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New Reflection Principles for Maxwell's Equations and Their Applications

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Abstract. Some new reflection principles for Maxwell's equations are first established, which are then applied to derive two novel identifiability results in inverse electromagnetic obstacle scattering problems with polyhedral scatterers.

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1. Introduction

The main goal of this work is to establish some new reflection principles for Maxwell's equations when general mixed perfect and imperfect scatterers are involved, and then to apply the reflection principles to the unique determination of the scatterers in inverse electromagnetic scattering problems by either the electric far field patterns or the magnetic far field patterns.

Consider an impenetrable scatterer **D**, which is assumed to be a compact domain in $\mathbb{R}^3$ and may consist of finitely many pairwise disjoint bounded polyhedra. Suppose the incident fields are taken to be the normalized time-harmonic electromagnetic plane waves of the form (cf. [10])

$$\mathbf{E}^{\mathbf{i}}(\mathbf{x}) := \frac{\mathrm{i}}{k}\mathrm{curl}\,\mathrm{curl}\, p\, e^{\mathrm{i}k\mathbf{x}\cdot d} = \mathrm{i}k(d\times p)\times d e^{\mathrm{i}k\mathbf{x}\cdot d}, \tag{1.1}$$

$$\mathbf{H}^{\mathbf{i}}(\mathbf{x}) := \mathrm{curl}\, p\, e^{\mathrm{i}k\mathbf{x}\cdot d} = \mathrm{i}kd\times p e^{\mathrm{i}k\mathbf{x}\cdot d}, \tag{1.2}$$

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where $\mathrm{i}=\sqrt{-1}$, and $p\in\mathbb{R}^3$, $k>0$ and $d\in\mathbb{S}^2:=\{\mathbf{x}\in\mathbb{R}^3; |\mathbf{x}|=1\}$ represent respectively a polarization, a wave number and a direction of propagation. Then the associated forward scattering problem is described by the following time-harmonic Maxwell's equations (see [10]):

$$\operatorname{curl}\mathbf{E}-\mathrm{i}k\mathbf{H}=0,\quad \operatorname{curl}\mathbf{H}+\mathrm{i}k\mathbf{E}=0\quad \text{in}\quad \mathbf{G}:=\mathbb{R}^3\backslash\mathbf{D}, \tag{1.3}$$

$$\lim_{|\mathbf{x}|\to\infty}(\mathbf{H}^{\mathbf{s}}\times\mathbf{x}-|\mathbf{x}|\mathbf{E}^{\mathbf{s}})=0, \tag{1.4}$$

where $\mathbf{E}=(E_1,E_2,E_3)$ and $\mathbf{H}=(H_1,H_2,H_3)$ are respectively the total electric and magnetic fields formed by the incident fields $\mathbf{E}^{\mathbf{i}}(\mathbf{x})$, $\mathbf{H}^{\mathbf{i}}(\mathbf{x})$ and the scattered fields $\mathbf{E}^{\mathbf{s}}(\mathbf{x})$ and $\mathbf{H}^{\mathbf{s}}(\mathbf{x})$:

$$\mathbf{E}(\mathbf{x})=\mathbf{E}^{\mathbf{i}}(\mathbf{x})+\mathbf{E}^{\mathbf{s}}(\mathbf{x}),\quad \mathbf{H}(\mathbf{x})=\mathbf{H}^{\mathbf{i}}(\mathbf{x})+\mathbf{H}^{\mathbf{s}}(\mathbf{x}). \tag{1.5}$$

We shall assume that the boundary $\partial\mathbf{D}$ of the scatterer $\mathbf{D}$ has a Lipschitz dissection, i.e., $\partial\mathbf{D}=\Gamma_D\cup\Sigma\cup\Gamma_I$, where Γ_D and Γ_I are two disjoint and relatively open subsets of $\partial\mathbf{D}$, having Σ as their common boundary (see [21]). Then we shall further complement the system (1.3)-(1.5) with the following general mixed boundary condition:

$$\nu\times\mathbf{E}=0\qquad \text{on}\quad \Gamma_D, \tag{1.6}$$

$$\nu\times\operatorname{curl}\mathbf{E}-\mathrm{i}\lambda(\nu\times\mathbf{E})\times\nu=0\qquad \text{on}\quad \Gamma_I, \tag{1.7}$$

where ν is the unit outward normal to $\partial\mathbf{D}$, and $\lambda\in C^{0,\alpha}(\Gamma_I)$ is a non-negative Hölder continuous function, with $0<\alpha<1$. Scattering problems with the mixed boundary conditions (1.6)-(1.7) are widely encountered in military and engineering applications. For instance, in order to avoid the detection by radar, hostile objects may be partially coated by some special material designed to reduce the radar cross section of the scattered wave. Boundary conditions (1.6)-(1.7) correspond to the case where the perfect conductor $\mathbf{D}$ is partially coated on the part $\mathbf{\Gamma}_I$ of its boundary with a dielectric. We refer to [3], [4] and [5] for the physical relevance and practical implications of the electromagnetic scattering problems in this setting.

It is known that the forward scattering system (1.3)-(1.7) has a unique solution $(\mathbf{E},\mathbf{H})\in H_{loc}(\operatorname{curl};\mathbf{G})\times H_{loc}(\operatorname{curl};\mathbf{G})$ (see [4] and [7]). And the singular behavior of the weak solution occurs only around the corners and edges, that is, $(\mathbf{E},\mathbf{H})$ satisfies (1.3) in the classical sense in any subdomain of $\mathbf{G}$, which does not meet any corner or edge of $\mathbf{D}$ (see [15]). By the regularity of the strong solution for the forward scattering problem (see [9] and [10]), we know that both $\mathbf{E}$ and $\mathbf{H}$ are $C^{0,\alpha}$-continuous up to the regular points, namely, points lying in the interior of the open faces of $\mathbf{D}$. Moreover, $\mathbf{E}$ and $\mathbf{H}$ are analytic in $\mathbf{G}$ and the asymptotic behavior of the scattered fields $\mathbf{E}^{\mathbf{s}}$ and $\mathbf{H}^{\mathbf{s}}$ is governed by (see [10])

$$\mathbf{E}^{\mathbf{s}}(\mathbf{x};\mathbf{D},p,k,d)=\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\left\{\mathbf{E}_\infty(\hat{\mathbf{x}};\mathbf{D},p,k,d)+\mathscr{O}(\frac{1}{|\mathbf{x}|})\right\}\quad \text{as}\quad |\mathbf{x}|\to\infty, \tag{1.8}$$

$$\mathbf{H}^{\mathbf{s}}(\mathbf{x};\mathbf{D},p,k,d)=\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\left\{\mathbf{H}_\infty(\hat{\mathbf{x}};\mathbf{D},p,k,d)+\mathscr{O}(\frac{1}{|\mathbf{x}|})\right\}\quad \text{as}\quad |\mathbf{x}|\to\infty, \tag{1.9}$$

uniformly for all $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^2$. The functions $\mathbf{E}_\infty(\hat{\mathbf{x}})$ and $\mathbf{H}_\infty(\hat{\mathbf{x}})$ in (1.8) and (1.9) are called, respectively, the electric and magnetic far field patterns, and both are analytic on the entire unit sphere $\mathbb{S}^2$. We note that the notation $\mathbf{E}^s(\mathbf{x};\mathbf{D},p,k,d)$ or $\mathbf{E}_\infty(\hat{\mathbf{x}};\mathbf{D},p,k,d)$ used above will be frequently adopted to specify their dependence on the polarization p, the wave number k and the incident direction d.

