Numer. Math. Theor. Meth. Appl. Vol. **2**, No. 3, pp. 341-352
doi: 10.4208/nmtma.2009.m88037 August 2009

Error Estimate of the Fourier Collocation Method for the Benjamin-Ono Equation

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Received 20 August 2008; Accepted (in revised version) 23 March 2009

Abstract. In this paper, the Fourier collocation method for solving the generalized Benjamin-Ono equation with periodic boundary conditions is analyzed. Stability of the semi-discrete scheme is proved and error estimate in $H^{1/2}$-norm is obtained.

AMS subject classifications: 65M12, 65M70, 76B15

Key words: Fourier collocation, Benjamin-Ono equation, error estimate.

1. Introduction

In this paper, we analyze the Fourier collocation (FC) approximation to the generalized Benjamin–Ono (BO) equation with periodic boundary conditions:

$$\begin{cases} \partial_t U(x,t) + \partial_x F(U)(x,t) + \mathscr{H}\partial_x^2 U(x,t) = 0, & x \in \mathbb{R}, 0 < t \le T, \\ U(x+2\pi,t) = U(x,t), & x \in \mathbb{R}, 0 < t \le T, \\ U(x,0) = U_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where U_0 is 2π-periodic in space, $F(z) \in C^1(\mathbb{R})$, and $\mathscr{H}$ is the periodic Hilbert transform [1]

$$\mathscr{H}u(x) = -\frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \cot\Big(\frac{\pi(x-y)}{2\pi}\Big)u(y)\mathrm{d}y.$$

The problem (1.1) arises in the propagation of internal waves in a stratified fluid of great depth. The special case $F(U) = U^2$ is the BO equation. Fourier methods for the BO equation have been studied by many authors [4, 9–12]. In recent work [11], it is proved that error of the Fourier Galerkin (FG) method for the BO equation is of the order $\mathscr{O}(N^{1-r})$ in L^2-norm for the analytic solution in H^r. An optimal error bound $\mathscr{O}(N^{1/2-r})$ of the method in $H^{1/2}$-norm is obtained in [4].

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As suggested in [6, 9, 10], the FC methods for the BO equation are efficient, but no error analysis has been provided. The aim of this work is to give rigorous proof of error estimate of the FC method for (1.1). In particular, it will be shown that the error is of the order $\mathscr{O}(N^{3/2-r})$ in $H^{1/2}$-norm.

In Section 2, the FC method for (1.1) is presented. In Section 3, some lemmas needed in error analysis are given. In Section 4, the stability and convergence of the semi-discrete FC method are analyzed. This paper does not give the analysis for the fully discrete scheme, but the accuracy of the fully discrete scheme will be demonstrated by using an example for the BO equation in Section 5.

2. The Fourier collocation method

Let $I=(-\pi,\pi)$. The inner product of $L^2(I)$ is denoted by $(\cdot,\cdot)$. For a positive integer N, the approximation space V_N of the real trigonometric polynomials of degree N is defined by

$$V_N=\left\{u(x)=\sum_{l=-N}^{N}{}'' a_l e^{ilx} \ :\ \overline{a_l}=a_{-l},\ |l|\le N;\ \ a_N=a_{-N}\right\},$$

where the notation $\sum''$ denotes halving the terms a_{-N} and a_N in the series. Let $h=2\pi/2N$, $x_j=jh-\pi$ $(j=0,\cdots,2N-1)$ be the collocation points so that the base 2 Fast Fourier Transform (FFT) can be directly adopted. Let $I_N: C(\bar{I})\rightarrow V_N$ be the Fourier interpolation operator defined by

$$I_N u(x_j)=u(x_j),\quad j=0,\cdots,2N-1.$$

We define the discrete product and norm as follows:

$$(u,v)_N=h\sum_{j=0}^{2N-1}u(x_j)\overline{v(x_j)},\qquad \|u\|_N=(u,u)_N^{1/2}.$$

Let $P_N: L^2(I)\rightarrow V_N$ be the L^2-orthogonal projection operator, i.e.,

$$(P_N u-u,v)=0,\qquad v\in V_N.$$

The semi-discrete FC method for (1.1) is to find $u_c(t)\in V_N$ such that for $0\le j\le 2N-1$,

$$\left\{\begin{array}{ll}(\partial_t u_c+\partial_x I_N F(u_c)+\mathscr{H}\partial_x^2 u_c)(x_j,t)=0, & 0<t\le T,\\ u_c(x_j,0)=P_N U_0(x_j). & \end{array}\right. \tag{2.1}$$

