\xi}(y,\xi)|.|r(y)|\,G(x,y)\,dy\\
&=\varepsilon^{-1}\int_{y=0}^{1}|r(y)|\,G(x,y)\,dy. \qquad \text{(A.7)}
\end{aligned}$$

Recalling (A.6), we have shown that

$$\int_{\xi=0}^{1}|G_\xi(x,\xi)|\,d\xi \le \int_{\xi=0}^{1}|\hat{G}_\xi(x,\xi)|\,d\xi+\frac{2R}{\varepsilon}\int_{\xi=0}^{1}G(x,\xi)\,d\xi. \qquad \text{(A.8)}$$

Here

$$\int_{\xi=0}^{1}|\hat{G}_\xi(x,\xi)|\,d\xi \le 1/(\varepsilon\underline{q})$$

by (A.5), and the maximum principle bound $\|v\|_\infty \le \underline{q}^{-2}\|Lv\|_\infty$ implies that

$$\int_{\xi=0}^{1}G(x,\xi)\,d\xi \le \underline{q}^{-2}$$

(take $Lv\equiv 1$). Thus

$$\int_{\xi=0}^{1}|G_\xi(x,\xi)|\,d\xi \le \frac{1}{\varepsilon\underline{q}}+\frac{2R}{\varepsilon\underline{q}^2}. \qquad \text{(A.9)}$$

But by definition of G, for each $x\in[0,1]$ we have

$$v(x)=\int_{\xi=0}^{1}G(x,\xi)h(\xi)\,d\xi=-\int_{\xi=0}^{1}G_\xi(x,\xi)H(\xi)\,d\xi,$$

where H is any antiderivative of h, and invoking (A.9) we get the desired inequality (A.2).

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