An important topic which we shall address is the inverse electromagnetic scattering problem, where one intends to determine the scatterer $\mathbf{D}$ by using some measurement data of the electric or magnetic far-field patterns. This inverse problem is fundamental in exploring objects by electromagnetic waves, for which we refer to [10] for a detailed discussion. One of the most important topics in the inverse scattering problem is on the uniqueness, that is, how much far field data can uniquely determine a scatterer $\mathbf{D}$. Mathematically, this can be formulated as follows:

If two scatterers $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$ *produce the same far field data, i.e.,*

$$\mathbf{E}_\infty(\hat{\mathbf{x}};\mathbf{D},p,k,d) = \mathbf{E}_\infty(\hat{\mathbf{x}};\widetilde{\mathbf{D}},p,k,d) \quad \textit{for all } \hat{\mathbf{x}} \in \mathbb{S}^2,$$

does $\mathbf{D}$ *have to be the same as* $\widetilde{\mathbf{D}}$ *?*

The electric far field data above can be replaced by the magnetic far field data. But noting that the discussion about the latter is completely parallel to the one with the former, we shall only focus on the electric far field data throughout this paper. As is widely known, the uniqueness in inverse problems always plays an indispensable role (see [16]). Moreover, the uniqueness for the inverse electromagnetic scattering problem with optimal measurement data has been remaining to be a longstanding open problem (see [6] and [11]). One can easily see that this inverse problem is formally determined with all $\hat{x} \in \mathbb{S}^2$ and fixed $p_0 \in \mathbb{R}^3$, $k_0 > 0$ and $d_0 \in \mathbb{S}^2$, since the far field data depend on the same number of variables, as does the object which is to be recovered. Hence, one may anticipate the uniqueness by using the far field data from only one or at most a few incident waves. Unfortunately, not much has been done on the topic in the literature. This uniqueness in inverse electromagnetic scattering is quite similar to the one in inverse acoustic obstacle scattering, where one utilizes acoustic far field measurements to identify the unknown object. But for the acoustic scattering, significant progress has been made in the past few years on the unique determination of polyhedral type scatterers by means of a single or several incident waves (see, e.g., [2,8,12,20]). The fundamental tools leading to the progress lie on various reflection principles for the Helmholtz equation, as well as suitably devised techniques such as the path argument developed in [20]. Along this line, a novel reflection principle was derived in [19] for time-harmonic Maxwell's equations associated with perfect conductors. This is the first time to derive and justify a reflection principle for the Maxwell system. In combination of this new reflection principle with the path argument from [20], the uniqueness result was established for determining the perfect polyhedral conductors in the inverse electromagnetic scattering by the far field measurements from two incident waves.

This current work is a continuation of our efforts in [19]. We shall first establish some new reflection principles (Theorems 2.1-2.4) for the Maxwell's equations with very general mixed boundary conditions, then apply (Section 3) the principles to derive the uniqueness results (Theorems 3.1 and 3.2) for the inverse electromagnetic obstacle scattering under general physical boundary conditions (1.6)-(1.7).

2. Reflection principles for Maxwell's equations

In this section, we shall establish some reflection principles for Maxwell's equations (1.3) when **D** is a polyhedral scatterer as described in Section 1. For the first time, the reflection principles are rigorously justified to be valid also for Maxwell's system with general mixed physical boundary conditions (1.6)-(1.7). In [19], the reflection principle was derived for Maxwell's equations with the simpler perfect boundary condition. As we shall see, the derivation for the new reflection principles is derived via Maxwell's equations, unlike that in [19] via the vector Helmholtz equation.

We start with a definition of some special planes in **G**. For any two-dimensional plane Π in $\mathbb{R}^3$, we shall use ν_Π and R_Π to denote respectively the unit normal to Π and the reflection with respect to Π in $\mathbb{R}^3$.

Definition 2.1. *Let Π be a two-dimensional plane in $\mathbb{R}^3$ and $\widetilde{\Pi} \subset \Pi$ be an open connected subset. $\widetilde{\Pi}$ is called a* perfect plane *corresponding to the electric field* **E** *in* (1.3)*, if*

$$\nu_\Pi \times \mathbf{E} = 0 \quad on \quad \widetilde{\Pi}; \tag{2.1}$$

while it is called an imperfect plane *corresponding to the electric field* **E** *in* (1.3)*, if*

$$\nu_\Pi \times \operatorname{curl} \mathbf{E} - \mathrm{i}\lambda(\nu_\Pi \times \mathbf{E}) \times \nu_\Pi = 0 \quad on \quad \widetilde{\Pi}, \tag{2.2}$$

where $\lambda \in C^{0,\alpha}(\widetilde{\Pi})$ is some non-negative function.

It is noted that a *perfect* or *imperfect plane* is not necessarily a plane in $\mathbb{R}^3$. Similarly, one can define the perfect and the imperfect planes corresponding to the magnetic field **H** in (1.3).

Now, we are ready to state a general reflection principle.

Theorem 2.1. *For a connected polyhedral domain Ω in $\mathbf{G} := \mathbb{R}^3 \backslash \mathbf{D}$, let $\widetilde{\Pi} \subset \partial\Omega$ be one of its faces and be a perfect plane associated with* **E**. *Furthermore, let Π be the plane in $\mathbb{R}^3$ containing $\widetilde{\Pi}$ and $\Omega \cup R_\Pi \Omega \subset \mathbf{G}$. Suppose that a different face Γ of Ω from $\widetilde{\Pi}$ is an imperfect plane corresponding to* **E**, *namely,*

$$\nu_\Gamma \times \operatorname{curl} \mathbf{E} - \mathrm{i}\lambda(\nu_\Gamma \times \mathbf{E}) \times \nu_\Gamma = 0 \quad on \quad \Gamma, \tag{2.3}$$

where ν_Γ is the unit normal to Γ directed to the interior of Ω. Then $\Gamma' = R_\Pi \Gamma$ is also an imperfect plane corresponding to **E**, *i.e.,*

$$\nu_{\Gamma'} \times \operatorname{curl} \mathbf{E} - \mathrm{i}\eta(\nu_{\Gamma'} \times \mathbf{E}) \times \nu_{\Gamma'} = 0 \quad on \quad \Gamma', \tag{2.4}$$

where $\nu_{\Gamma'}$ is the unit normal to Γ' directed to the interior of $R_\Pi\Omega$ and $\eta(\mathbf{x}) = \lambda(R_\Pi\mathbf{x})$ for $\mathbf{x}\in\Gamma'$.

The proof of Theorem 2.1 will be divided into the following several lemmata.

Lemma 2.1. *Let Ω and $\widetilde{\Pi}$ be the same as stated in Theorem 2.1 and c be a real constant and $\mathbf{C}=(0,0,2c)\in\mathbb{R}^3$. If the face $\widetilde{\Pi}$ lies on the plane Π: $x_3=c$, then it holds that*

$$\mathbf{E}(\mathbf{x})+R_\Pi\mathbf{E}(R_\Pi\mathbf{x})=\mathbf{C} \quad \text{in} \ \ \Omega\cup\widetilde{\Pi}\cup R_\Pi\Omega, \tag{2.5}$$

and

$$\mathbf{H}(\mathbf{x})-R_\Pi\mathbf{H}(R_\Pi\mathbf{x})=\mathbf{C} \quad \text{in} \ \ \Omega\cup\widetilde{\Pi}\cup R_\Pi\Omega. \tag{2.6}$$

Proof. We note that $\nu_\Pi=(0,0,1)$ and $R_\Pi(x_1,x_2,x_3)=(x_1,x_2,2c-x_3)$. Setting

$$\mathbf{V}(\mathbf{x})=\mathbf{E}(\mathbf{x})+R_\Pi\mathbf{E}(R_\Pi\mathbf{x})-\mathbf{C}, \quad \mathbf{W}(\mathbf{x})=\mathbf{H}(\mathbf{x})-R_\Pi\mathbf{H}(R_\Pi\mathbf{x})-\mathbf{C},$$

we can easily check that

$$\begin{aligned}\mathbf{V}(\mathbf{x})=\Big(&E_1(x_1,x_2,x_3)+E_1(x_1,x_2,2c-x_3),E_2(x_1,x_2,x_3)+E_2(x_1,x_2,2c-x_3),\\ &E_3(x_1,x_2,x_3)-E_3(x_1,x_2,2c-x_3)\Big),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\mathbf{W}(\mathbf{x})=\Big(&H_1(x_1,x_2,x_3)-H_1(x_1,x_2,2c-x_3),H_2(x_1,x_2,x_3)-H_2(x_1,x_2,2c-x_3),\\ &H_3(x_1,x_2,x_3)+H_3(x_1,x_2,2c-x_3)\Big).\end{aligned} \tag{2.8}$$

Using these two relations and the fact that ($\mathbf{E}$, $\mathbf{H}$) satisfies (1.3), it is straightforward to verify that

$$\operatorname{curl}\mathbf{V}-ik\mathbf{W}=0, \quad \operatorname{curl}\mathbf{W}+ik\mathbf{V}=0 \quad \text{in} \quad \Omega\cup\widetilde{\Pi}\cup R_\Pi\Omega. \tag{2.9}$$

Noting that $\widetilde{\Pi}$ is a perfect plane, we see from condition (2.1) that $E_1=E_2=0$ on $\widetilde{\Pi}$, thus

$$V_1=V_2=0 \quad \text{on} \quad \widetilde{\Pi}, \tag{2.10}$$

which implies

$$\frac{\partial V_l}{\partial x_l}=0 \quad \text{on} \quad \widetilde{\Pi} \quad \text{for} \quad l=1,2.$$