For the time advance, we use the second-order leapfrog-Crank-Nicolson scheme. Let τ be the step size in time and $t_k=k\tau$ $(k=0,1,\cdots,n_{_T};\ T=n_{_T}\tau)$. Denote $u^k(x):=u(x,t_k)$ by u^k and

$$u_{\hat{t}}^k=\frac{1}{2\tau}(u^{k+1}-u^{k-1}),\qquad \hat{u}^k=\frac{1}{2}(u^{k+1}+u^{k-1}).$$

The fully discrete FC method for (1.1) is to find $u_C^k \in V_N$ such that for $0 \le j \le 2N-1$,

$$\begin{cases} (u_{C\hat{t}}^k + \partial_x I_N F(u_C^k) + \mathscr{H}\partial_x^2 \hat{u}_C^k)(x_j) = 0, \quad 1 \le k \le n_T - 1, \\ u_C^1(x_j) = P_N[U_0 + \tau\partial_t U(0) + \frac{1}{2}\tau^2\partial_t^2 U(0)](x_j), \\ u_C^0(x_j) = P_N U_0(x_j). \end{cases} \tag{2.2}$$

3. Some lemmas

In this section, some lemmas needed in error analysis are given. Throughout this paper C will denote a generic positive constant. For any real number $r \ge 0$, $H^r(I) := W^{r,2}(I)$ is the Sobolev space with the norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$, where the subscript r will be dropped whenever $r = 0$. Let $H_p^r(I)$ be the subspace of $H^r(I)$ consisting of all periodic functions of the period 2π equipped with the following equivalent norm and semi-norm:

$$\|u\|_r = \left\{ \sum_{l=-\infty}^{\infty} (1+|l|^2)^r |a_l|^2 \right\}^{1/2}, \qquad |u|_r = \left\{ \sum_{l=-\infty}^{\infty} |l|^{2r} |a_l|^2 \right\}^{1/2},$$

where

$$u(x) = \sum_{l=-\infty}^{\infty} a_l e^{ilx}, \qquad a_l = \frac{1}{2\pi}\int_I u(x) e^{-ilx}\, dx.$$

For $0 < r < 1$, the equivalent semi-norm $|\cdot|_r$ on $H^r(I)$ is defined by [2]

$$|u|_r = \left\{ \int_I\int_I \frac{|u(x)-u(y)|^2}{|x-y|^{2r+1}}\, dx\, dy \right\}^{1/2}.$$

For $r > 1$, the equivalent norm $\|\cdot\|_r$ on $H^r(I)$ is defined by

$$\|u\|_r = \left(\|u\|_m^2 + |u^{(m)}|_\mu^2 \right)^{1/2},$$

where $m = [r]$ and $\mu = r - [r]$.

Lemma 3.1 ([5,7]). *If $0 \le \mu \le r$ and $u \in H_p^r(I)$, then*

$$\|P_N u - u\|_\mu \le C N^{\mu - r} |u|_r; \tag{3.1}$$

and if $r > 1/2$, $v \in H^r(I)$ and $C(r)$ denotes a constant depending on r, then

$$\|I_N u - u\|_\mu \le C N^{\mu - r} |u|_r, \tag{3.2}$$

$$\|uv\|_r \le C(r)\|u\|_r \|v\|_r. \tag{3.3}$$

Lemma 3.2 ([2,4,7]). *If $u \in H^1(I)$, then*

$$\|u\|_{L^\infty(I)} \le C\|u\|^{1/2}\|u\|_1^{1/2}; \tag{3.4}$$

and if $u \in V_N$, *then*

$$\|u\|_{L^\infty(I)} \le C(\ln N)^{1/2}\|u\|_{1/2}, \tag{3.5}$$

$$|u|_r \le N^{r-\mu}|u|_\mu, \qquad 0 \le \mu \le r. \tag{3.6}$$

Lemma 3.3 ([4,7]). *Let* $\varepsilon > 0$. *We have*

$$|uv|_{1/2} \le C(\varepsilon)\|u\|_{1/2}\|v\|_{1/2+\varepsilon}, \qquad \forall u \in H^{1/2}(I), \;\; v \in H^{1/2+\varepsilon}(I), \tag{3.7}$$

