NUMERICAL MATHEMATICS: Theory, Methods and Applications
Numer. Math. Theor. Meth. Appl., Vol. **1**, No. 2, pp. 214-234 (2008)

A Finite Difference Scheme on *a Priori* Adapted Meshes for a Singularly Perturbed Parabolic Convection-Diffusion Equation

Grigory I. Shishkin*

Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg 620219, Russia.

Received 3 August, 2007; Accepted (in revised version) 19 December, 2007

Abstract. A boundary value problem is considered for a singularly perturbed parabolic convection-diffusion equation; we construct a finite difference scheme on *a priori* (sequentially) *adapted* meshes and study its convergence. The scheme on *a priori* adapted meshes is constructed using a majorant function for the singular component of the discrete solution, which allows us to find *a priori* a subdomain where the computed solution requires a further improvement. This subdomain is defined by the perturbation parameter ε, the step-size of a uniform mesh in x, and also by the required accuracy of the discrete solution and the prescribed number of refinement iterations K for improving the solution. To solve the discrete problems aimed at the improvement of the solution, we use uniform meshes on the subdomains. The error of the numerical solution depends weakly on the parameter ε. The scheme converges almost ε-uniformly, precisely, under the condition $N^{-1} = o(\varepsilon^{\nu})$, where N denotes the number of nodes in the spatial mesh, and the value $\nu = \nu(K)$ can be chosen arbitrarily small for suitable K.

AMS subject classifications: 65M06, 65M12, 65M15, 65M50

Key words: Singular perturbations, convection-diffusion problem, piecewise-uniform mesh, *a priori* adapted mesh, almost ε-uniform convergence.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

At present, there are fairly well-developed methods for constructing ε-uniformly convergent schemes on meshes that are *a priori* adapted in a boundary layer region and not changing in the computational process, or, in short, schemes on meshes condensing in boundary layers *a priori* (see, e.g., [1–5] for partial differential equations and [6] for ordinary differential equations). The methods based on piecewise-uniform meshes condensing in boundary layers received fairly widespread use due to their simplicity and convenience in application (see, e.g., [2–5] and the references therein). The disadvantageous property

*Corresponding author. *Email addresses:* g.shishkin@cs.ucc.ie, shishkin@imm.uran.ru (G. I. Shishkin)

of these numerical methods on *a priori* adapted meshes is the necessity to solve the difference equations on meshes whose step-size changes sharply in a neighbourhood of the boundary layer.

Another alternative approach to the construction of numerical methods for singularly perturbed boundary value problems developed, for example, in [7–10] leads to methods on sequentially *a posteriori* adapted meshes that are fitted (refined) in the computational process depending on the computed solution, or briefly, methods on *a posteriori adapted* meshes. In this approach, classical finite difference approximations of the boundary value problem are used; the discrete solution is corrected on a finer mesh in that subdomain where errors in the solution turn to be intolerably large. The subdomain in which the solution should be locally improved is determined using an indicator that is a functional of the solution (for example, the solution gradient) of the discrete problem. In these methods, the discrete problems on the subdomains where the solution is *a posteriori* improved are solved on uniform meshes.

In this respect, it would be of interest to consider such numerical methods on *a priori* adapted meshes in which the discrete problems in the subdomains where the computed solution is *a priori* corrected are solved on uniform meshes. Methods of this kind are unknown in the literature.

In the present paper, we consider the Dirichlet problem for a parabolic convection-diffusion equation with a small parameter ε multiplying the highest-order derivative. For the boundary value problem, we construct finite difference schemes on *locally uniform* meshes (namely, uniform meshes on the subdomains where the solution should be improved) that are adapted *a priori*, and study their convergence. To construct the schemes, a standard finite difference approximation of the differential equation is used. Note that the scheme on *a priori* condensing (in the layer) *piecewise-uniform meshes* converges ε-uniformly. The "standard" scheme on uniform meshes converges only under the condition $N^{-1} \ll \varepsilon$, where the value N defines the number of mesh points in x.

For the scheme on *a priori* adapted meshes, boundaries of the subdomains where it requires to improve the solution are determined by a *majorant for the singular component of the discrete solution*, which is specified in its turn by the perturbation parameter ε, the step-size of a mesh used in x, and also by the required accuracy of the discrete solution. On the meshes adapted with respect to the majorant function of the discrete solution, we construct a sufficiently simple finite difference scheme for which the error in the solution is weakly depending on the parameter ε. The scheme constructed on *a priori* adapted meshes converges *"almost ε-uniformly"*, precisely, under the condition $N^{-1} \ll \varepsilon^{\nu}$, where the value ν defining the scheme (the number of refinement iterations required for the discrete solution to be improved) can be chosen arbitrary in $(0, 1]$.

2. Problem formulation and aim of research

2.1. Problem formulation

On the set $\overline{G}$

$$\overline{G} = G \cup S, \quad G = D \times (0, T], \tag{2.1}$$

where $D=(0,d)$, we consider the Dirichlet problem for the singularly perturbed parabolic convection-diffusion equation

$$\begin{aligned} L\,u(x,t) &= f(x,t), \quad (x,t)\in G,\\ u(x,t) &= \varphi(x,t), \quad (x,t)\in S. \end{aligned} \tag{2.2}$$

Here

$$L=\varepsilon\, a(x,t)\frac{\partial^2}{\partial x^2}+b(x,t)\frac{\partial}{\partial x}-c(x,t)-p(x,t)\frac{\partial}{\partial t}, \quad (x,t)\in G,$$

the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ and $\varphi(x,t)$ are assumed to be sufficiently smooth on $\overline{G}$ and on smooth parts of S respectively, and satisfy*

$$\begin{aligned} &a_0\le a(x,t)\le a^0, \quad b_0\le b(x,t)\le b^0, \quad 0\le c(x,t)\le c^0,\\ &p_0\le p(x,t)\le p^0, \quad (x,t)\in\overline{G};\\ &|f(x,t)|\le M,\ \ (x,t)\in\overline{G}; \quad |\varphi(x,t)|\le M,\ \ (x,t)\in S; \end{aligned} \tag{2.3}$$

a_0, b_0, $p_0>0$. The parameter ε takes arbitrary values in the half-open interval $(0,1]$.

For small values of the parameter ε, a regular boundary layer appears in a neighbourhood of the set $S_1^L=\{(x,t):\ x=0,\ 0<t\le T\}$. Here S_1^L and S_2^L are the left and right parts of the lateral boundary S^L; $S=S^L\cup S_0$, $S^L=S_1^L\cup S_2^L$, and $S_0=\overline{S}_0$ is the lower part of the boundary.

2.2. Some definitions

In the case of the boundary value problem (2.2), (2.1), we are interested in numerical methods whose solutions converge uniformly with respect to the parameterε (or, briefly, ε-uniformly) in the maximum discrete norm. However, in the case of singularly perturbed problems, the ε-uniform convergence of the numerical solution $z(x,t)$ at the points of the mesh $\overline{G}_h$ is, in general, not adequate to give a representation about ε-uniform convergence of an approximation constructed on the whole set $\overline{G}$. For example, the solution of a difference scheme obtained by the classical approximation of problem (2.2), (2.1) on a uniform mesh like $\overline{G}_{h(3.3)}=\overline{G}_h^{\,u}$ converges on the mesh $\overline{G}_h$ when $\varepsilon^{-1}h\to\infty$ for $h\to 0$, i.e., when the typical width of the boundary layer defined by ε is much less than the mesh size in x. However, even the simplest *interpolant*

$$\overline{z}(x,t)=\overline{z}\big(x,t;\ z(\cdot),\overline{G}_h\big), \quad (x,t)\in\overline{G} \tag{2.4}$$

does not converge on $\overline{G}$, where $\overline{z}(x,t)$ is a linear interpolant on triangular elements (triangulation of elementary rectangles from $\overline{G}$ generated by the points of the mesh $\overline{G}_h$) that is constructed using the discrete function $z(x,t)$, $(x,t)\in\overline{G}_h$. Under the given requirements on the mesh $\overline{G}_{h(3.3)}$, the interpolant

$$\overline{u}^h(x,t)=\overline{z}_{(2.4)}\big(x,t;\ u^h(\cdot),\overline{G}_h\big), \quad (x,t)\in\overline{G},$$

* By M (by m) we denote sufficiently large (small) positive constants that are independent of ε and the scheme parameters.

constructed using the discrete function

$$u^h(x,t) = u(x,t), \quad (x,t) \in \overline{G}_h,$$

where $u(x,t)$ is the solution of problem (2.2), (2.1), does not converge on $\overline{G}$ as well.