Next, by the definition of $\mathbf{V}$ we know

$$\partial_3V_1=\partial_3V_2=0 \quad \text{and} \quad \partial_1V_3=\partial_2V_3=0 \quad \text{on} \quad \widetilde{\Pi}.$$

Then by direct calculations we obtain

$$(\operatorname{curl}\mathbf{V})_1=(\operatorname{curl}\mathbf{V})_2=0 \quad \text{on} \quad \widetilde{\Pi}, \tag{2.11}$$

where $(\mathrm{curl}\,\mathbf{V})_1$ and $(\mathrm{curl}\,\mathbf{V})_2$ are, respectively, the first and second Cartesian components of curl $\mathbf{V}$. Clearly, (2.10) implies that $\nu_\Pi \times \mathbf{V} = 0$ on $\widetilde{\Pi}$ while (2.11) implies that $\nu_\Pi \times \mathbf{W} = 0$ on $\widetilde{\Pi}$. Then, by the unique continuation (see Lemma 3.2, [1]), we have $\mathbf{V} = \mathbf{W} = 0$ in $\Omega \cup \widetilde{\Pi} \cup R_\Pi \Omega$. The proof is completed. □

Now, we can show that Theorem 2.1 holds in a special case.

Lemma 2.2. *Theorem* 2.1 *holds when* $\widetilde{\Pi}$ *is contained in the plane* Π*:* $x_3 = c$.

Proof. Let $\nu_\Gamma = (a_1, a_2, a_2) := \mathbf{a}$. Then $\nu_{\Gamma'} = (a_1, a_2, -a_3) := \mathbf{a}'$. By straightforward calculations, we obtain from (2.3) that for any $\mathbf{x} = (x_1, x_2, x_3) \in \Gamma$,

$$\begin{aligned} 0 =& \nu_\Gamma \times \mathrm{curl}\,\mathbf{E}(\mathbf{x}) - \mathrm{i}\lambda(\mathbf{x})\Big(\nu_\Gamma \times \mathbf{E}(\mathbf{x})\Big) \times \nu_\Gamma \\ =& \mathrm{i}k \begin{bmatrix} a_2H_3 - a_3H_2 \\ a_3H_1 - a_1H_3 \\ a_1H_2 - a_2H_1 \end{bmatrix} (\mathbf{x}) - \mathrm{i}\lambda(\mathbf{x}) \begin{bmatrix} E_1 - (\mathbf{E}\cdot\mathbf{a})a_1 \\ E_2 - (\mathbf{E}\cdot\mathbf{a})a_2 \\ E_3 - (\mathbf{E}\cdot\mathbf{a})a_3 \end{bmatrix} (\mathbf{x}). \end{aligned} \tag{2.12}$$

From the proof of Lemma 2.1, we know that E_1 and E_2 are odd symmetric with respect to Π and E_3 is even symmetric with respect to Π, whereas H_1 and H_2 are even symmetric with respect to Π and H_3 is odd symmetric with respect to Π. Then a direct algebraic manipulation along with (2.12) further gives

$$\begin{aligned} &\mathrm{i}k \begin{bmatrix} -a_2H_3 - a_3H_2 \\ a_3H_1 + a_1H_3 \\ a_1H_2 - a_2H_1 \end{bmatrix} (x_1, x_2, 2c - x_3) \\ &\quad - \mathrm{i}\lambda(\mathbf{x}) \begin{bmatrix} -E_1 + (\mathbf{E}\cdot\mathbf{a}')a_1 \\ -E_2 + (\mathbf{E}\cdot\mathbf{a}')a_2 \\ E_3 + (\mathbf{E}\cdot\mathbf{a}')a_3 \end{bmatrix} (x_1, x_2, 2c - x_3) = 0 \end{aligned} \tag{2.13}$$

for $\mathbf{x} \in \Gamma$. Noting that $(x_1, x_2, 2c - x_3) \in \Gamma'$ for $\mathbf{x} \in \Gamma$, we can reformulate (2.13) to give

$$\mathrm{i}k\nu_{\Gamma'} \times \mathbf{H}(x_1, x_2, 2c - x_3) - \mathrm{i}\lambda(x_1, x_2, x_3)\Big(\nu_{\Gamma'} \times \mathbf{E}(x_1, x_2, 2c - x_3)\Big) \times \nu_{\Gamma'} = 0$$

for $\mathbf{x} \in \Gamma$, which is actually (2.4). □

In order to prove Theorem 2.1 in the general case, we need some further auxiliary results in the following Lemmata 2.3 and 2.4. Henceforth, we shall use $U = (u_{kl})_{1\le k,l\le 3}$ to denote a rotation matrix in $\mathbb{R}^3$.

Lemma 2.3. *For arbitrary constant vectors* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$*, we have*

$$U(\mathbf{a} \times \mathbf{b}) = U\mathbf{a} \times U\mathbf{b}. \tag{2.14}$$

Proof. This is clear from the geometric interpretation of vector product. □

Lemma 2.4. *Using the transformation* $\mathbf{y} = U^{-1}\mathbf{x}$ *for any* $\mathbf{x} \in \mathbb{R}^3$*, or* $\mathbf{x} = U\mathbf{y}$ *for any* $\mathbf{y} \in \mathbb{R}^3$*, we define*

$$\mathbf{J}(\mathbf{y}) = U^{-1}\mathbf{E}(\mathbf{x}) = U^{-1}\mathbf{E}(U\mathbf{y}) \quad \forall \mathbf{y} \in U^{-1}\mathbf{G}, \tag{2.15}$$

$$\mathbf{K}(\mathbf{y}) = U^{-1}\mathbf{H}(\mathbf{x}) = U^{-1}\mathbf{H}(U\mathbf{y}) \quad \forall \mathbf{y} \in U^{-1}\mathbf{G}. \tag{2.16}$$

If $(\mathbf{E}(\mathbf{x}), \mathbf{H}(\mathbf{x}))$ *satisfies the Maxwell's equations* (1.3) *in* $\mathbf{G}$*, then* $(\mathbf{J}(\mathbf{y}), \mathbf{K}(\mathbf{y}))$ *satisfies the Maxwell's equations in* $U^{-1}\mathbf{G}$*, namely*

$$\mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y}) - \mathrm{i}k\mathbf{K}(\mathbf{y}) = 0, \quad \mathrm{curl}_{\mathbf{y}}\mathbf{K}(\mathbf{y}) + \mathrm{i}k\mathbf{J}(\mathbf{y}) = 0 \quad \textit{in} \quad U^{-1}\mathbf{G}. \tag{2.17}$$

Proof. By a coordinate transformation and Lemma 2.3, one can easily find that

$$\nabla_{\mathbf{y}} \times \mathbf{J}(\mathbf{y}) = U^T \nabla_{\mathbf{x}} \times U^T \mathbf{E}(\mathbf{x}) = U^T(\nabla_{\mathbf{x}} \times \mathbf{E}(\mathbf{x})). \tag{2.18}$$

That is, $\mathrm{curl}_{\mathbf{x}}\mathbf{E}(\mathbf{x}) = U\mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y})$. Similarly, $\mathrm{curl}_{\mathbf{x}}\mathbf{H}(\mathbf{x}) = U\mathrm{curl}_{\mathbf{y}}\mathbf{K}(\mathbf{y})$. Then using (1.3) and (2.15)-(2.16), we have

$$U\mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y}) - \mathrm{i}kU\mathbf{K}(\mathbf{y}) = 0, \quad U\mathrm{curl}_{\mathbf{y}}\mathbf{K}(\mathbf{y}) + \mathrm{i}kU\mathbf{J}(\mathbf{y}) = 0 \quad \text{for} \;\; \mathbf{y} \in U^{-1}\mathbf{G},$$

which exactly implies (2.17). □

Now, we are in a position to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Let U be a rotation matrix such that $U^{-1}\Pi = \{x_3 = c\}$ with a constant c; namely, $\mathbf{y} = U^{-1}\mathbf{x} \in \{y_3 = c\}$ for $\mathbf{x} \in \widetilde{\Pi}$. Set

$$\mathbf{J}(\mathbf{y}) = U^{-1}\mathbf{E}(U\mathbf{y}), \quad \mathbf{K}(\mathbf{y}) = U^{-1}\mathbf{H}(U\mathbf{y}).$$