$$|(u, \partial_x v)| \le |u|_{1/2}|v|_{1/2}, \qquad \forall u \in H_p^{1/2}(I), \;\; v \in H_p^1(I), \tag{3.8}$$

$$(u,v)_N = (I_N u, I_N v)_N = (I_N u, I_N v), \qquad \forall u, v \in C(\bar{I}), \tag{3.9}$$

$$(\partial_x u, v)_N = (P_{N-1}\partial_x u, v) = (\partial_x P_{N-1} u, P_{N-1} v), \qquad \forall u, v \in V_N. \tag{3.10}$$

Lemma 3.4 ([8]). *Suppose that*

(i) $E_i(t)$, $\rho_i(t)$, $i = 1,2$, *are non-negative functions continuous on* $[0,T]$, $\rho_i(t)$ *is increasing with respect to* t, *and* M, C *are positive constants;*

(ii) *for any* $t \in [0,T]$, *if* $\max\limits_{0\le s\le t}\{E_1(s), E_2(s)\} \le M$,

$$E_i(t) \le \rho_i(t) + C\int_0^t E_i(s)\,\mathrm{d}s;$$

(iii) $E_i(0) \le \rho_i(0)$ *and* $\rho_i(T)\mathrm{e}^{CT} \le M$.

Then for any $t \in [0,T]$, $E_i(t) \le \rho_i(t)\mathrm{e}^{CT}$.

4. Error estimates of the semi-discrete scheme

By the definition of the discrete product, (2.1) is equivalent to, for any $v \in V_N$,

$$\begin{cases} (\partial_t u_c(t) + \partial_x I_N F(u_c(t)) + \mathscr{H}\partial_x^2 u_c(t), v)_N = 0, \quad 0 < t \le T, \\ u_c(0) = P_N U_0. \end{cases} \tag{4.1}$$

We first consider the stability. Assume that $u_c(t)$ and the term on the right-hand side in (4.1) have errors $\tilde{u}(t)$ and $\tilde{f}(t)$, respectively. By (4.1), we have for any $v \in V_N$ that

$$\begin{cases} (\partial_t \tilde{u}(t) + \partial_x I_N \tilde{F}(t) + \mathscr{H}\partial_x^2 \tilde{u}(t) - \tilde{f}(t), v)_N = 0, \quad 0 < t \le T, \\ \tilde{u}(0) = \tilde{u}_0, \end{cases} \tag{4.2}$$

where

$$\tilde{F} = F(u_c + \tilde{u}) - F(u_c) := \tilde{G}\,\tilde{u}, \qquad \tilde{G} = \int_0^1 F'(u_c + \theta\tilde{u})\,\mathrm{d}\theta.$$

In what follows, for given $0 < t \le T$, assume

$$\max_{0\le s\le t} \|\tilde{u}(s)\|_{1/2} \le N^{-1/2}, \qquad \max_{0\le s\le t} \|\partial_s \tilde{u}(s)\|_{1/2} \le N^{-1/2}. \tag{4.3}$$

Thus, we have by (3.6) that for any $0 \le s \le t$,

$$\|\tilde{u}(s)\|_{L^\infty(I)} \le M_1(\ln N)^{1/2}\|\tilde{u}(s)\|_{1/2} \le M_1, \tag{4.4}$$

$$\|\partial_s\tilde{u}(s)\|_{L^\infty(I)} \le M_1(\ln N)^{1/2}\|\partial_s\tilde{u}(s)\|_{1/2} \le M_1, \tag{4.5}$$

where M_1 is the constant in (3.6). We note that

$$\mathscr{H} v(x) = i\sum_{l=-\infty}^{\infty} \mathrm{sign}(l)\, a_l \mathrm{e}^{ilx}, \qquad \text{for } v(x) = \sum_{l=-\infty}^{\infty} a_l \mathrm{e}^{ilx},$$

where $i\,\mathrm{sign}(l)$ is the symbol of $\mathscr{H}$. Then we have

$$|v|_{1/2}^2 = \sum_{l=-\infty}^{\infty} \mathrm{sign}(l) l |a_l|^2 = -(\partial_x v, \mathscr{H} v). \tag{4.6}$$

It is easy to check the three properties of $\mathscr{H}$: (i)$\mathscr{H}$ is skew-symmetric on L^2; (ii)$\mathscr{H}$ is bounded; (iii)$\mathscr{H}$ commutes with differentiation.