Let us give some *definitions*. In that case when the interpolant $\overline{z}_{(2.4)}(x,t)$, $(x,t) \in \overline{G}$, converges on $\overline{G}$, we say that the difference *scheme solves the boundary value problem* (converges on $\overline{G}$); otherwise, we say that the difference scheme does not solve the boundary value problem. In that case when the interpolant $\overline{z}(x,t)$, $(x,t) \in \overline{G}$, converges on $\overline{G}$ ε-uniformly, we say that the *difference scheme* converges (*solves the boundary value problem*) *ε-uniformly*.

But if the interpolant $\overline{u}^h(x,t)$, $(x,t) \in \overline{G}$, converges (converges ε-uniformly) on $\overline{G}$, we say that the *mesh* $\overline{G}_h$ is *informative (ε-uniformly informative) for the solution of the boundary value problem*; if not, we say that the mesh $\overline{G}_h$ is not informative for the solution.

We are interested in difference schemes that converge on $\overline{G}$ ε-uniformly or schemes close to ε-uniformly convergent schemes on $\overline{G}$.

2.3. Aim of the paper

Let us formulate the aim of this research.

The error bound (3.4) for the discrete solution in Section 3 implies that the solution of the classical difference scheme (3.2) on the uniform mesh (3.3) converges under the rather restrictive condition $(h \ll \varepsilon)$ $\varepsilon^{-1} = o(N)$, where $N+1$ is the number of mesh points in x. If this condition is violated, for example, for $\varepsilon^{-1} = \mathcal{O}(N)$, then, in general, the solution of the difference scheme (3.2), (3.3) does not converge to the solution of problem (2.2), (2.1) as $N, N_0 \to \infty$; $N_0 + 1$ is the number of mesh points in t.

Let us give some *definitions* [7]. Suppose that $z(x,t)$, $(x,t) \in \overline{G}_h$, is a solution of some difference scheme, and let the function $z(x,t)$ satisfy the estimate

$$|u(x,t) - z(x,t)| \le M\lambda\left(\varepsilon^{-\nu}N^{-1}, N_0^{-1}\right), \quad (x,t) \in \overline{G}_h, \tag{2.5}$$

where $\lambda(\xi_1, \xi_2) \to 0$ as $\xi_1, \xi_2 \to 0$, uniformly in the parameter ε; and $\nu \ge 0$. By definition, the solution of this scheme converges on the set $\overline{G}_h$ *uniformly with respect to the parameter ε* (or, briefly, *ε-uniformly*) if $\nu = 0$ in estimate (2.5). Otherwise, we say that the scheme does not converge ε-uniformly on $\overline{G}_h$. But if the scheme converges for $N^{-1} = o(\varepsilon^\nu)$, where the constant M in estimate (2.5), generally speaking, depends on ν, however in general, there is no convergence for $N^{-1} = \mathcal{O}(\varepsilon^\nu)$, we say that the scheme converges with *defect* (of ε-uniform convergence) ν. In that case when the value ν can be chosen arbitrarily small, and also the solution of the difference scheme *controlled by the value ν* satisfies estimate (2.5), we say that the scheme converges on $\overline{G}_h$ *almost ε-uniformly with defect ν* (or, briefly, *almost ε-uniformly*).

In a similar way, the convergence defect of the scheme on the set $\overline{G}$ can be defined.

The defect of scheme (3.2), (3.3) is equal to 1.

For problem (2.2), (2.1), the difference scheme from [2] in the case $n = 1$ (that is, the scheme on *a priori* adapted piecewise-uniform mesh with a single transition point at which the mesh changes its step-size) converges ε-uniformly. Note that, in schemes on piecewise-uniform meshes (see, e.g., [2–5] and the references therein), the mesh size changes sharply at the transition points where the mesh switches from coarse to fine (the ratio of the mesh sizes is not ε-uniformly bounded; see Remark 3.1 in Section 3), which, in general, can lead to restrictions in using efficient numerical approaches to the computation of discrete solutions and improvement of their accuracy (see, e.g., [11–15] and the references therein).

Schemes on a *a posteriori* adapted meshes that converge almost ε-uniformly were considered in [7–9]. An adapted mesh is constructed using meshes that are uniform on the subdomains in which the computed solution is corrected. The advantage of this scheme is that its solution is "synthesized" using the components of solutions of auxiliary intermediate problems that are solved on the corresponding subdomains having uniform meshes with the same numbers of mesh point in x, t on each subdomain.

It should be noted that in order to solve discrete problems on uniform meshes, highly efficient numerical methods have been developed, which require the number of operations for computing the solution of the same order as the number of mesh points (see, e.g., [11–13] and the references therein). Due to this, it would be of interest to construct and examine almost ε-uniformly convergent schemes on *a priori* adapted meshes based on locally uniform meshes, that is, uniform meshes on each of the subdomains.

Our aim for the boundary value problem (2.2), (2.1) is to construct an almost ε-uniformly converging scheme on *a priori* adapted locally uniform meshes.

3. Schemes on uniform and piecewise-uniform meshes

In this section, we give a finite difference scheme on a piecewise-uniform mesh that converges ε-uniformly.

3.1. A finite difference scheme on an arbitrary grid

On the set $\overline{G}$, we introduce the rectangular grid

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are, in general, arbitrary nonuniform meshes on the closed intervals $[0, d]$ and $[0, T]$, respectively. Let $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$, $h = \max_i h^i$, and $h_t^k = t^{k+1} - t^k$, t^k, $t^{k+1} \in \overline{\omega}_0$, $h_t = \max_k h_t^k$. Assume that $h \le M N^{-1}$ and $h_t \le M N_0^{-1}$, where $N + 1$ and $N_0 + 1$ are the numbers of points in the meshes in $\overline{\omega}$ and $\overline{\omega}_0$, respectively.

Problem (2.2), (2.1) is approximated by the finite difference scheme [11]

$$\begin{aligned} \Lambda z(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h, \end{aligned} \tag{3.2}$$

where $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$, and

$$\Lambda \equiv \varepsilon\, a(x,t)\, \delta_{\overline{x}\widehat{x}} + b(x,t)\, \delta_x - c(x,t) - p(x,t)\, \delta_{\overline{t}}, \quad (x,t) \in G_h.$$

Here $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the central difference derivative on the nonuniform mesh,

$$\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x,t) - \delta_{\overline{x}} z(x,t)], \quad (x,t) = (x^i, t) \in G_h;$$

$\delta_x\, z(x,t)$ and $\delta_{\overline{x}} z(x,t)$ are the first-order (forward and backward) difference derivatives. Scheme (3.2), (3.1) is monotone ε-uniformly. A definition of monotonicity for regular problems can be found in [11].

3.2. An error bound on a uniform grid

In the case of uniform meshes in both variables

$$\overline{G}_h = \overline{G}_h^u = \overline{\omega} \times \overline{\omega}_0, \tag{3.3}$$

using the maximum principle, we obtain the error bound

$$|u(x,t) - z(x,t)| \le M\left[\left(\varepsilon + N^{-1}\right)^{-1} N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h; \tag{3.4}$$

bound (3.4) is unimprovable with respect to N, N_0, ε.

The interpolant $\overline{z}(x,t) = \overline{z}_{(2.4)}\big(x,t;\, z_{(3.2,3.3)}(\cdot),\, \overline{G}_h^u\big)$ satisfies the error bound

$$|u(x,t) - \overline{z}(x,t)| \le M\left[\left(\varepsilon + N^{-1}\right)^{-1} N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}. \tag{3.5}$$

Scheme (3.2), (3.3) converges under the condition ($N^{-1} \ll \varepsilon$):

$$\varepsilon^{-1} = o(N), \quad N \to \infty, \quad \varepsilon \in (0,1]. \tag{3.6}$$

3.3. An ε-uniform convergent scheme

Let us give a scheme that converges ε-uniformly (see, for example, [2,3]). On the set $\overline{G}$, we introduce the grid

$$\overline{G}_h = \overline{G}_h^s \equiv \overline{\omega}^s \times \overline{\omega}_0, \tag{3.7}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(3.3)}$, $\overline{\omega}^s$ is a piecewise uniform mesh that is constructed as follows. The interval $[0,d]$ is divided into two parts $[0,\sigma]$ and $[\sigma,d]$. The mesh sizes on the intervals $[0,\sigma]$ and $[\sigma,d]$ are constant and equal to

$$h^{(1)} = 2\,\sigma\, N^{-1} \quad \text{and} \quad h^{(2)} = 2(d-\sigma)N^{-1},$$

respectively. The parameter σ is chosen so as to satisfy the condition

$$\sigma = \sigma(\varepsilon, N) = \min\left[2^{-1}\, d,\, m^{-1}\,\varepsilon\, \ln N\right],$$

where m is an arbitrary number in the interval $(0, m_0)$, and $m_0 = m_{0(A.2)}$.