Then it follows from Lemma 2.4 that

$$\mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y}) - \mathrm{i}k\mathbf{K}(\mathbf{y}) = 0, \quad \mathrm{curl}_{\mathbf{y}}\mathbf{K}(\mathbf{y}) + \mathrm{i}k\mathbf{J}(\mathbf{y}) = 0 \quad \text{for} \;\; \mathbf{y} \in U^{-1}\Omega. \tag{2.19}$$

Noting that $\widetilde{\Pi}$ is a perfect plane, we have

$$\nu_\Pi \times U\mathbf{J}(\mathbf{y}) = 0 \quad \text{for} \;\; \mathbf{y} \in U^{-1}\widetilde{\Pi}, \tag{2.20}$$

which along with Lemma 2.3 further leads to

$$U^{-1}\nu_\Pi \times \mathbf{J}(\mathbf{y}) = 0 \quad \text{for} \;\; \mathbf{y} \in U^{-1}\Pi. \tag{2.21}$$

On the other hand, by noting that Γ is an imperfect plane, we have

$$\nu_\Gamma \times \mathrm{curl}_{\mathbf{x}} U\mathbf{J}(\mathbf{y}) - \mathrm{i}\lambda(U\mathbf{y})\Big(\nu_\Gamma \times U\mathbf{J}(\mathbf{y})\Big) \times \nu_\Gamma = 0 \quad \text{for } \mathbf{y} \in U^{-1}\Gamma. \tag{2.22}$$

Equations (2.14), (2.18) and (2.22) give

$$\begin{aligned}
0 =& \nu_\Gamma \times \mathrm{curl}_{\mathbf{x}} U\mathbf{J}(\mathbf{y}) - \mathrm{i}\lambda(U\mathbf{y})\Big(\nu_\Gamma \times U\mathbf{J}(\mathbf{y})\Big) \times \nu_\Gamma \\
=& UU^{-1}\nu_\Gamma \times \mathrm{curl}_{\mathbf{x}} U\mathbf{J}(\mathbf{y}) - \mathrm{i}\lambda(U\mathbf{y})\Big(UU^{-1}\nu_\Gamma \times U\mathbf{J}(\mathbf{y})\Big) \times UU^{-1}\nu_\Gamma \\
=& U\Big(U^{-1}\nu_\Gamma \times \mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y})\Big) - \mathrm{i}\lambda(U\mathbf{y})U\Big[\Big(U^{-1}\nu_\Gamma \times \mathbf{J}(\mathbf{y})\Big) \times U^{-1}\nu_\Gamma\Big]
\end{aligned}$$

for all $\mathbf{y} \in U^{-1}\Gamma$, that is,

$$U^{-1}\nu_\Gamma \times \mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y}) - \mathrm{i}\lambda(U\mathbf{y})\Big(U^{-1}\nu_\Gamma \times \mathbf{J}(\mathbf{y})\Big) \times U^{-1}\nu_\Gamma = 0 \quad \text{for } \mathbf{y} \in U^{-1}\Gamma. \tag{2.23}$$

Noting that $U^{-1}\widetilde{\Pi} \subset \{x_3 = c\}$, by (2.21) and (2.23), we can apply Lemma 2.2 to obtain for any $\mathbf{y} \in (U^{-1}\Gamma)'$ that

$$(U^{-1}\nu_\Gamma)' \times \mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y}) - \mathrm{i}\lambda(U\mathbf{y}')\Big((U^{-1}\nu_\Gamma)' \times \mathbf{J}(\mathbf{y})\Big) \times (U^{-1}\nu_\Gamma)' = 0, \tag{2.24}$$

where $(U^{-1}\Gamma)'$, $(U^{-1}\nu_\Gamma)'$ and $\mathbf{y}'$ are respectively, the reflections of $U^{-1}\Gamma$, $U^{-1}\nu_\Gamma$ and $\mathbf{y}$ with respect to $U^{-1}\Pi = \{y_3 = c\}$. Noting that

$$(U^{-1}\Gamma)' = U^{-1}\Gamma', \quad (U^{-1}\nu_\Gamma)' = U^{-1}\nu_{\Gamma'},$$

we derive from (2.24) that for any $\mathbf{y} \in U^{-1}\Gamma'$,

$$U^{-1}\nu_{\Gamma'} \times \mathrm{curl}_{\mathbf{y}}\mathbf{J}(\mathbf{y}) - \mathrm{i}\lambda(U\mathbf{y}')\Big((U^{-1}\nu_{\Gamma'}) \times \mathbf{J}(\mathbf{y})\Big) \times (U^{-1}\nu_{\Gamma'}) = 0. \tag{2.25}$$

Multiplying the both sides of (2.25) by U and using relations in (2.14) and (2.18) again, we finally come to

$$\nu_{\Gamma'} \times \mathrm{curl}_{\mathbf{x}}\mathbf{E}(\mathbf{x}) - \mathrm{i}\lambda(R_\Pi\mathbf{x})\Big(\nu_{\Gamma'} \times \mathbf{E}(\mathbf{x})\Big) \times \nu_{\Gamma'} = 0 \quad \text{for } \mathbf{x} = U\mathbf{y} \in \Gamma'. \tag{2.26}$$

This completes the proof of Theorem 2.1. □

Remark 2.1. One can easily see from the proof of Theorem 2.1 that Theorem 2.1 holds also for Γ which may not be a face of Ω. In fact, it has been shown that if $\mathbf{x} \in \Gamma$ is an interior point such that (2.3) holds, then (2.4) holds for $\mathbf{x}' := R_\Pi\mathbf{x} \in \Gamma'$. Hence, Theorem 2.1 also holds when Γ is a subset of one (open) face of Ω.

Setting $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) := (-\mathbf{H}, \mathbf{E})$, one can easily see that $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ satisfy the Maxwell's equations. Therefore we have the following reflection principle on the magnetic field $\mathbf{H}$ which is the counterpart to Theorem 2.1.

Theorem 2.2. *For a connected polyhedral domain Ω in $\mathbf{G} := \mathbb{R}^3\backslash\mathbf{D}$, let $\widetilde{\Pi} \subset \partial\Omega$ be one of its faces and be a perfect plane associated with $\mathbf{H}$. Furthermore, let Π be the plane in $\mathbb{R}^3$ containing $\widetilde{\Pi}$ and $\Omega \cup R_\Pi\Omega \subset \mathbf{G}$. Suppose that Γ is a subset of one face of Ω other than $\widetilde{\Pi}$, and the following condition holds,*

$$\nu_\Gamma \times \mathrm{curl}\,\mathbf{H} - \mathrm{i}\lambda(\nu_\Gamma \times \mathbf{H}) \times \nu_\Gamma = 0 \quad \textit{on} \quad \Gamma, \tag{2.27}$$

where ν_Γ is the unit normal to Γ directed to the interior of Ω. Then we have

$$\nu_{\Gamma'} \times \mathrm{curl}\,\mathbf{H} - \mathrm{i}\eta(\nu_{\Gamma'} \times \mathbf{H}) \times \nu_{\Gamma'} = 0 \quad \textit{on} \quad \Gamma', \tag{2.28}$$

where $\Gamma' = R_\Pi\Gamma$, $\nu_{\Gamma'}$ is the unit normal to Γ' directed to the interior of $R_\Pi\Omega$ and $\eta(\mathbf{x}) = \lambda(R_\Pi\mathbf{x})$ for $\mathbf{x} \in \Gamma'$.

In order to prove our next reflection principle, we need the following theorem, which was given in [19].

Theorem 2.3. *For a polyhedral domain* Ω *in* $\mathbf{G} := \mathbb{R}^3 \backslash \mathbf{D}$*, let* $\widetilde{\Pi} \subset \partial\Omega$ *be one of its faces and be a perfect plane associated with* $\mathbf{E}$ *in* (1.3)*. Furthermore, let* Π *be the plane in* $\mathbb{R}^3$ *containing* $\widetilde{\Pi}$ *and* $\Omega \cup R_{\Pi}\Omega \subset \mathbf{G}$*. Suppose that a different face* Γ *of* Ω *from* $\widetilde{\Pi}$ *is another perfect plane associated with* $\mathbf{E}$*. Then the reflection* Γ' *of* Γ *with respect to* Π *is also a perfect plane associated with* $\mathbf{E}$*.*

Similarly to Remark 2.1, we would like to emphasize that Theorem 2.3 still holds when Γ is not necessarily a face of Ω but only a subset of one face of Ω.