First, taking $v = I_N\tilde{F}(t) + \mathscr{H}\partial_x\tilde{u}(t)$ in (4.2) and integrating it in time lead to

$$|\tilde{u}(t)|_{1/2}^2 \le |\tilde{u}(0)|_{1/2}^2 + 2\int_0^t \Big\{(\partial_x I_N\tilde{f}(s), \mathscr{H}\tilde{u}(s)) + (\partial_s\tilde{u}(s) - I_N\tilde{f}(s), I_N\tilde{F}(s))\Big\}\mathrm{d}s, \tag{4.7}$$

where we have used (3.9), (3.10), (4.6) and the properties (i), (iii) of $\mathscr{H}$. By the Cauchy-Schwarz inequality,

$$2|(I_N\tilde{f}(s), I_N\tilde{F}(s))| \le \|I_N\tilde{f}(s)\|^2 + \|I_N\tilde{F}(s)\|^2. \tag{4.8}$$

Using (3.9), (4.4) and the notation

$$C_F(z_1,z_2) = (|z_1|+|z_2|)\max_{|z|\le|z_1|+|z_2|}\Big(|\partial_z F(z)|, |\partial_z^2 F(z)|, |\partial_z^3 F(z)|\Big), \tag{4.9}$$

we obtain

$$\begin{aligned}
\|I_N\tilde{F}(s)\|^2 = \|\tilde{F}(s)\|_N^2 &= h\sum_{j=0}^{2N-1}((\tilde{G}\,\tilde{u})(x_j,s))^2 \\
&\le \max_{0\le j<2N}(\tilde{G}(x_j,s))^2 h\sum_{j=0}^{2N-1}(\tilde{u}(x_j,s))^2 \\
&\le C_F(\|u_c(s)\|_{C(I)}, M_1)\|\tilde{u}(s)\|^2.
\end{aligned}$$

By (3.8) and the property (ii) of $\mathscr{H}$, we have

$$2|(\partial_x I_N \tilde{f}(s), \mathscr{H}\tilde{u}(s))| \leq C|I_N \tilde{f}(s)|_{1/2}^2 + |\tilde{u}(s)|_{1/2}^2. \tag{4.10}$$

By integration by parts in time, we get

$$\begin{aligned}
&2\Big|\int_0^t (\partial_s \tilde{u}(s), I_N \tilde{F}(s))\, ds\Big| \\
&= 2\Big|\int_0^t (\partial_s \tilde{u}(s), \tilde{G}(s)\tilde{u}(s))_N \,\mathrm{d}s\Big| \\
&= \Big|(\tilde{u}(s), \tilde{G}(s)\tilde{u}(s))_N\Big|_0^t - \int_0^t (\tilde{u}(s), \partial_s \tilde{G}(s)\tilde{u}(s))_N \,\mathrm{d}s\Big| \\
&= \max_{0\leq j<2N} |\tilde{G}(x_j,s)|\|\tilde{u}(s)\|^2\Big|_0^t + \int_0^t \max_{0\leq j<2N} |\partial_s \tilde{G}(x_j,s)|\|\tilde{u}(s)\|^2 \,\mathrm{d}s \\
&\leq C_F(\|u_c\|_{C^1(0,T;C(I))}, M_1)\Big\{\|\tilde{u}(t)\|^2 + \|\tilde{u}(0)\|^2 + \int_0^t \|\tilde{u}(s)\|^2 \,\mathrm{d}s\Big\}.
\end{aligned} \tag{4.11}$$

Now we have by (4.5) that

$$\begin{aligned}
\max_{0\leq j<2N} |\partial_s \tilde{G}(x_j,s)| &\leq \max_{0\leq j<2N} \int_0^1 |F''(u_c + \theta\tilde{u})\partial_s(u_c + \theta\tilde{u})(x_j,s)| \,\mathrm{d}\theta \\
&\leq C_F(\|u_c\|_{C^1(0,T;C(I))}, M_1).
\end{aligned}$$