For the solution of the difference scheme (3.2), (3.7), we obtain the ε-uniform bound

$$|u(x,t) - z(x,t)| \le M\,\left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h; \tag{3.8}$$

the bound is unimprovable with respect to N, N_0.

The interpolant $\overline{z}(x,t) = \overline{z}_{(2.4)}\big(x,t;\, z_{(3.2,3.7)}(\cdot),\, \overline{G}_h^s\big)$ satisfies the error bound

$$|u(x,t) - \overline{z}(x,t)| \le M\,\left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}. \tag{3.9}$$

Theorem 3.1. *Let the components of the solution $u(x,t)$ of the boundary value problem* (2.2)*,* (2.1) *in representation* (A.1) *satisfy the estimates of Theorem A.*1*. Then the difference scheme* (3.2)*,* (3.7) *(scheme* (3.2)*,* (3.3)*) converges ε-uniformly (under condition* (3.6)*). The discrete solutions and their interpolants satisfy bounds* (3.4)*,* (3.8) *and* (3.5)*,* (3.9)*, respectively.*

Remark 3.1. For the mesh $\overline{G}_{h(3.7)}$, the ratio of $h^{(2)}$ and $h^{(1)}$, i.e., the mesh sizes in x on the mesh intervals with constant step-size, is of the order $\mathcal{O}(\varepsilon^{-1}\ln^{-1}N)$.

4. Grid approximations on locally refined meshes that are uniform in subdomains

In this section, we present an algorithm for constructing a locally refined mesh (adapted in the boundary layer) and a grid solution on it. In each of subdomains subjected to mesh refinement, this algorithm uses uniform meshes with respect to space and time (the temporal mesh is not refined).

4.1. A formal iterative algorithm

First, we describe a formal iterative algorithm for constructing approximate solutions for the boundary value problem (2.2), (2.1) [7].

On the set $\overline{G}$, we introduce the coarsened (initial) mesh

$$\overline{G}_{1h} = \overline{\omega}_1 \times \overline{\omega}_0, \tag{4.1a}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform meshes, $\overline{\omega}_0 = \overline{\omega}_{0(3.3)}$; the mesh size of $\overline{\omega}_1$ is $h_1 = d\,N^{-1}$. We denote the solution of problem (3.2), (4.1a) by $z_1(x,t)$, $(x,t)\in\overline{G}_{1h}$, where $\overline{G}_{1h} = \overline{G}_{1h(4.1)}$. Note that $\overline{G}_{1h(4.1)} = \overline{G}_{h(3.3)}$.

Let the value $d_1 \in \overline{\omega}_1$ be found in such a way that for $x \ge d_1$, the discrete solution $z_1(x,t)$, $(x,t) \in \overline{G}_{1h}$, is a good approximation of the solution of problem (2.2), (2.1), moreover,

$$|u(x,t) - z_1(x,t)| \le M\,\delta, \quad (x,t)\in\overline{G}_{1h}, \quad x \ge d_1, \tag{4.2a}$$

where $\delta > 0$ is an arbitrary sufficiently small number specifying the required accuracy of the discrete solution, and M is a constant independent of δ, and $d_1 \in [0,d)$.

If it turns out that $d_1 > 0$, then we define the subdomain

$$\overline{G}_{(2)} = G_{(2)} \cup S_{(2)}, \;\; G_{(2)} = G_{(2)}(d_1), \;\; G_{(2)} = D_{(2)} \times (0,T], \;\; D_{(2)} = (0,d_1),$$

where we shall refine the grid. On the subdomain $\overline{G}_{(2)}$, we introduce the grid

$$\overline{G}_{(2)h} = \overline{\omega}_{(2)} \times \overline{\omega}_0,$$

where $\overline{\omega}_{(2)}$ is a uniform mesh with $N+1$ points and the mesh size $h_{(2)}$.

On the set $\overline{G}_{(2)h}$, we find the solution $z_{(2)}(x,t)$ of the discrete problem

$$\Lambda_{(3.2)}\, z_{(2)}(x,t) = f(x,t), \qquad (x,t) \in G_{(2)h},$$
$$z_{(2)}(x,t) = \begin{cases} z_1(x,t), & (x,t) \in S_{(2)h} \setminus S, \\ \varphi(x,t), & (x,t) \in S_{(2)h} \bigcap S, \end{cases}$$

where

$$G_{(2)h} = G_{(2)} \bigcap \overline{G}_{(2)h}, \quad S_{(2)h} = S_{(2)} \bigcap \overline{G}_{(2)h}.$$

The grid set $\overline{G}_{2h}$ on $\overline{G}$ and the function $z_2(x,t)$, $(x,t) \in \overline{G}_{2h}$, are defined by

$$\overline{G}_{2h} = \overline{G}_{(2)h} \cup \left\{ \overline{G}_{1h} \setminus \overline{G}_{(2)} \right\}, \quad z_2(x,t) = \begin{cases} z_{(2)}(x,t), & (x,t) \in \overline{G}_{(2)h}, \\ z_1(x,t), & (x,t) \in \overline{G}_{1h} \setminus \overline{G}_{(2)}. \end{cases}$$

For $k \geq 3$ at the $(k-1)$th iteration, suppose that the grid set $\overline{G}_{k-1,h}$ and the grid function $z_{k-1}(x,t)$ on this set have already been constructed. Furthermore, let the value $d_{k-1} \in \omega_{k-1}$ be found in such a way that for $x \geq d_{k-1}$ the discrete solution $z_{k-1}(x,t)$, $(x,t) \in \overline{G}_{k-1,h}$, is a good approximation of the solution of the problem (2.2), (2.1). To be precise, we require

$$|u(x,t) - z_{k-1}(x,t)| \leq M\,\delta, \quad (x,t) \in \overline{G}_{k-1,h}, \quad x \geq d_{k-1}. \tag{4.2b}$$

The constant M depends on k, $M_{(4.2b)} = M_{(4.2b)}(k-1)$, where $M(k) = M^*\,k$.[†] Here

$$\overline{G}_{k-1,h} = \overline{\omega}_{k-1} \times \overline{\omega}_0,$$

$\overline{\omega}_{k-1}$ is a mesh generating the mesh $\overline{G}_{k-1,h}$; $N_k + 1$ is the number of nodes in the mesh $\overline{\omega}_k$, $k \geq 2$; and $N_1 = N$.

If it happens that $d_{k-1} > 0$, then we define the subdomain

$$\overline{G}_{(k)} = G_{(k)} \cup S_{(k)}, \;\; G_{(k)} = G_{(k)}(d_{k-1}), \;\; G_{(k)} = D_{(k)} \times (0,T], \;\; D_{(k)} = (0, d_{k-1}). \tag{4.1b}$$

On the set $\overline{G}_{(k)}$, we introduce the grid

$$\overline{G}_{(k)h} = \overline{\omega}_{(k)} \times \overline{\omega}_0, \tag{4.1c}$$

where $\overline{\omega}_{(k)}$ is the uniform mesh with $N+1$ points and $h_{(k)}$ is the step-size in the mesh $\overline{\omega}_{(k)}$; and $h_{(k)} = d_{k-1}N^{-1}$, $d_{k-1} = d$ for $k = 1$. Let $z_{(k)}(x,t)$, $(x,t) \in \overline{G}_{(k)h}$, be the solution of the grid problem

$$\Lambda_{(3.2)}\, z_{(k)}(x,t) = f(x,t), \qquad (x,t) \in G_{(k)h},$$
$$z_{(k)}(x,t) = \begin{cases} z_{k-1}(x,t), & (x,t) \in S_{(k)h} \setminus S, \\ \varphi(x,t), & (x,t) \in S_{(k)h} \bigcap S. \end{cases} \tag{4.1d}$$

[†] Here and in what follows, M^* denote constants independent of k.

We set

$$\overline{G}_{kh} = \overline{G}_{(k)h} \bigcup \left\{ \overline{G}_{k-1,h} \setminus \overline{G}_{(k)} \right\},$$

$$z_k(x,t) = \begin{cases} z_{(k)}(x,t), & (x,t) \in \overline{G}_{(k)h}, \\ z_{k-1}(x,t), & (x,t) \in \overline{G}_{k-1,h} \setminus \overline{G}_{(k)}. \end{cases}$$

If for some value $k = K_0$ it turns out that $d_{K_0} = 0$, then we set $d_k = 0$ for $k \geq K_0$. For $k \geq K_0 + 1$, the sets $\overline{G}_{(k)}$ are assumed to be empty, and we do not compute the functions $z_{(k)}(x,t)$. For example, for $k \geq K_0$ we have $z_k(x,t) = z_{K_0}(x,t)$, $\overline{G}_{kh} = \overline{G}_{K_0 h}$.