Theorem 2.4. *For a connected polyhedral domain* Ω *in* $\mathbf{G} := \mathbb{R}^3 \backslash \mathbf{D}$*, let* $\widetilde{\Pi} \subset \partial\Omega$ *be one of its faces and be a perfect plane associated with* $\mathbf{H}$*. Furthermore, let* Π *be the plane in* $\mathbb{R}^3$ *containing* $\widetilde{\Pi}$ *and* $\Omega \cup R_{\Pi}\Omega \subset \mathbf{G}$*. Suppose that* $\Gamma \subset \partial\Omega$ *is a subset of one face of* Ω *other than* $\widetilde{\Pi}$*, and the following condition holds,*

$$\nu_\Gamma \times \operatorname{curl} \mathbf{E} - \mathrm{i}\lambda(\nu_\Gamma \times \mathbf{E}) \times \nu_\Gamma = 0 \quad on \quad \Gamma, \tag{2.29}$$

where ν_Γ *is the unit normal to* Γ *directed to the interior of* Ω *and* $\lambda \in C^{0,\alpha}(\Gamma)$ *is non-negative. Then we have*

$$\nu_{\Gamma'} \times \operatorname{curl} \mathbf{E} - \mathrm{i}\eta(\nu_{\Gamma'} \times \mathbf{E}) \times \nu_{\Gamma'} = 0 \quad on \quad \Gamma', \tag{2.30}$$

where $\Gamma' = R_\Pi\Gamma$*,* $\nu_{\Gamma'}$ *is the unit normal to* Γ' *directed to the interior of* $R_\Pi\Omega$ *and* $\eta(\mathbf{x}) = \lambda(R_\Pi\mathbf{x})$ *for* $\mathbf{x} \in \Gamma'$*.*

Proof. We first consider the two cases where $\lambda \equiv 0$ on Γ and $\lambda > 0$ on Γ.

For the first case when $\lambda \equiv 0$ on Γ, i.e., $\nu_\Gamma \times \mathbf{H} = 0$ on Γ, the theorem follows directly from Theorem 2.3 by setting $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}) := (-\mathbf{H}, \mathbf{E})$.

Now, we treat the second case where λ is strictly positive on Γ. By Maxwell's equations (1.3), we can reformulate condition (2.29) as

$$(\nu_\Gamma \times \operatorname{curl} \mathbf{H}) \times \nu_\Gamma + \mathrm{i}\frac{k^2}{\lambda}(\nu_\Gamma \times \mathbf{H}) = 0 \quad \text{on} \quad \Gamma, \tag{2.31}$$

which implies that

$$\Big[(\nu_\Gamma \times \operatorname{curl} \mathbf{H}) \times \nu_\Gamma\Big] \times \nu_\Gamma + \mathrm{i}\frac{k^2}{\lambda}(\nu_\Gamma \times \mathbf{H}) \times \nu_\Gamma = 0 \quad \text{on} \quad \Gamma, \tag{2.32}$$

and by direct calculations we have

$$\nu_\Gamma \times \operatorname{curl} \mathbf{H} - \mathrm{i}\frac{k^2}{\lambda}(\nu_\Gamma \times \mathbf{H}) \times \nu_\Gamma = 0 \quad \text{on} \quad \Gamma. \tag{2.33}$$

Now, noting $\nu_\Pi \times \mathbf{H} = 0$ on $\widetilde{\Pi}$ and (2.33), we can apply the reflection principle in Theorem 2.2 to obtain that

$$\nu_{\Gamma'} \times \operatorname{curl} \mathbf{H} - \mathrm{i}\frac{k^2}{\eta}(\nu_{\Gamma'} \times \mathbf{H}) \times \nu_{\Gamma'} = 0 \quad \text{on} \quad \Gamma'. \tag{2.34}$$

By an argument similar to deriving (2.33), one can show that (2.34) implies (2.30).

Finally, in the case where

$$\Gamma = \{\mathbf{x} \in \Gamma; \lambda(x) > 0\} \cup \{\mathbf{x} \in \Gamma; \lambda(x) = 0\},$$

the theorem follows by combining the above results for two cases. □

3. Uniqueness for inverse electromagnetic obstacle scattering

In this section, we will present some uniqueness results for determining a polyhedral scatterer $\mathbf{D}$ in $\mathbb{R}^3$ as described in Section 1. We start with the introduction of some notations. For any fixed $k_0 > 0$, $d_0 \in \mathbb{S}^2$ and two polarizations p_1, p_2 such that p_1, p_2, d_0 are linearly independent, we shall write

$$\mathscr{E}(\mathbf{x}; \mathbf{D}) = \Big\{\mathbf{E}(\mathbf{x}; \mathbf{D}, p_1, k_0, d_0), \mathbf{E}(\mathbf{x}; \mathbf{D}, p_2, k_0, d_0)\Big\}, \tag{3.1}$$

$$\mathscr{H}(\mathbf{x}; \mathbf{D}) = \Big\{\mathbf{H}(\mathbf{x}; \mathbf{D}, p_1, k_0, d_0), \mathbf{H}(\mathbf{x}; \mathbf{D}, p_2, k_0, d_0)\Big\}, \tag{3.2}$$

where $\mathbf{E}(\mathbf{x}; \mathbf{D}, p_l, k_0, d_0)$ and $\mathbf{H}(\mathbf{x}; \mathbf{D}, p_l, k_0, d_0)$, $l = 1, 2$, are respectively the total electric and magnetic fields corresponding to the scatterer $\mathbf{D}$. Starting from now on, the operations on $\mathscr{E}(\mathbf{x}; \mathbf{D})$ and $\mathscr{H}(\mathbf{x}; \mathbf{D})$ are always understood to be elementwise. An open ball in $\mathbb{R}^3$ with center $\mathbf{x}$ and radius r will be written as $B_r(\mathbf{x})$, the closure of $B_r(\mathbf{x})$ as $\bar{B}_r(\mathbf{x})$ and the boundary of $B_r(\mathbf{x})$ as $S_r(\mathbf{x})$. Now we define

Definition 3.1. *$\mathscr{P}_{\mathscr{E}}$ is called the perfect set of $\mathscr{E}$ in $\mathbf{G}$ if*

$$\mathscr{P}_{\mathscr{E}} = \Big\{x \in \mathbf{G};\ \exists \text{ a perfect plane } \widetilde{\Pi} \text{ associated with } \mathscr{E} \text{ passing through } x\Big\}.$$

Similarly, the perfect set $\mathscr{P}_{\mathscr{H}}$ of $\mathscr{H}$ in $\mathbf{G}$, is defined.

Next, we shall recall some crucial properties of perfect sets and perfect planes. Since the Cartesian components of $\mathscr{E}$ and $\mathscr{H}$ are analytic in $\mathbf{G}$, by analytic continuation, we know that for any perfect plane $\widetilde{\Pi}$, its maximally connected extension is still a perfect plane. That is, let Π be the plane in $\mathbb{R}^3$ containing $\widetilde{\Pi}$. Then the open connected component of $\Pi\backslash\mathbf{D}$ which contains $\widetilde{\Pi}$ is also a perfect plane. Hence from now on, without loss of generality, we will always assume that a perfect plane is meant to have been maximally connectedly extended in $\mathbf{G}$. Moreover, for an integer l, by Π_l we always denote a plane in $\mathbb{R}^3$ which contains some perfect plane $\widetilde{\Pi}_l$. Finally, both perfect sets with respect to $\mathscr{E}$ or $\mathscr{H}$ and perfect planes are bounded (see Lemma 3.2 in [19]).

Now, we are in a position to present our main uniqueness results. First, we consider the determination of a polyhedral scatterer $\mathbf{D}$ associated with the mixed boundary conditions (1.6)-(1.7) with the surface impedance λ which is identically zero; namely

$$\nu \times \mathbf{E} = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \nu \times \mathbf{H} = 0 \quad \text{on} \quad \Gamma_N, \tag{3.3}$$

where Γ_D and Γ_N form a Lipschitz dissection of $\partial\mathbf{D}$. Such a boundary value problem often arises for example in the semiconductor modeling (see [17] and [18]), where Γ_D is the electric contact while Γ_N is the insulating part. In the following, we shall write $\mathfrak{B}[\mathbf{E},\mathbf{H}]=0$ to denote the boundary condition of type (3.3).

Theorem 3.1. *Let $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ be two polyhedral scatterers with respective boundary conditions $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$. For any fixed $k_0>0$ and $d_0\in\mathbb{S}^2$, and two different polarizations p_1 and p_2 such that p_1,p_2,d_0 are linearly independent, we have $\mathbf{D}=\widetilde{\mathbf{D}}$ and $\mathfrak{B}=\widetilde{\mathfrak{B}}$ as long as $\mathscr{E}_\infty(\hat{\mathbf{x}};\mathbf{D})=\mathscr{E}_\infty(\hat{\mathbf{x}};\widetilde{\mathbf{D}})$ for $\hat{\mathbf{x}}\in\mathbb{S}^2$.*

Proof. With the new reflection principles established in Section 2, the theorem can be proved by using a path argument similar to Theorem 3.3 in [19]. We shall outline the basic ideas and emphasize some necessary modifications.