Putting (4.8), (4.10) and (4.11) in (4.7), we obtain

$$\begin{aligned}
|\tilde{u}(t)|_{1/2}^2 &\leq |\tilde{u}(0)|_{1/2}^2 + C_F(\|u_c\|_{C^1(0,T;C(I))}, M_1) \\
&\quad \cdot \Big\{\|\tilde{u}(t)\|^2 + \|\tilde{u}(0)\|^2 + \int_0^t (\|I_N \tilde{f}(s)\|_{1/2}^2 + |\tilde{u}(s)|_{1/2}^2) \,\mathrm{d}s\Big\}.
\end{aligned} \tag{4.12}$$

Second, taking $v = \tilde{u}(t)$ in (4.2) and integrating the resulting equation in time yield

$$\|\tilde{u}(t)\|^2 = \|\tilde{u}(0)\|^2 + 2\int_0^t \Big\{(I_N \tilde{f}(s), \tilde{u}(s)) + (P_{N-1} I_N \tilde{F}(s), \partial_x P_{N-1} \tilde{u}(s))\Big\} \,\mathrm{d}s. \tag{4.13}$$

By (3.6), (4.3) and (4.4), we have

$$\begin{aligned}
\|\tilde{G}\|_1 &= \Big\|\int_0^1 \big(F''(u_c + \theta\tilde{u})\partial_x(u_c + \theta\tilde{u}) + F'(u_c + \theta\tilde{u})\big) \,\mathrm{d}\theta\Big\| \\
&\leq C_F(\|u_c\|_1, M_1).
\end{aligned} \tag{4.14}$$

Then by (3.8) and (3.7) we have

$$\begin{aligned}
&2|(P_{N-1}\tilde{F}(s), \partial_x P_{N-1}\tilde{u}(s))| \\
&\leq C|\tilde{F}(s)|_{1/2}^2 + |\tilde{u}(s)|_{1/2}^2 \leq C\|\tilde{G}(s)\|_1^2 \|\tilde{u}(s)\|_{1/2}^2 + |\tilde{u}(s)|_{1/2}^2 \\
&\leq C_F(\|u_c\|_1, M_1)\|\tilde{u}(s)\|_{1/2}^2.
\end{aligned} \tag{4.15}$$

It follows from (4.3), (4.4) and (3.6) that

$$\begin{aligned}|\tilde{F}|_1 &= \|\partial_x \tilde{G}\tilde{u} + \tilde{G}\partial_x \tilde{u}\| \\ &\le \left\| \int_0^1 F''(u_c + \theta\tilde{u})\partial_x(u_c + \theta\tilde{u})\,\mathrm{d}\theta\tilde{u}\| + \|\tilde{G}\|_{L^\infty(I)}\|\partial_x \tilde{u}\right\| \\ &\le C(\|u_c\|_{C^1(I)}, M_1)N^{1/2}\|\tilde{u}\|_{1/2}.\end{aligned}$$

Thus, we have by (3.3) and (3.6) that

$$\begin{aligned}&2|(P_{N-1}(I_N\tilde{F}(s) - \tilde{F}(s)), \partial_x P_{N-1}\tilde{u}(s))| \\ \le\ & C|I_N\tilde{F}(s) - \tilde{F}(s)|_{1/2}|\tilde{u}(s)|_{1/2} \le CN^{-1/2}|\tilde{F}(s)|_1|\tilde{u}(s)|_{1/2} \\ \le\ & C_F(\|u_c\|_{C^1(I)}, M_1)\|\tilde{u}(s)\|^2_{1/2}.\end{aligned}$$

By (4.15) and the above inequality, we have

$$\begin{aligned}&2|(P_{N-1}I_N\tilde{F}(s), \partial_x P_{N-1}\tilde{u}(s))| \\ \le\ & 2|(P_{N-1}(I_N\tilde{F}(s) - \tilde{F}(s) + \tilde{F}(s)), \partial_x P_{N-1}\tilde{u}(s))| \\ \le\ & C_F(\|u_c\|_{C^1(I)}, M_1)\|\tilde{u}(s)\|^2_{1/2}.\end{aligned}$$

Therefore, by (4.13) we have

$$\|\tilde{u}(t)\|^2 \le \|\tilde{u}(0)\|^2 + C_F(\|u_c\|_{C(0,T;C^1_p(I))}, M_1)\int_0^t \left(\|I_N\tilde{f}(s)\|^2 + \|\tilde{u}(s)\|^2_{1/2}\right)\mathrm{d}s. \tag{4.16}$$