For $k = K$, where K is a given fixed number (the number of iterations for improving the grid solution), $K \geq 1$, we assume

$$\overline{G}_h^K = \overline{G}_{Kh} \equiv \overline{G}_h, \quad z^K(x,t) = z_K(x,t) \equiv z(x,t). \tag{4.1e}$$

The grid $\overline{G}_h$ and the function $z(x,t)$ in (4.1e) are constructed using the grid sets $\overline{G}_{(k)h}$ and the functions $z_{(k)}(x,t)$, $(x,t) \in \overline{G}_{(k)h}$, $k = 1, \cdots, K$.

We call the function $z_{(4.1)}(x,t)$, $(x,t) \in \overline{G}_{h(4.1)}$, the solution of scheme (3.2), (4.1) (of the scheme on refined meshes being uniform on local subdomains). The functions $z_k(x,t)$, $(x,t) \in \overline{G}_{kh}$, $k = 1, \cdots, K$ are called the components of the solution of the difference scheme.

Let the value $d^K \in \overline{\omega}_K$, $d^K = d_K$, be found so that for $x \geq d_K$ the solution $z_K(x,t)$ approximates the solution of problem (2.2), (2.1); in this case we have

$$|u(x,t) - z(x,t)| \leq M\,\delta, \quad (x,t) \in \overline{G}_h, \quad x \geq d^K, \tag{4.2c}$$

where $z(x,t) = z_{(4.1)}(x,t)$, $\overline{G}_h = \overline{G}_{h(4.1)}$.

The difference scheme (3.2), (4.1) is the scheme on locally refined meshes that are uniform on the subdomains $\overline{G}_{(k)h}$ in which the computed solution is corrected. In that case when the values d_k are determined in the process of the numerical solution of problem (3.2), (4.1) depending on the results of computations, the scheme (3.2), (4.1) is the scheme on *a posteriori* adapted meshes (we say, the *a posteriori* adapted scheme). But if the values d_k are given before the start of computations regardless of the computational results obtained, scheme (3.2), (4.1) is the scheme on *a priori* adapted meshes (we say, the *a priori* adapted scheme).

4.2. Maximum principle for the algorithm $A_{(4.1)}$

The given algorithm (we call it $A_{(4.1)}$) allows us to construct the solution of problem (3.2), (4.1) on the basis of the sequence of values d_k, $k = 1, \cdots, K$. The value $N_K + 1$ is the number of nodes in the mesh $\overline{\omega}^K = \overline{\omega}_K$ used for the construction of the function $z^K(x,t)$. For the value N_K, we have the estimate

$$N_K \leq K(N-1) + 1 \leq KN.$$

The ratio of $h_{(k)}$ and $h_{(k+1)}$, i.e., the mesh step-sizes with respect to x in neighbouring subregions of the adaptive mesh, does not exceed the value N.

In schemes (3.2), (4.1) when solving the intermediate problems (4.1d), an interpolation is not required to find values of the functions $z_{(k)}(x,t)$ on the boundary $S_{(k)h}$.

For the scheme (3.2), (4.1), the maximum principle holds. The following comparison theorem is valid.

Theorem 4.1. *Let the functions $z_k^1(x,t)$ and $z_k^2(x,t)$, $(x,t)\in\overline{G}_{kh}$, $\overline{G}_{kh}=\overline{G}_{kh(4.1)}$, $k=1,2,\cdots,K$, satisfy the conditions*

$$
\begin{aligned}
&\Lambda z_1^1(x,t)\le\Lambda z_1^2(x,t),\ \ (x,t)\in G_{(1)h},\\
&z_1^1(x,t)\ge z_1^2(x,t),\ \ (x,t)\in S_{(1)h};\\
&\Lambda z_k^1(x,t)\le\Lambda z_k^2(x,t),\ \ (x,t)\in G_{(k)h},\\
&z_k^1(x,t)\ge z_k^2(x,t),\ \ (x,t)\in S_{(k)h}\cap S,\\
&z_k^1(x,t)\ge z_{k-1}^1(x,t),\ \ z_{k-1}^2(x,t)\ge z_k^2(x,t),\\
&(x,t)\in\overline{G}_{kh}\setminus\{G_k\cup\{S_{(k)}\cap S\}\},\ \ k=2,\cdots,K.
\end{aligned}
$$

Then $z_K^1(x,t)\ge z_K^2(x,t)$, $(x,t)\in\overline{G}_{Kh}$.

The theorem can be proved by the induction with respect to k, where k is the number of the current iteration in the iterative process.

The meshes $\overline{G}_{kh}$, $k=1,\cdots,K$, obtained by the algorithm $A_{(4.1)}$, are defined by the choice of the values d_k, $k=1,2,\cdots,K$, and also by the values K and N,N_0. In the meshes obtained by the algorithm $A_{(4.1)}$, the values d_k will be determined regardless of the results obtained in the computational process, i.e., the meshes $\overline{G}_{kh}$ belong to *a priori* condensing meshes.

Note that there exist no schemes in this class of difference schemes whose solutions converge ε-uniformly to the solution of the boundary value problem (2.2), (2.1).

5. Difference scheme on *a priori* adapted mesh

In this section, we consider a difference scheme on *a priori* adapted meshes constructed using a majorant function for the singular component of the grid solution.

5.1. Auxiliary constructions

We present a number of auxiliary constructions. For the differential and the difference problems, we introduce the width of the boundary layer specified by majorant functions for the singular components of their solutions.

The function

$$W^c(x)=W^c(x;\,\varepsilon)=\exp(-m^0\varepsilon^{-1}x),\qquad x\in\overline{D}^{\infty},\tag{5.1a}$$

is a majorant function (up to a constant factor) for the singular component $V(x,t)$ in representation (A.1) of the solution of the boundary value problem (2.2), (2.1). Here $m^0 = \min_{\overline{G}}[a^{-1}(x,t)b(x,t)]$, and

$$\overline{D}^{\infty} = [0, \infty). \tag{5.1b}$$

Based on the function $W^c(x)$, we introduce the width of the boundary layer for problem (2.2), (2.1). We say that the value

$$\eta^c = \eta^c(\delta;\, \varepsilon), \tag{5.2a}$$

where $\delta > 0$ is a sufficiently small number, is the *width of the boundary layer* (*defined by the majorant function for the singular component* $V(x,t)$) *with the threshold of order* δ (or, briefly, the *width of the boundary layer defined by the majorant function*), if η^c is the minimum value of η^0 for which the following estimate is fulfilled:

$$W^c(x;\, \varepsilon) \le \delta, \quad x \in \overline{D}^{\infty}, \quad r(x, \Gamma_1) \ge \eta^0, \tag{5.2b}$$

where Γ_1 is the boundary of the set $\overline{D}^{\infty}$; $\overline{D}^{\infty} = D^{\infty} \cup \Gamma$, $\Gamma = \Gamma_1$. The value η^c may take magnitudes exceeding $d_{(2.1)}$ (for sufficiently small values δ such that $\delta \le \delta(\varepsilon)$); η^c is defined by the formula

$$\eta^c = (m^0)^{-1} \varepsilon \ln \delta^{-1}. \tag{5.2c}$$

Let us introduce the width of the discrete boundary layer defined on the basis of the majorant function for the discrete singular component. By $z_v(x,t)$, $(x,t) \in \overline{G}$, we denote the solution of the difference scheme

$$\Lambda_{(3.2)}\, z(x,t) = L_{(2.2)}\, v(x,t), \;\; (x,t) \in G_h, \quad z(x,t) = v(x,t), \;\; (x,t) \in S_h,$$

where $v(x,t)$ is an arbitrary sufficiently smooth function, and $v \in C^{2,1}(G) \cap C(\overline{G})$. The solution of problem (3.2), (3.1) can be represented in the form of the sum of functions

$$z(x,t) = z_U(x,t) + z_V(x,t), \quad (x,t) \in \overline{G}_h, \tag{5.3}$$

where $z_U(x,t)$ and $z_V(x,t)$ are the grid functions approximating the components $U(x,t)$ and $V(x,t)$ in representation (A.1), and $z_V(x,t)$ is the function of the discrete boundary layer.