By contradiction, let $\widetilde{\mathbf{D}}$ be another polyhedral scatterer associated with the following boundary condition

$$\nu\times\mathbf{E}=0 \quad \text{on } \widetilde{\Gamma}_D \quad \text{and} \quad \nu\times\mathbf{H}=0 \quad \text{on } \widetilde{\Gamma}_N,$$

where $\widetilde{\Gamma}_D$ and $\widetilde{\Gamma}_N$ are a Lipschitz dissection of $\partial\widetilde{\mathbf{D}}$, such that

$$\mathscr{E}_\infty(\hat{\mathbf{x}};\mathbf{D})=\mathscr{E}_\infty(\hat{\mathbf{x}};\widetilde{\mathbf{D}}) \qquad \hat{\mathbf{x}}\in\mathbb{S}^2. \tag{3.4}$$

First, we show that it must have $\mathbf{D}=\widetilde{\mathbf{D}}$. Otherwise, by a standard argument, we can always assume that, without loss of generality, there exists a perfect plane $\widetilde{\Pi}_1$ in $\mathbf{G}=\mathbb{R}^3\backslash\mathbf{D}$ associated with $\mathscr{E}(\mathbf{x};\mathbf{D})$ or $\mathscr{H}(\mathbf{x};\mathbf{D})$. Then, we construct a curve $\gamma(t)(t\geq 0)$ in $\mathbf{G}$ such that

(a) $\gamma(t)$ is regular, namely, $\gamma(t)$ is C^1-smooth and $\frac{d}{dt}\gamma(t)\neq 0$;

(b) $\gamma(0)=\mathbf{x}_1\in\widetilde{\Pi}_1$ and $\gamma(t)(t>0)$ is contained entirely in the unbounded connected component of $\mathbf{G}\backslash\widetilde{\Pi}$ and $\lim_{t\to\infty}|\gamma(t)|=\infty$;

(c) The distance between the curve γ and the scatterer $\mathbf{D}$ is larger than a positive constant r_0, i.e., $\mathbf{d}(\gamma,\mathbf{D})>r_0>0$. Note that γ is a closed set in $\mathbb{R}^3$ while $\mathbf{D}$ is a compact set, this is always possible.

By the construction of the curve γ, we easily see that $\bar{B}_{r_0}(\mathbf{x})\subset\mathbf{G}$ for any point $\mathbf{x}\in\gamma(t)$. Now we set $t_1=0$ and let R_l denote the reflection in $\mathbb{R}^3$ with respect to a plane Π_l. Let

$$\tilde{t}_2=\max\{t>0;\gamma(t)\in S_{r_0}(\mathbf{x}_1)\}, \quad \tilde{\mathbf{x}}_2^+=\gamma(\tilde{t}_2)$$

be the 'last' intersection point between γ and $S_{r_0}(\mathbf{x}_1)$. Then we let $\tilde{\mathbf{x}}_2^-\in S_{r_0}(\mathbf{x}_1)$ be the symmetric point of $\tilde{\mathbf{x}}_2^+$ with respect to Π_1. Now, let $\mathbf{G}_1^+$ be the connected component of $\mathbf{G}\backslash\widetilde{\Pi}_1$ containing $\tilde{\mathbf{x}}_2^+$, and $\mathbf{G}_1^-$ be the connected component of $\mathbf{G}\backslash\widetilde{\Pi}_1$ containing $\tilde{\mathbf{x}}_2^-$. Then let $\mathbf{\Lambda}_1^+$ be the connected component of $\mathbf{G}_1^+\cap R_1(\mathbf{G}_1^-)$ containing $\tilde{\mathbf{x}}_2^+$ and $\mathbf{\Lambda}_1^-$ be the connected component of $\mathbf{G}_1^-\cap R_1(\mathbf{G}_1^+)$ containing $\tilde{\mathbf{x}}_2^-$. It is observed that $\mathbf{\Lambda}_1^+=R_1(\mathbf{\Lambda}_1^-)$, and if we set $\mathbf{\Lambda}_1=\mathbf{\Lambda}_1^+\cup\widetilde{\Pi}_1\cup\mathbf{\Lambda}_1^-$, then $\mathbf{\Lambda}_1$ contains the closed ball $\bar{B}_{r_0}(\mathbf{x}_1)$ and is symmetric with respect to Π_1. Clearly, $\mathbf{\Lambda}_1$ is an open connected set with the boundary composed of subsets on

$\partial\mathbf{D}$ and $R_1(\partial\mathbf{D})$. Since $\mathbf{D}$ is a polyhedral scatterer with the boundary condition (3.3), in terms of the reflection principles in Theorems 2.1-2.4, we know that at each regular point of $\partial\mathbf{\Lambda}_1$, either

$$\nu\times\mathscr{E}=0\quad\text{or}\quad\nu\times\mathscr{H}=0,$$

where ν is the inward normal to $\partial\mathbf{\Lambda}_1$. Next, we show that $\mathbf{\Lambda}_1$ must be bounded. Let $U\in\mathbb{R}^{3\times3}$ be a rotation matrix such that $U^{-1}\Pi_1\subset\{x_3=c\}$ with a constant c. Without loss of generality, we may assume that $\widetilde{\Pi}_1$ is a perfect plane corresponding to $\mathscr{E}$, namely, $\nu_{\Pi_1}\times\mathscr{E}=0$ on $\widetilde{\Pi}_1$. Then by the proof of Theorem 2.1 (see (2.21)), we see

$$U^{-1}\nu_{\Pi_1}\times\mathscr{J}(\mathbf{y})=0\quad\text{for }\mathbf{y}\in U^{-1}\Pi,$$

where $\mathscr{J}(\mathbf{y})=U^{-1}\mathscr{E}(U\mathbf{y})$. Next, set $\mathscr{J}(\mathbf{y})=(J_1(\mathbf{y}),J_2(\mathbf{y}),J_3(\mathbf{y}))$. Then by the proof of Lemma 2.1, we have that $J_1(\mathbf{y})$ and $J_2(\mathbf{y})$ are odd symmetric with respect to $U^{-1}\Pi$ while $J_3(\mathbf{y})$ is even symmetric with respect to $U^{-1}\Pi$ in $U^{-1}\mathbf{\Lambda}_1$. Hence,

$$J_1(\mathbf{y})=J_2(\mathbf{y})=0\quad\text{for }\ \mathbf{y}\in U^{-1}\Pi_1\cap U^{-1}\mathbf{\Lambda}_1;$$

namely,

$$U^{-1}\nu_{\Pi_1}\times\mathscr{J}(\mathbf{y})=0\quad\text{for }\ \mathbf{y}\in U^{-1}\Pi_1\cap U^{-1}\mathbf{\Lambda}_1.$$

By using Lemma 2.3, we then deduce that $\nu_{\Pi_1}\times\mathscr{E}(\mathbf{x})=0$ for $\mathbf{x}\in\Pi_1\cap\mathbf{\Lambda}_1$. Next, it is noted that $\partial\mathbf{\Lambda}_1$, $\partial\mathbf{G}_1^{\pm}$ and $R_1(\partial\mathbf{G}_1^{\pm})$ are bounded by our construction. If $\mathbf{\Lambda}_1$ is unbounded, then $\mathbf{\Lambda}_1$ would contain $\mathbb{R}^3\backslash B_r(\mathbf{x}_1)$ for some sufficiently large $r>0$. Using $\nu_{\Pi_1}\times\mathscr{E}\,|_{\mathbf{\Lambda}_1\cap\Pi_1}=0$ and analytic continuation, we know that $\Pi_1\backslash B_r(\mathbf{x}_1)$ are parts of some perfect planes associated with $\mathscr{E}(\mathbf{x};\mathbf{D})$. However $\Pi_1\backslash B_r(\mathbf{x}_1)$ is unbounded, so it contradicts the boundedness of any perfect planes. Hence $\mathbf{\Lambda}_1$ is bounded and it forms a polyhedral domain in $\mathbf{G}$. Now by the unboundedness of γ, there must exist a $t_2>\tilde{t}_2$ such that $\mathbf{x}_2=\gamma(t_2)\in\partial\mathbf{\Lambda}_1$. Noting the fact that on $\partial\mathbf{\Lambda}_1$ either $\mathscr{E}$ or $\mathscr{H}$ takes the perfect condition, then by analytic continuation $\mathbf{x}_2\in\partial\mathbf{\Lambda}_1$ implies the existence of a perfect plane passing through $\mathbf{x}_2$, which we denote by $\widetilde{\Pi}_2$. Clearly, we have $\mathbf{x}_2=\gamma(t_2)\in\mathscr{P}_{\mathscr{E}}\cup\mathscr{P}_{\mathscr{H}}$. Again, we assume that $t_2=\max\{t>0;\gamma(t)\in\widetilde{\Pi}_2\}<\infty$.