Combining (4.12) with (4.16) yields

$$\|\tilde{u}(t)\|^2_{1/2} \le C\left\{\|\tilde{u}(0)\|^2_{1/2} + \int_0^t \left(\|I_N\tilde{f}(s)\|^2_{1/2} + \|\tilde{u}(s)\|^2_{1/2}\right)\mathrm{d}s\right\}. \tag{4.17}$$

It remains to estimate $\|\partial_t\tilde{u}(t)\|_{1/2}$ for completing the stability analysis. For this, assume that $\partial_t\tilde{f}(t)$ exists. Differentiating (4.2) with respect to t yields

$$(\partial_t^2\tilde{u}(t) + \partial_x I_N\partial_t\tilde{F}(t) + \mathscr{H}\partial_x^2\partial_t\tilde{u}(t) - \partial_t\tilde{f}(t), v)_N = 0. \tag{4.18}$$

Third, taking $v = I_N\partial_t\tilde{F}(t) + \mathscr{H}\partial_x\partial_t\tilde{u}(t)$ in (4.18) and integrating in time yield

$$\begin{aligned}|\partial_t\tilde{u}(t)|^2_{1/2} \le |\partial_t\tilde{u}(0)|^2_{1/2} + 2\int_0^t \Big\{ &\left(\partial_x I_N\partial_s\tilde{f}(s), \mathscr{H}\partial_s\tilde{u}(s)\right) \\ &+ \left(\partial_s^2\tilde{u}(s) - I_N\partial_s\tilde{f}(s), I_N\partial_s\tilde{F}(s)\right)\Big\}\mathrm{d}s,\end{aligned} \tag{4.19}$$

which can pass into

$$\begin{aligned}|\partial_t\tilde{u}(t)|^2_{1/2} \le\ & |\partial_t\tilde{u}(0)|^2_{1/2} + C_F(\|u_c\|_{C^1(0,T;C(I))}, M_1)\int_0^t \left(\|\tilde{u}(s)\|^2 + \|\partial_s\tilde{u}(s)\|^2_{1/2}\right)\mathrm{d}s \\ &+ \int_0^t \left(\|I_N\partial_s\tilde{f}(s)\|^2_{1/2} + (\partial_s^2\tilde{u}(s), I_N\partial_s\tilde{F}(s))\right)\mathrm{d}s.\end{aligned}$$

Denote $\partial_s\tilde{F}=G\partial_s\tilde{u}+G_1\tilde{u}$, where $G=F'(u_c+\tilde{u}), G_1=\int_0^1 F''(u_c+\theta\tilde{u})\,d\theta\,\partial_s u_c$. By integration by parts in time, we have

$$
\begin{aligned}
&2\Big|\int_0^t(\partial_s^2\tilde{u}(s),I_N\partial_s\tilde{F}(s))\,\mathrm{d}s\Big|\\
=&2\Big|\int_0^t(\partial_s^2\tilde{u}(s),G(s)\partial_s\tilde{u}(s)+G_1(s)\tilde{u}(s))_N\,\mathrm{d}s\Big|\\
=&\Big|(\partial_s\tilde{u}(s),G(s)\partial_s\tilde{u}(s))_N\Big|_0^t-\int_0^t(\partial_s\tilde{u}(s),\partial_s G(s)\partial_s\tilde{u}(s))_N\,ds\\
&+2(\partial_s\tilde{u}(s),G_1(s)\tilde{u}(s))_N\Big|_0^t-2\int_0^t(\partial_s\tilde{u}(s),\partial_s G_1(s)\tilde{u}(s)+G_1(s)\partial_s\tilde{u}(s))_N\,\mathrm{d}s\Big|\\
\le& C_F(\|u_c\|_{C^2(0,T;C(I))},M_1)\Big\{\|\tilde{u}(t)\|^2+\|\partial_t\tilde{u}(t)\|^2+\|\tilde{u}(0)\|^2+\|\partial_t\tilde{u}(0)\|^2\\
&+\int_0^t\left(\|\tilde{u}(s)\|^2+\|\partial_s\tilde{u}(s)\|^2\right)\,\mathrm{d}s\Big\},
\end{aligned}
$$

where the terms $G,\partial_s G, G_1$ and $\partial_s G_1$ can be bounded in the maximal norm by the same argument as in [4]. Note that if $u\in H^1(0,T;L^2(I))$, then

$$
\|u(t)\|^2\le 2\|u(0)\|^2+2T\int_0^t\|\partial_s u(s)\|^2\mathrm{d}s. \tag{4.20}
$$