The function

$$W(x) = W(x;\, \varepsilon, h) = (1 + m^0 \varepsilon^{-1} h)^{-n}, \quad x = x^n \in \overline{D}_h^{\infty}, \;\; x^n = nh, \tag{5.4}$$

where $\overline{D}_h^{\infty}$ is the uniform mesh on the semi-axis $\overline{D}_{(5.1)}^{\infty}$ with the step-size h, $m^0 = m^0_{(5.1)}$, is a majorant function (up to a constant factor) for the singular component $z_V(x,t)$ in representation (5.3) of the solution of the difference scheme (3.2) on the mesh (3.3), where $h_{(3.3)} = h_{(5.4)}$. We say that the value

$$\eta = \eta(\delta;\, \varepsilon, h), \tag{5.5a}$$

where $\delta > 0$ is a sufficiently small number, is the *width of the discrete boundary layer (defined by the majorant function $W(x)$ for the singular component $z_V(x,t)$) with the threshold of the order of δ* (or, briefly, the *width of the discrete boundary layer defined by the majorant function*) if η is the minimum value of η_0 for which the estimate

$$W(x;\,\varepsilon,h) \le \delta, \quad x \in \overline{D}_h^{\infty}, \quad r(x,\Gamma_1) \ge \eta_0 \tag{5.5b}$$

holds. The quantity η may take values exceeding $d_{(2.1)}$; η is determined by the formula

$$\eta = \eta(\delta;\,\varepsilon,h) = \begin{cases} h\ln\delta^{-1}\ln^{-1}(1+m^0\varepsilon^{-1}h) \quad \text{for} \\ \qquad \left[\ln\delta^{-1}\ln^{-1}(1+m^0\varepsilon^{-1}h)\right]^e = \ln\delta^{-1}\ln^{-1}(1+m^0\varepsilon^{-1}h), \\ h\left\{\left[\ln\delta^{-1}\ln^{-1}(1+m^0\varepsilon^{-1}h)\right]^e + 1\right\} \quad \text{for} \\ \qquad \left[\ln\delta^{-1}\ln^{-1}(1+m^0\varepsilon^{-1}h)\right]^e < \ln\delta^{-1}\ln^{-1}(1+m^0\varepsilon^{-1}h), \end{cases} \tag{5.5c}$$

with $\delta \in (0,1)$, $\varepsilon \in (0,1]$, where $h = h_{(5.4)}$, and $[\,a\,]^e$ is the integer part of a number a.

It is convenient to use the following notation. We associate the value $a \ge 0$, on the uniform mesh $\overline{D}_{h(5.4)}^{\infty}$ with the step-size h, with the value $\{a;\,h\}^e$ defined by the relation

$$\{a;\,h\}^e = \begin{cases} a \quad \text{for} \quad h\left[h^{-1}a\right]^e = a, \\ h\left\{h\left[h^{-1}a\right]^e + 1\right\} \quad \text{for} \quad h\left[h^{-1}a\right]^e < a, \end{cases}$$

where $[\,a\,]^e = [\,a\,]^e_{(5.5)}$. So, the value η is representable in the form

$$\eta = \eta(\delta;\,\varepsilon,h) = \{a;\,h\}^e, \tag{5.5d}$$

where

$$a = h\ln\delta^{-1}\ln^{-1}(1+m^0\varepsilon^{-1}h).$$

On the set $\overline{G}_{(k)}$, $k \ge 1$, the mesh $\overline{G}_{(k)h}$ with its step-size $h_{(k)}$ in x is defined. Let us define the values d_k in (4.1) by the relation

$$d_k = d_k(\delta;\,\varepsilon,N) \equiv \min\left[\eta(\delta;\,\varepsilon,h_{(k)}),\, d\right], \quad k = 1,\cdots,K, \tag{5.6a}$$

where $h_{(1)} = dN^{-1}$, $h_{(k)} = d_{k-1}N^{-1}$, $k \ge 2$. Assume

$$\delta = \delta(N) \to 0 \quad \text{for} \quad N \to \infty. \tag{5.6b}$$

The difference scheme (3.2), (4.1), (5.6) is the scheme on *a priori* adapted meshes. The values d_k are computed using an indicator based on the majorant function of the discrete boundary layer controlled by the parameters δ, ε, and h.

5.2. A nonconstructive error bound

For the solution of the difference scheme (3.2), (4.1), (5.6), using the maximum principle we establish the error bound

$$|u(x,t)-z(x,t)| \le \begin{cases} M\left[\delta(N)+N^{-1}+N_0^{-1}\right], & (x,t)\in\overline{G}_h,\ \ r(x,\Gamma_1)\ge d_K; \\ M\left[(\varepsilon+d_{K-1}N^{-1})^{-1}d_{K-1}N^{-1}+\delta(N)+N^{-1}+N_0^{-1}\right], & \\ & (x,t)\in\overline{G}_h. \end{cases} \tag{5.7}$$

Thus, the difference scheme (3.2), (4.1), (5.6) converges ε-uniformly outside the d_K-neighbourhood of the boundary S_1^L, and also on the whole set $\overline{G}_h$ under the condition that $h_{(K)} \ll \varepsilon$:

$$\varepsilon^{-1}=o\big(d_{K-1}^{-1}N\big),$$

which is essentially weaker in comparison with the convergence condition (3.6).

Bound (5.7) is nonconstructive since the values $d_{K-1(5.6)}$ and $d_{K(5.6)}$ depend on ε, N, and K implicitly, which complicates the investigation of scheme (3.2), (4.1), (5.6) depending on the values of N, ε, and K.

Theorem 5.1. *Let the solution of problem* (2.2), (2.1) *satisfy the hypothesis of Theorem* 3.1. *Then the solution of the difference scheme* (3.2), (4.1), (5.6) *satisfies the bound* (5.7).

5.3. A difference scheme on a priori adapted meshes

We now consider a version of the difference scheme on *a priori* adapted meshes that allows us to write out efficient estimates for $\eta(\delta;\,\varepsilon,h_{(k)})$. These estimates make it possible to study convergence properties of the scheme on *a priori* adapted meshes.

Let us note some properties of the value $\eta_{(5.5)}$ implied by its explicit form. The function $\eta(\delta;\varepsilon,h)$ for fixed values of δ and h is a piecewise-constant nondecreasing function with respect to the variable ε.

We assume that the following condition is fulfilled:

$$\delta = N^{-\alpha},\quad \alpha\in(0,1]. \tag{5.8}$$

For the value η, we have the estimate

$$\eta(\delta;\,\varepsilon,h_1) > \eta^c(\delta;\,\varepsilon),$$

where $h_1 = h_{1(4.1a)}$. However,

$$\eta(\delta;\,\varepsilon,h_1) \le M_1\eta^c(\delta;\,\varepsilon) \tag{5.9a}$$

under the condition

$$h_1 \le m_1(m^0)^{-1}\varepsilon;\quad M_1 = M_1(m_1),\quad m^0 = m^0_{(5.1)}, \tag{5.9b}$$

where $M_1(m_1)$ is evaluated by the inequality

$$\alpha_1 \ln^{-1}(1+\alpha_1) \le M_1 \quad \text{for} \quad \alpha_1 \le m_1; \ \text{ e.g., } \ M_1(m_1=1)=2. \tag{5.9c}$$

In the case of the condition

$$\varepsilon \ge \varepsilon^{(0)},$$

$\eta(\delta;\, \varepsilon, h)$ satisfies the lower bound

$$\eta(\delta;\, \varepsilon, h_1) \ge h_1;$$

moreover, under the condition

$$\varepsilon \le \varepsilon^{(1)},$$

for $\eta(\delta;\, \varepsilon, h)$ we have the upper bound

$$\eta(\delta;\, \varepsilon, h_1) \le h_1.$$

Here the values $\varepsilon^{(j)}$ are defined by the relations

$$\begin{aligned} &\varepsilon^{(j)} = \varepsilon^{(j)}(\delta, N) = \varepsilon^{(j)}(\delta, N;\, d), \quad j \ge -1; \\ &\varepsilon^{(-1)} = M_2 m^0 d \, \ln^{-1} \delta^{-1}, \quad \varepsilon^{(0)} = M_1 m^0 d N^{-1}, \\ &\varepsilon^{(j)} = m^0 d\, \delta(1-\delta)^{-1} N^{-j}, \ \ j \ge 1, \end{aligned} \tag{5.10}$$

where $d = d_{(2.1)}$, $N = N_{(4.1a)}$, $m^0 = m^0_{(5.1)}$, $M_1 = M_{1(5.9)}$, and M_2 is an arbitrary constant satisfying the inequality

$$M_2 \le M_1^{-1},$$

$j \ge -1$ is an integer. By this choice of the constants M_1, M_2, we have $\eta(\delta;\varepsilon, h_1) \le d$ for $\delta = \delta_{(5.8)}$ and $\varepsilon \le \varepsilon^{(-1)}$.