Up to now, we see the following facts: $\widetilde{\Pi}_2$ is different from $\widetilde{\Pi}_1$, since $\widetilde{\Pi}_1$ intersects γ lastly at $\mathbf{x}_1$; the length of $\gamma(t)$ from t_1 to t_2 is larger than r_0, i.e.,

$$|\gamma(t_1\le t\le t_2)|\ge|\gamma(t_1\le t\le\tilde{t}_2)|\ge r_0.$$

Next, from the perfect plane $\widetilde{\Pi}_2$, by exactly the same argument as we derived the point $\mathbf{x}_2=\gamma(t_2)$ and $\widetilde{\Pi}_2$ with $\widetilde{\Pi}_1$, we can find a point $\mathbf{x}_3\in\gamma(t_3)$ $(t_3>t_2)$ and a perfect plane $\widetilde{\Pi}_3$ passing through $\mathbf{x}_3$ such that $\widetilde{\Pi}_3$ is different from $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$, and

$$|\gamma(t_2\le t\le t_3)|\ge r_0.$$

Continuing with the above procedure, we can construct a strictly increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that for any n,

$$\mathbf{x}_n=\gamma(t_n)\in\mathscr{P}_{\mathscr{E}}\cup\mathscr{P}_{\mathscr{H}},$$

and $\widetilde{\Pi}_n$ is a perfect plane with respect to $\mathscr{E}$ or $\mathscr{H}$ passing through $\mathbf{x}_n$. Moreover, those perfect planes are different from each other, and the length of $\gamma(t)$ from t_n to t_{n+1} is not less than r_0, i.e.,

$$|\gamma(t_n \le t \le t_{n+1})| \ge r_0. \tag{3.5}$$

Since both $\mathscr{P}_{\mathscr{E}}$ and $\mathscr{P}_{\mathscr{H}}$ are bounded and $\lim_{t\to\infty}|\gamma(t)| = +\infty$, we must have $\lim_{n\to\infty} t_n = t_0$ for some finite t_0. Finally, because $\gamma(t)$ is a C^1-smooth curve, we further have

$$\lim_{n\to\infty}|\gamma(t_n \le t \le t_{n+1})| = \lim_{n\to\infty}\int_{t_n}^{t_{n+1}} |\gamma'(t)| dt = 0, \tag{3.6}$$

which contradicts inequality (3.5). Thus the proof that $\mathbf{D} = \widetilde{\mathbf{D}}$, is completed.

Finally, we show that $\Gamma_D = \widetilde{\Gamma}_D$ and $\Gamma_N = \widetilde{\Gamma}_N$. In fact, if $\Gamma_D \neq \widetilde{\Gamma}_D$, then one can easily see that there exists an open portion of $\partial\mathbf{D}$ and we write as Γ_0, on which

$$\nu \times \mathscr{E} = \nu \times \mathscr{H} = 0.$$

Then, by the unique continuation principle (see Lemma 3.2 in [1]), we obtain $\mathscr{E} = \mathscr{H} = 0$ in $\mathbf{G}$, which is certainly not true. The proof is completed. □

Next, we consider the uniqueness for partially coated polyhedral scatterers. Let $\mathbf{D}$ be a polyhedral scatterer on which the following boundary condition is enforced:

$$\begin{aligned} \nu \times \mathbf{E} &= 0 \quad \text{on} \quad \Gamma_D, \\ \nu \times \mathbf{H} &= 0 \quad \text{on} \quad \Gamma_N, \\ \nu \times \operatorname{curl} \mathbf{E} - i\lambda(\nu \times \mathbf{E}) \times \nu &= 0 \quad \text{on} \quad \Gamma_I, \end{aligned} \tag{3.7}$$

where Γ_D, Γ_N and Γ_I form a Lipschitz dissection of $\partial\mathbf{D}$, and $\lambda \in C^{0,\alpha}(\Gamma_I)$ satisfies that $\lambda \ge \lambda_0$ with λ_0 being a positive constant. If $\Gamma_I \neq \emptyset$, the scatterer is said to be *partially coated*, otherwise it is *non-coated*. Apparently, partial coating or non-coating is an intrinsic physical property of the underlying scatterers, which one usually does not know a priori in practical applications. The following theorem provides a uniqueness result in determining such coating properties for polyhedral scatterers.

Theorem 3.2. *Let $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ be two polyhedral scatterers such that $\mathbf{D}$ is partially coated while $\widetilde{\mathbf{D}}$ is non-coated. Then for any fixed $k_0 > 0$ and $d_0 \in \mathbb{S}^2$, and two different polarizations p_1 and p_2 such that p_1, p_2, d_0 are linearly independent, there cannot hold that*

$$\mathscr{E}_\infty(\hat{\mathbf{x}}; \mathbf{D}) = \mathscr{E}_\infty(\hat{\mathbf{x}}; \widetilde{\mathbf{D}}) \quad \textit{for } \hat{\mathbf{x}} \in \mathbb{S}^2. \tag{3.8}$$

Remark 3.1. Theorem 3.2 tells that two scatterers of the different coating properties can not produce the same far-field patterns. In this sense, the corresponding far-field patterns can uniquely determine such coating properties. However except for such coating properties, we are still unable to completely prove or disprove the unique determination of the underlying scatterers by finitely many incident waves.

In order to prove Theorem 3.2, we need the following lemma.

Lemma 3.1. *Let $\mathbf{\Lambda} \subset \mathbf{G}$ be a bounded polyhedral domain and $\partial\mathbf{\Lambda} = \overline{\partial\mathbf{\Lambda}_1} \cup \overline{\partial\mathbf{\Lambda}_2} \cup \overline{\partial\mathbf{\Lambda}_3}$ where $\partial\mathbf{\Lambda}_l, l = 1,2,3$ are three disjoint relative open subsets of $\partial\mathbf{\Lambda}$. Let $(\mathscr{E},\mathscr{H})$ be the solution to the Maxwell's system (1.3)-(1.5) and (3.7), and suppose that $\nu \times \mathscr{E} = 0$ on $\partial\mathbf{\Lambda}_1$, $\nu \times \mathscr{H} = 0$ on $\partial\mathbf{\Lambda}_2$ and $\nu \times \text{curl } \mathscr{E} - \mathrm{i}\lambda(\nu \times \mathscr{E}) \times \nu = 0$ (or $\nu \times \text{curl } \mathscr{E} + \mathrm{i}\lambda(\nu \times \mathscr{E}) \times \nu = 0$) on $\partial\mathbf{\Lambda}_3$, where ν is the unit normal to $\partial\mathbf{\Lambda}$ directed to the interior of $\mathbf{\Lambda}$ and $\lambda \in C^{0,\alpha}(\partial\mathbf{\Lambda}_3)$ is nonnegative. Then $\lambda = 0$ on $\partial\mathbf{\Lambda}_3$.*

Proof. It suffices to consider the case where

$$\nu \times \text{curl } \mathscr{E} - \mathrm{i}\lambda(\nu \times \mathscr{E}) \times \nu = 0 \quad \text{on} \ \ \partial\mathbf{\Lambda}_3,$$

while the case where

$$\nu \times \text{curl } \mathscr{E} + \mathrm{i}\lambda(\nu \times \mathscr{E}) \times \nu = 0 \quad \text{on} \ \ \partial\mathbf{\Lambda}_3$$

can be treated exactly in the same manner. First, by the Maxwell's equations (1.3) we have

$$\text{curl curl } \mathscr{E} - k^2\mathscr{E} = 0 \quad \text{in} \quad \mathbf{\Lambda}. \tag{3.9}$$

Then by Green's formula, we can deduce

$$\int_{\mathbf{\Lambda}} \text{curl curl } \mathscr{E} \cdot \overline{\mathscr{E}} - \int_{\mathbf{\Lambda}} \text{curl } \mathscr{E} \cdot \text{curl } \overline{\mathscr{E}} = -\int_{\partial\mathbf{\Lambda}} (\nu \times \text{curl } \mathscr{E}) \cdot \overline{\mathscr{E}}. \tag{3.10}$$