It follows that

$$
\begin{aligned}
|\partial_t\tilde{u}(t)|_{1/2}^2\le& |\partial_t\tilde{u}(0)|_{1/2}^2+C_F(\|u_c\|_{C^2(0,T;C(I))},M_1)\Big\{\|\partial_t\tilde{u}(t)\|^2+\|\tilde{u}(0)\|^2\\
&+\|\partial_t\tilde{u}(0)\|^2+\int_0^t\left(\|I_N\partial_s\tilde{f}(s)\|_{1/2}^2+\|\partial_s\tilde{u}(s)\|_{1/2}^2\right)\,\mathrm{d}s\Big\}.
\end{aligned} \tag{4.21}
$$

Forth, taking $v=\partial_t\tilde{u}(t)$ in (4.18) and integrating in time yield

$$
\begin{aligned}
&\|\partial_t\tilde{u}(t)\|^2\\
=&\|\partial_t\tilde{u}(0)\|^2+2\int_0^t\left((I_N\partial_s\tilde{f}(s),\partial_s\tilde{u}(s))+(P_{N-1}I_N\partial_s\tilde{F}(s),\partial_x P_{N-1}\partial_s\tilde{u}(s))\right)\,\mathrm{d}s\\
\le&\|\partial_t\tilde{u}(0)\|^2+2\int_0^t\left(\|I_N\partial_s\tilde{f}(s)\|^2+\|\partial_s\tilde{u}(s)\|^2+(P_{N-1}I_N\partial_s\tilde{F}(s),\partial_x P_{N-1}\partial_s\tilde{u}(s))\right)\,\mathrm{d}s.
\end{aligned}
$$

By (3.8) and (3.7), we have

$$
\begin{aligned}
&2|(P_{N-1}\partial_s\tilde{F}(s),\partial_x P_{N-1}\partial_s\tilde{u}(s))|\\
\le& C\left(|G(s)\partial_s\tilde{u}(s)|_{1/2}^2+|G_1(s)\tilde{u}(s)|_{1/2}^2+|\partial_s\tilde{u}(s)|_{1/2}^2\right)\\
\le& C\left(\|G_1(s)\|_1^2\|\tilde{u}(s)\|_{1/2}^2+\|G(s)\|_1^2\|\partial_s\tilde{u}(s)\|_{1/2}^2\right)\\
\le& C_F(\|u_c\|_{C^1(0,T;H_p^1(I))},M_1)\left(\|\tilde{u}(s)\|_{1/2}^2+\|\partial_s\tilde{u}(s)\|_{1/2}^2\right),
\end{aligned} \tag{4.22}
$$

where the terms G, G_1 can be bounded in H^1-norm by the same argument as in [4]. It follows from (4.3), (4.4) and (3.6) that

$$
\begin{aligned}
|\partial_s \tilde{F}|_1 &= |G\partial_s\partial_s\tilde{u} + G_1\tilde{u}|_1 \\
&= \|\partial_x G\partial_s\tilde{u} + G\partial_x\partial_s\tilde{u} + \partial_x G_1\tilde{u} + G_1\partial_x\tilde{u}\| \\
&\le C_F(\|u_c\|_{C^1(\bar{I}\times[0,T])}, M_1)N^{1/2}\left(\|\tilde{u}\|_{1/2} + \|\partial_s\tilde{u}\|_{1/2}\right).
\end{aligned}
$$

Then by (3.3) and (3.6) we have

$$
\begin{aligned}
&2|(P_{N-1}(I_N\partial_s\tilde{F}(s) - \partial_s\tilde{F}(s)), \partial_x P_{N-1}\partial_s\tilde{u}(s))| \\
\le\ & C|I_N\partial_s\tilde{F}(s) - \partial_s\tilde{F}(s)|_{1/2}|\partial_s\tilde{u}(s)|_{1/2} \\
\le\ & CN^{-1/2}|\partial_s\tilde{F}(s)|_1|\partial_s\tilde{u}(s)|_{1/2} \\
\le\ & C_F(\|u_c\|_{C^1(\bar{I}\times[0,T])}, M_1)\left(\|\tilde{u}\|_{1/2}^2 + \|\partial_s\tilde{u}\|_{1/2}^2\right).
\end{aligned}
$$