We describe the rule for determining the values $d_{k(4.1)}$ in the grid construction (3.2), (4.1) for the given values of K and ε, considering that the parameter ε belongs to the prescribed fixed intervals defined by the values $\varepsilon^{(j)}$. To construct the scheme on adapted meshes for given K, it is necessary to prescribe the values d_k for $k \le K-1$. However, when studying the schemes, we will need the values d_k for $k \le K$.

We suppose that the parameter ε belongs to one of the following intervals defined by the value j

$$\varepsilon \in \big[\, \varepsilon^{(j)}, 1 \,\big] \quad \text{for} \ \ j = -1, \quad \text{or} \quad \varepsilon \in \big[\, \varepsilon^{(j)},\, \varepsilon^{(j-1)} \big) \quad \text{for} \ \ j \ge 0, \tag{5.11a}$$

where $\varepsilon^{(j)} = \varepsilon^{(j)}_{(5.10)}(\delta, N)$, $j \ge -1$. The value d_k depends on K, j and also on δ, ε, k, and it is chosen in the set $\overline{G}_{(k)h}$ so that the value of $\eta(\delta;\, \varepsilon, h_{(k)})$, i.e., the width of the discrete boundary layer, satisfies the estimate

$$\eta(\delta;\, \varepsilon, h_{(k)}) \le d_k \quad \text{for} \ \ 1 \le k \le K$$

in that case when the parameter ε belongs to one of the intervals in (5.11a).

Consider the case when the following relation is fulfilled:

$$K = K_{(5.11b)}(j) \equiv j + 2, \quad j \ge -1, \tag{5.11b}$$

where $j = j_{(5.11a)}$ defines the interval of varying the parameter ε. Let $\varepsilon \in [\varepsilon^{(j)}, 1]$ for $j = -1$. In this case $K = 1$; we set

$$d_1 = \min\left[\left\{M_1(m^0)^{-1}\varepsilon \ln \delta^{-1};\ h_{(1)}\right\}^e,\ d\,\right]. \tag{5.11c}$$

Let $\varepsilon \in \left[\varepsilon^{(j)},\ \varepsilon^{(j-1)}\right),\ j \ge 0$. Assume

$$\begin{aligned}
&d_1 = d_2 = \left\{M_1(m^0)^{-1}\varepsilon \ln \delta^{-1};\ h_{(1)}\right\}^e, \quad \text{if } j = 0; \qquad (5.11d)\\
&d_1 = \left\{h_{(1)} \ln \delta^{-1} \ln^{-1}(1 + m^0 \varepsilon^{-1} h_{(1)});\ h_{(1)}\right\}^e,\\
&\quad d_2 = d_3 = \left\{M_1(m^0)^{-1}\varepsilon \ln \delta^{-1};\ h_{(2)}\right\}^e, \quad \text{if } j = 1;\\
&d_1 = h_{(1)}, \cdots,\ d_k = h_{(k)}, \quad k \le j - 1,\\
&\quad d_k = \left\{h_{(k)} \ln \delta^{-1} \ln^{-1}(1 + m^0 \varepsilon^{-1} h_{(k)});\ h_{(k)}\right\}^e, \quad k = j,\\
&\quad d_k = d_{k+1} = \left\{M_1(m^0)^{-1}\varepsilon \ln \delta^{-1};\ h_{(k)}\right\}^e, \quad k = j + 1, \quad \text{if } j \ge 2.
\end{aligned}$$

Here $h_{(i)} = d_{i-1} N^{-1}$, $1 \le i \le j+1$, $d_0 = d_{(2.1)}$, $h_{(1)} = h_{1(4.1)}$, $m^0 = m^0_{(5.1)}$, and $M_1 = M_{1(5.9)}$.

The relations (5.11 b, c, d) prescribe the values d_k depending on the values δ, ε, $h_{(k)}$ and on the ratio between j and k for $k \le K,\ K = j + 2$.

In that case when

$$K > j + 2, \quad j \ge -1, 5 \tag{5.11e}$$

we set

$$d_k = d_{k(5.11d)} \ \text{ for } \ k \le j + 2, \ \ d_k = d_{j+2(5.11d)} \ \text{ for } \ j + 2 < k \le K, \ \ j \ge -1; \tag{5.11f}$$

here $K > K_{(5.11b)}(j)$. But if

$$K \le j + 1, \quad K \ge 1, \quad j \ge 0, \tag{5.11g}$$

then we assume that

$$d_k = d_{k(5.11d)} \quad \text{for } \ 1 \le k \le K; \tag{5.11h}$$

here $K < K_{(5.11b)}(j)$.

Thus, for the parameter ε chosen in one of the intervals in (5.11a) and for given K, formulas (5.11) depending on the relation between K and $j = j_{(5.11a)}$ give us the set of the values $d_k = d_k(\delta;\ \varepsilon,\ h_{(k)})$.

As follows from the formulas (5.11b–h), the values d_k, by virtue of the relation $h_{(k)} = d_{k-1}N^{-1}$, are defined only by the parameters j, k and δ, ε, N; we have

$$d_k = d_{k(5.11)}(\delta;\ \varepsilon, N) = d_k^j(\delta;\ \varepsilon, N), \quad 1 \le k \le K, \quad j \ge -1. \tag{5.11i}$$

The difference scheme (3.2), (4.1), (5.11) is the scheme on *a priori* adapted meshes refined sequentially in a neighbourhood of the boundary layer. With the choice of the values d_k, as the indicator we use the majorant function of the discrete boundary layer controlled by the parameters δ, ε, h, taking into account that the parameter ε belongs to the prescribed intervals from (5.11a); $\varepsilon \in (0, 1]$.

5.4. Some estimates for the width η

Under the above choice of the values $d_{k(5.11)}$, taking into account the explicit form of the width of the discrete boundary layer $\eta(\delta;\, \varepsilon, h)$, we find the estimates

$$\begin{aligned} &\eta(\delta;\varepsilon,h_{(1)}) \ge m \quad \text{for} \ \ \varepsilon \in \left[\, \varepsilon^{(-1)},\, 1\, \right]; \\ &\eta(\delta;\varepsilon,h_{(k)}) \le d_k, \quad 1 \le k \le K, \\ &\eta(\delta;\varepsilon,h_{(k)}) \ge m\, d_k, \quad j+1 \le k \le K \quad \text{for} \ \ \varepsilon \in \left[\varepsilon^{(j)},\, \varepsilon^{(j-1)}\right), \quad j \ge 0, \end{aligned} \tag{5.12}$$

where $h_{(k)} = d_{k-1}N^{-1}$. The smallest mesh size attained in this process is not less than dN^{-K}.

Lemma 5.1. *In the case of the difference scheme* (3.2), (4.1), (5.11)*, the estimates* (5.12) *hold for the values* $\eta(\delta;\varepsilon,h_{(k)})$ *and* $d_{k(5.11\mathrm{i})}$.

Lemma 5.2. *In the case of the difference scheme* (3.2), (4.1), (5.6)*, the values* $d_{k(5.6\mathrm{a})}$ *and* $d^{\,j}_{k(5.11\mathrm{i})}$ *satisfy the estimate*

$$d_{k\,(3.2,\,4.1,\,5.6\mathrm{a})} \le d^{\,j}_{k\,(3.2,\,4.1,\,5.11\mathrm{i})}\,, \quad 1 \le k \le K, \tag{5.13}$$

where $j = j_{(5.11\mathrm{a})}$ *defines the interval in* (5.11a) *to which the parameter* ε *belongs.*