Noting that $\nu \times \mathscr{E} = 0$ on $\partial\mathbf{\Lambda}_1$, $\nu \times \text{curl } \mathscr{E} = 0$ on $\partial\mathbf{\Lambda}_2$ and $\nu \times \text{curl } \mathscr{E} - \mathrm{i}\lambda(\nu \times \mathscr{E}) \times \nu = 0$ on $\partial\mathbf{\Lambda}_3$, we have from (3.9) and (3.10) that

$$\int_{\partial\mathbf{\Lambda}_3} (\nu \times \text{curl } \mathscr{E}) \cdot \overline{\mathscr{E}} = k^2 \int_{\mathbf{\Lambda}} -|\mathscr{E}|^2 + |\text{curl } \mathscr{E}|^2, \tag{3.11}$$

which further gives by the corresponding boundary condition on $\partial\mathbf{\Lambda}_3$ that

$$\mathrm{i}\int_{\partial\mathbf{\Lambda}_3} \lambda|\nu \times \mathscr{E}|^2 = k^2 \int_{\mathbf{\Lambda}} -|\mathscr{E}|^2 + |\text{curl } \mathscr{E}|^2. \tag{3.12}$$

By taking the imaginary parts, we have from (3.12) that

$$\int_{\partial\mathbf{\Lambda}_3} \lambda|\nu \times \mathscr{E}|^2 = 0. \tag{3.13}$$

Since λ is non-negative, we cannot have an open portion Σ of $\partial\mathbf{\Lambda}_3$ on which λ is non-zero. Otherwise, we easily see from (3.13) and the boundary condition on $\partial\mathbf{\Lambda}_3$ that $\nu \times \mathscr{E} = 0$ and $\nu \times \mathscr{H} = 0$ on Σ which implies by the continuation principle that $\mathscr{E} = \mathscr{H} = 0$ in $\mathbf{G}$. This is a contradiction with the boundedness of $\mathscr{P}_{\mathscr{E}}$. Now, by the continuity of λ, it must vanish identically on $\partial\mathbf{\Lambda}_3$. The proof is completed. □

Proof of Theorem 3.2. With Lemma 3.1, the proof follows basically by an argument similar to that for Theorem 3.1. In the following, we only sketch the main difference.

By contradiction, we assume that there exist two scatterers as stated in Theorem 3.2 such that (3.8) holds. Clearly, we must have $\mathbf{D} \neq \widetilde{\mathbf{D}}$. Otherwise, we can easily show that there exists an open subset on $\partial\mathbf{D}$ where $\nu \times \mathscr{E} = \nu \times \mathscr{H} = 0$, which implies by the unique continuation that $\mathscr{E} = \mathscr{H} = 0$ in $\mathbf{G}$, hence leading to a contradiction. Therefore, without loss of generality, we can assume that $\mathbf{D} \neq \widetilde{\mathbf{D}}$. Let Ω be the unbounded connected component of $\mathbb{R}^3\backslash(\mathbf{D}\cup\widetilde{\mathbf{D}})$. Then, we must have either

$$\mathbf{\Lambda} := (\mathbb{R}^3\backslash\Omega)\backslash\widetilde{\mathbf{D}} \neq \emptyset \quad \text{or} \quad \widetilde{\mathbf{\Lambda}} := (\mathbb{R}^3\backslash\Omega)\backslash\mathbf{D} \neq \emptyset.$$

Moreover, by choosing some connected components of $\mathbf{\Lambda}$ and $\widetilde{\mathbf{\Lambda}}$ if necessary, we may assume that both $\mathbf{\Lambda}$ and $\widetilde{\mathbf{\Lambda}}$ are connected. Clearly, they form two bounded polyhedral domains in $\mathbb{R}^3$.

First, we consider the case where $\widetilde{\mathbf{\Lambda}} \neq \emptyset$. Apparently, $\partial\widetilde{\mathbf{\Lambda}}$ consists of two parts, one of which lies on $\partial\widetilde{\mathbf{D}}$ and the other part lies on $\partial\mathbf{D}$. By the boundary conditions together with the help of Rellich's theorem, we see that on the part lying on $\partial\widetilde{\mathbf{D}}$, $(\mathscr{E}(\widetilde{\mathbf{D}}), \mathscr{H}(\widetilde{\mathbf{D}}))$ assumes the boundary condition of type (3.3), while on the part lying on $\partial\mathbf{D}$, $(\mathscr{E}(\widetilde{\mathbf{D}}), \mathscr{H}(\widetilde{\mathbf{D}}))$ assumes the boundary condition of type (3.7). Then we have by Lemma 3.1 that either

$$\nu \times \mathscr{E} = 0 \quad \text{or} \quad \nu \times \mathscr{H} = 0$$

at each regular point of $\partial\widetilde{\mathbf{\Lambda}}$. Hence, by analytic continuation, we easily deduce that there exists a perfect plane in $\mathbb{R}^3\backslash\widetilde{\mathbf{D}}$, leading us to the same situation as that in the proof of Theorem 3.1, where we eventually arrive at a contradiction.

Whereas for the case with $\mathbf{\Lambda} \neq \emptyset$, by the exactly same argument as treating the case for $\widetilde{\mathbf{\Lambda}} \neq \emptyset$, we see that there exists a perfect plane (corresponding to $\mathscr{E}(\mathbf{D})$ or $\mathscr{H}(\mathbf{D})$) in $\mathbf{G} = \mathbb{R}^3\backslash\mathbf{D}$. In the following, let us first see what will happen if we make a reflection argument in $\mathbf{G}$. Let $\widetilde{\Pi}_1$ be the perfect plane obtained above, and let $\mathbf{\Lambda}_1, \mathbf{\Lambda}_1^+$ and $\mathbf{\Lambda}_1^-$ be respectively those bounded polyhedral domains in the proof of Theorem 3.1, which are derived by making reflection with respect to Π_1 in $\mathbf{G}$. In terms of the reflection principles in Theorems 2.1-2.4, we know that at all the regular points of $\partial\mathbf{\Lambda}_1$, either

$$\nu \times \mathscr{E} = 0, \quad \text{or} \quad \nu \times \mathscr{H} = 0, \quad \text{or} \quad \nu \times \operatorname{curl} \mathscr{E} - i\eta(\nu \times \mathscr{E}) \times \nu = 0,$$

where $\eta(\mathbf{x}) = \lambda(\mathbf{x})$ for $\mathbf{x} \in \partial\mathbf{D}$ and $\eta(\mathbf{x}) = \lambda(R_{\Pi_1}\mathbf{x})$ for $\mathbf{x} \in R_{\Pi_1}(\partial\mathbf{D})$, and ν denotes the inward unit normal to $\partial\mathbf{\Lambda}_1$. Using Lemma 3.1 in $\mathbf{\Lambda}_1^+$ and $\mathbf{\Lambda}_1^-$ respectively, we further have that the impedance boundary condition must be excluded on $\partial\mathbf{\Lambda}_1$. Hence, we can still make use of the path argument as that in the proof of Theorem 3.1, and from $\widetilde{\Pi}_1$ we can find another perfect plane $\widetilde{\Pi}_2$ and eventually we are led to a contradiction. This completes the proof. $\square$

4. Concluding remarks

Up to now, we have only considered polyhedral scatterers consisting of finitely many pairwise disjoint polyhedra. However, almost all the previous results can be easily extended to scatterers of much more general type as that considered in [19].

Let us first follow [19] to prescribe exactly those general polyhedral scatterer. In the following, a cell is defined to be the closure of an open subset of a 2-dimensional plane. Moreover, an obstacle $\mathbf{D}$ is said to be a *general polyhedral scatterer* if it is a compact subset of $\mathbb{R}^3$ with connected complement $\mathbf{G} = \mathbb{R}^3\backslash\mathbf{D}$, and the boundary of $\mathbf{G}$ is the union of a finite number of cells, i.e.,

$$\partial\mathbf{G} = \bigcup_{j=1}^{m} C_j, \tag{4.1}$$

where each C_j is a cell. Clearly, scatterers of this type admit the simultaneous presence of both solid- and crack-type components. Under the assumption that there exists a unique solution in $H_{loc}(\mathrm{curl};\mathbf{G}) \times H_{loc}(\mathrm{curl};\mathbf{G})$ to the forward scattering problem associated with such general obstacles, then one can show, with some natural minor modifications, that all our previous reflection principles and uniqueness results, except for the one in Theorem 3.2, still hold for such general polyhedral scatterers.

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