6. The convergence of the scheme on *a priori* adapted mesh

We consider the difference scheme (3.2), (4.1), (5.11) assuming the following condition to be fulfilled:

$$\delta = N^{-1}. \tag{6.1}$$

6.1. Estimates of solutions on subdomains

Let $z_{[k]}(x,t)$, $(x,t) \in \overline{G}_{(k)h}$, be a solution of the difference scheme (3.2), (4.1c) approximating the boundary value problem

$$L\,u(x,t) = f(x,t), \ \ (x,t) \in G_{(k)}, \quad u(x,t) = \varphi(x,t), \ \ (x,t) \in S_{(k)}, \tag{6.2}$$

where $\overline{G}_{(k)} = \overline{G}_{(k)(4.1\mathrm{b})}$, $\overline{G}_{(k)h} = \overline{G}_{(k)h(4.1\mathrm{c})}$, $k \ge 1$. For the solution $z_{[k]}(x,t)$, we have the error bound

$$\begin{aligned} &|u(x,t) - z_{[k]}(x,t)| \\ &\le \begin{cases} M\left[h_{(1)}(\varepsilon + h_{(1)})^{-1} + N^{-1} + N_0^{-1}\right], & k \ge 1, \ \ j = -1, 0; \\ M\left[h_{(k)}(\varepsilon + h_{(k)})^{-1} + N^{-1} + N_0^{-1}\right], & k,\, j \ge 1; \quad (x,t) \in \overline{G}_{(k)h}, \end{cases} \end{aligned} \tag{6.3}$$

where the parameter ε belongs to one of the intervals in (5.11a), and

$$h_{(k)} = h_{(k)}(j) = h_{(k)}(j;N) \le \begin{cases} MN^{-1} & \text{for } j=-1,0, \ k \ge 1; \\ MN^{-j-1}\ln N, & j \le k-1, \\ MN^{-k}, & k \le j \quad \text{for } j \ge 1, \ k \ge 1. \end{cases}$$

For $k \ge j+2$, the function $z_{[k]}(x,t)$ satisfies the estimate

$$|u(x,t) - z_{[k]}(x,t)| \le M\left[N^{-1}\ln N + N_0^{-1}\right], \ (x,t) \in \overline{G}_{(k)h}, \ k \ge j+2, \ j \ge -1. \quad (6.4)$$

Outside the σ_k^j-neighbourhood of the boundary S_1^L, the following estimate holds for $z_{[k]}(x,t)$:

$$|u(x,t) - z_{[k]}(x,t)| \le M\left[N^{-1} + N_0^{-1}\right], \ (x,t) \in \overline{G}_{(k)h}, \ r(x,\Gamma_1) \ge \sigma_k^j, \ k \ge 1, \ j \ge 0, \quad (6.5)$$

where

$$\sigma_k^j = d_k^j, \ 1 \le k \le j+1; \quad \sigma_k^j = d_{j+2}^j, \ k \ge j+2; \quad d_k^j = d_{k(5.11i)}^j.$$

Lemma 6.1. *Let the hypothesis of Theorem* 3.1 *be fulfilled. Then the function* $z_{[k]}(x,t)$*,* $(x,t) \in \overline{G}_{(k)h(4.1c)}$*, i.e., the solution of the difference scheme* (3.2), (4.1c) *approximating the boundary value problem* (6.2)*, satisfies the estimates* (6.3)-(6.5)*.*

Remark 6.1. The interpolant $\overline{z}_{[k]}(x,t)$ constructed on $\overline{G}_{(k)}$ using the function $z_{[k]}(x,t)$, under the assumption of Theorem 3.1, satisfies the estimates (6.3)-(6.5), where $z_{[k]}(x,t)$ and $\overline{G}_{(k)h}$ are $\overline{z}_{[k]}(x,t)$ and $\overline{G}_{(k)}$ respectively.

6.2. Main convergence results

We now consider the difference scheme (3.2), (4.1), (5.11), (6.1).

Taking into account estimates (6.3)-(6.5), for the solution of the difference scheme (3.2), (4.1), (5.11), (6.1) for $\varepsilon \in (0,1]$, we obtain the estimate

$$\begin{aligned} &|u(x,t) - z(x,t)| \\ &\le \left\{ \begin{array}{ll} M\left\{\min\left[\varepsilon^{-1}N^{-1}, 1\right] + N_0^{-1}\right\}, & K=1 \\ M\left\{\min\left[\varepsilon^{-1}N^{-K}\ln N, 1\right] + N^{-1}\ln N + N_0^{-1}\right\}, & K \ge 2 \end{array} \right\}, \\ &\quad (x,t) \in \overline{G}_h, \quad K \ge 1, \quad \varepsilon \in (0,1]. \end{aligned} \quad (6.6)$$

Thus, the difference scheme converges on $\overline{G}_h$ under the condition $\left(N^{-K}\ln N \ll \varepsilon\right)$:

$$\varepsilon^{-1} = o\left(N^K \ln^{-1} N\right) \quad \text{for } K \ge 2, \ \varepsilon \in (0,1].$$

Let the parameter ε satisfies the condition

$$\varepsilon \in \left(0, \varepsilon^{(j)}\right], \ j \ge 2, \quad \varepsilon^{(j)} = \varepsilon^{(j)}_{(5.10)}. \quad (6.7)$$

For the error of the solution of the boundary value problem (2.2), (2.1) outside the σ_K-neighbourhood of the set S_1^L, we obtain the estimate

$$|u(x,t)-z(x,t)| \le M\left[N^{-1}+N_0^{-1}\right], \quad (x,t)\in \overline{G}_h, \;\; r(x,\Gamma_1)\ge \sigma_K, \tag{6.8a}$$

where

$$\sigma_K = d_K^j, \;\; d_K^j = d^j_{K(5.11i)}, \quad 1\le K\le j-1, \;\; j=j_{(6.7)}. \tag{6.8b}$$

The value σ_K satisfies the relation

$$\sigma_K = dN^{-K}. \tag{6.8c}$$

Thus, in the case of the condition (6.7), the solution of the difference scheme converges ε-uniformly with the first order of accuracy in x and t outside the σ_K-neighbourhood of the boundary S_1^L, where σ_K shrinks to zero at the rate $\mathscr{O}(N^{-K})$.

Let the parameter ε belongs to one of the intervals in (5.11a). In this case, depending on the relation between K and j, we obtain the estimate

$$\begin{aligned}&|u(x,t)-z(x,t)|\\ &\le \left\{\begin{array}{ll} \left.\begin{array}{ll} M\left\{\min\left[\varepsilon^{-1}N^{-1},1\right]+N_0^{-1}\right\}, & K=1\\ M\left\{\min\left[\varepsilon^{-1}N^{-K}\ln N,1\right]+N^{-1}\ln N+N_0^{-1}\right\}, & K\ge 2\end{array}\right\}, & K=j+1\\ M\left[N^{-1}\ln N+N_0^{-1}\right], & K\ge j+2\end{array}\right\},\\ &\qquad (x,t)\in\overline{G}_h, \quad K\ge j+1, \quad j\ge -1.\end{aligned} \tag{6.9}$$

Outside the σ_K^j-neighbourhood of the set S_1^L, we have the estimate

$$|u(x,t)-z(x,t)|\le M\left[N^{-1}+N_0^{-1}\right], \;\; (x,t)\in \overline{G}_h, \;\; r(x,\Gamma_1)\ge \sigma_K^j, \;\; K\ge 1, \;\; j\ge 0; \tag{6.10a}$$

the value σ_K^j, where

$$\sigma_K^j = d_K^j, \;\; 1\le K\le j+1; \qquad \sigma_K^j = d_{j+2}^j, \;\; K\ge j+2; \quad j\ge 0, \tag{6.10b}$$

satisfies the estimate

$$\sigma_K^j \le \left\{\begin{array}{ll} M\varepsilon\ln N, & K\ge j+1, \quad j\ge 0,\\ MN^{-K}\ln N, & K=j, \quad j\ge 1,\end{array}\right. \tag{6.10c}$$

and the relation

$$\sigma_K^j = dN^{-K}, \quad K\le j-1, \quad j\ge 2. \tag{6.10d}$$

Thus, in that case when the parameter ε belongs to one of the intervals in (5.11a), the convergence rate of the scheme on the set $\overline{G}_h$, as well as the size of that neighbourhood of the set S_1^L outside which the scheme converges at the rate $\mathscr{O}\left(N^{-1}+N_0^{-1}\right)$, depend essentially on the parameters K and j.

According to the estimate (6.9), it requires K iterations, where $K = j+2$, in order to obtain the solution of the difference scheme (3.2), (4.1), (5.11), (6.1) with the error bound

$$|u(x,t) - z(x,t)| \le M\left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h,$$

provided that the parameter ε belongs to one of the intervals in (5.11a).

By virtue of estimate (6.6), the difference scheme (3.2), (4.1), (5.11), (6.1) converges on $\overline{G}_h$ under the condition $\left(N^{-1} \ll \varepsilon \text{ for } K=1 \text{ and } N^{-K}\ln N \ll \varepsilon \text{ for } K \ge 2\right)$:

$$\varepsilon^{-1} = o(N) \text{ for } K=1 \text{ and } \varepsilon^{-1} = o\left(N^K \ln^{-1} N\right) \text{ for } K \ge 2, \qquad (6.11)$$
$$N \to \infty, \ \varepsilon \in (0,1].$$

In order that the difference scheme be convergent almost ε-uniformly with the convergence defect no greater than the value $\nu_{(2.5)}$, it is sufficient to choose the value K satisfying the condition

$$K > K(\nu), \quad K(\nu) = \nu^{-1}. \qquad (6.12)$$

Thus, the difference scheme (3.2), (4.1), (5.11), (6.1), (6.12) converges almost ε-uniformly, with the convergence defect ν.

Theorem 6.1. *Let the solution of the boundary value problem* (2.2), (2.1) *satisfy the hypothesis of Theorem* 3.1. *Then the difference scheme* (3.2), (4.1), (5.11), (6.1) *converges on* $\overline{G}_h$ *under condition* (6.11); *under condition* (6.12), *the scheme converges almost ε-uniformly with defect ν. The discrete solution satisfies the estimate* (6.6) *and, in the case of conditions* (6.7) *and* (5.11a), *it satisfies the estimates* (6.8) *and* (6.9), (6.10), *respectively.*

In the case of the difference scheme (3.2), (4.1), (5.6), (6.1), the following theorem established taking account of estimate (5.13) holds.

Theorem 6.2. *Let the solution of the boundary value problem* (2.2), (2.1) *satisfy the hypothesis of Theorem* 3.1. *Then the difference scheme* (3.2), (4.1), (5.6), (6.1) *converges on* $\overline{G}_h$ *under condition* (6.11); *under condition* (6.12), *the scheme converges almost ε-uniformly with defect ν. The discrete solution satisfies the estimate* (6.6) *and, in the case of conditions* (6.7) *and* (5.11a), *it satisfies, respectively, the estimates* (6.8) *and* (6.9), (6.10), *where in* (6.8)

$$\sigma_K = d_{K(5.6)}(\delta;\varepsilon,N), \quad \delta = \delta_{(6.1)}, \quad \varepsilon = \varepsilon_{(6.7)},$$

provided that $\varepsilon^{-1}h_{(K)} \ge (m^0)^{-1}N$; *and in* (6.10),

$$\sigma_K^j = d_{K(5.6)}(\delta;\varepsilon,N), \quad \delta = \delta_{(6.1)}, \quad \varepsilon = \varepsilon_{(5.11a)}, \quad j = j_{(5.11a)}.$$

Remark 6.2. Let the hypothesis of Theorem 6.1 (Theorem 6.2) be fulfilled. Then for the interpolants $\overline{z}(x,t)$ constructed on $\overline{G}$ using the functions $z(x,t)$, $(x,t) \in \overline{G}_h$, the estimates of Theorem 6.1 (Theorem 6.2) remain valid, where $z(x,t)$ and $\overline{G}_h$ are $\overline{z}(x,t)$ and $\overline{G}$, respectively.

Acknowledgments This research was supported by the Russian Foundation for Basic Research (grant No. 07-01-00729), the Boole Centre for Research in Informatics at the National University of Ireland, Cork and by the Mathematics Applications Consortium for Science and Industry in Ireland (MACSI) under the Science Foundation Ireland (SFI) Mathematics Initiative.

A. Appendix

In this section, we present estimates for the solution of the boundary value problem and its derivatives (the estimates can be derived analogously to the derivation of similar estimates in [16–18]). The solution of problem (2.2) is represented in the form of decomposition into the sum of functions

$$u(x,t) = U(x,t) + V(x,t), \qquad (x,t) \in \overline{G}, \tag{A.1}$$

where $U(x,t)$ and $V(x,t)$ are the regular and singular components of the solution.

The functions $U(x,t)$, $V(x,t)$ satisfy the estimates

$$\begin{aligned} &\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t) \right| \le M\left[1+\varepsilon^{2-k}\right], \\ &\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V(x,t) \right| \le M\varepsilon^{-k} \exp\left(-m\varepsilon^{-1} r(x,\Gamma_1)\right), \\ &\qquad (x,t) \in \overline{G}, \quad k+2k_0 \le 4, \ \ k \le 3, \end{aligned} \tag{A.2}$$

where m is an arbitrary number in the interval $(0, m_0)$, $m_0 = \min_{\overline{G}} \left[a^{-1}(x,t)\, b(x,t)\right]$; and $r(x,\Gamma_1)$ is the distance between the point x and the left boundary Γ_1 of the set D.

Theorem A.1. *Let the data of the boundary value problem* (2.2)*,* (2.1) *satisfy condition* (2.3)*, the condition* $a, b, c, p, f \in C^{6+\alpha}(\overline{G})$*,* $\varphi \in C^{6+\alpha}(S)$*,* $\alpha > 0$*, and also the condition*

$$\varphi(x,t) = 0, \ \ (x,t) \in S_0; \quad \frac{\partial^{k_0}}{\partial t^{k_0}} \varphi(x,t) = 0, \ \ \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} f(x,t) = 0, \ \ (x,t) \in S^c,$$

where k, $k_0 \le 6$, $S^c = \overline{S}^L \cap S_0$*. Then the components in representation* (*A*.1) *of the solution of the boundary value problem satisfy estimates* (*A*.2)*.*

References

[1] Bakhvalov N S. On the optimization of methods for boundary-value problems with boundary layers. Zh. Vychisl. Mat. Mat. Fiz., 1969, 9: 841–859 (in Russian).

[2] Shishkin G I. Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations. Ural Branch of Russian Acad. Sci., Ekaterinburg, 1992 (in Russian).

[3] Miller J J H, O'Riordan E, Shishkin G I. Fitted Numerical Methods for Singular Perturbation Problems. World Scientific, Singapore, 1996.

[4] Farrell P A, Hegarty A F, Miller J J H, O'Riordan E, Shishkin G I. Robust Computational Techniques for Boundary Layers. CRC Press, Boca Raton, 2000.

[5] Roos H-G, Stynes M, Tobiska L. Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion and Flow Problems. Heidelberg, Springer, 1996.

[6] Liseikin V D. Grid Generation Methods. Springer-Verlag, Berlin, 1999.

[7] Shishkin G I. A posteriori adapted (to the solution gradient) grids in the approximation of singularly perturbed convection-diffusion equations. Vychisl. Tekhnol. (Computational Technologies), 2001, 6: 72–87 (in Russian).

[8] Shishkin G I. Approximation of singularly perturbed reaction-diffusion equations on adaptive meshes. Mat. Model., 2001, 13: 103–118 (in Russian).

[9] Shishkin G I. The use of solutions on embedded grids for the approximation of singularly perturbed parabolic convection-diffusion equations on adapted grids. Comp. Maths. Math. Phys., 2006, 46: 1539–1559.

[10] Hemker P W, Shishkin G I, Shishkina L P. A class of singularly perturbed convection-diffusion problems with a moving interior layer. *A posteriori* adaptive mesh technique. Computational Methods in Applied Mathematics, 2004, 4: 105–127.

[11] Samarskii A A. The Theory of Difference Schemes. Nauka, Moscow, 1989 (in Russian). Translation: Marcel Dekker, New York, 2001.

[12] Samarskii A A, Nikolaev E S. Methods of Solving Grid Equations. Nauka, Moscow, 1978 (in Russian). Translation: Numerical Methods for Grid Equations. I, II. Birkhäuser, Basel, 1989.

[13] Marchuk G I. Methods of Numerical Mathematics. Nauka, Moscow, 1989 (in Russian). Translation: Springer, New York, 1982, second ed. (First ed. in 1975. The original Russian edition "Metody Vychislitel'noi Matematiki" was published in 1973 by Nauka, Novosibirsk).

[14] Marchuk G I, Shaidurov V V. Improving the Accuracy of Finite Difference Schemes. Nauka, Moscow, 1979 (in Russian).

[15] Bakhvalov N S. Numerical Methods. Nauka, Moscow, 1973 (in Russian). Translation: MIR, Moscow, 1977.

[16] Shishkin G I. Robust novel high-order accurate numerical methods for singularly perturbed convection-diffusion problems. Mathematical Modelling and Analysis, 2005, 10: 393–412.

[17] Hemker P W, Shishkin G I, Shishkina L P. ε-uniform schemes with high-order time-accuracy for parabolic singular perturbation problems. IMA J. Numer. Anal., 2000, 20: 99–121.

[18] Hemker P W, Shishkin G I, Shishkina L P. Novel defect-correction high-order, in space and time, accurate schemes for parabolic singularly perturbed convection-diffusion problems. Computational Methods in Applied Mathematics, 2003, 3: 387–404.