Similar to (4.20), we have that

$$
\begin{aligned}
\|\partial_t\tilde{u}(t)\|^2 \le\ & C_F(\|u_c\|_{C^1(\bar{I}\times[0,T])}, M_1) \\
&\cdot\left\{\|\tilde{u}(0)\|^2 + \|\partial_t\tilde{u}(0)\|^2 + \int_0^t \left(\|I_N\partial_s\tilde{f}(s)\|^2 + \|\partial_s\tilde{u}(s)\|_{1/2}^2\right) \mathrm{d}s\right\}.
\end{aligned}
\tag{4.23}
$$

Combining (4.21) with (4.23) yields

$$
\|\partial_t\tilde{u}(t)\|_{1/2}^2 \le C\left\{\|\tilde{u}(0)\|^2 + \|\partial_t\tilde{u}(0)\|_{1/2}^2 + \int_0^t \left(\|I_N\partial_s\tilde{f}(s)\|_{1/2}^2 + \|\partial_s\tilde{u}(s)\|_{1/2}^2\right) \mathrm{d}s\right\}. \tag{4.24}
$$

Define

$$
\begin{aligned}
&E_1(t) = \|\tilde{u}(t)\|_{1/2}^2, \quad \rho_1(t) = C\left(\|\tilde{u}(0)\|_{1/2}^2 + \int_0^t \|I_N\tilde{f}(s)\|_{1/2}^2\,\mathrm{d}s\right), \\
&E_2(t) = \|\partial_t\tilde{u}(t)\|_{1/2}^2, \quad \rho_2(t) = C\left(\|\tilde{u}(0)\|^2 + \|\partial_t\tilde{u}(0)\|_{1/2}^2 + \int_0^t \|I_N\partial_s\tilde{f}(s)\|_{1/2}^2\,\mathrm{d}s\right).
\end{aligned}
$$

By (4.17), (4.24) and Lemma 3.4, we obtain the following stability result.

Theorem 4.1. *Suppose that $F(z) \in C^3(\mathbb{R})$. Then there exists a constant C depending on $\|u_c\|_{C^1(\bar{I}\times[0,T])}$ and $\|u_c\|_{C^2(0,T;C(I))}$, such that if*

$$
\rho_i(t) \le M_1^2 \mathrm{e}^{-CT}(2N+1)^{-1}, \qquad i = 1, 2,
$$

then

$$
E_i(t) \le \rho_i(t)\mathrm{e}^{Ct}, \qquad 0 < t \le T.
$$

Next we consider the convergence of scheme (4.1). Let $e(t)=u_c(t)-u^*(t)$, where $u^*(t)=P_NU(t)$. From (1.1), (4.1) and (3.10), we have

$$\begin{cases} (\partial_t e(t)+\partial_x I_N\tilde{F}(t)+\mathscr{H}\partial_x^2 e(t)-\tilde{f}(t),v)_N=0, \qquad 0<t\le T, \\ e(0)=u_c(0)-u^*(0), \end{cases} \tag{4.25}$$

where

$$\tilde{F}=F(u_c)-F(u^*), \qquad \tilde{f}=\partial_x[P_NF(U)-I_NF(u^*)].$$

In terms of the stability analysis, we need to bound the terms $\|I_N\tilde{f}(s)\|_{1/2}$ and $\|I_N\partial_t\tilde{f}(s)\|_{1/2}$. By (3.1), we have

$$\begin{aligned} \|F(U)-F(u^*)\| &\le \left\| \int_0^1 F'(\theta U+(1-\theta)u^*)(U-u^*)\,\mathrm{d}\theta \right\| \\ &\le \max_{x\in I}\int_0^1 |F'(\theta U+(1-\theta)u^*)|\,\mathrm{d}\theta \|U-P_NU\| \\ &\le \max_{|z|\le\|U(s)\|_1} |\partial_z F(z)|CN^{-r}|U|_r. \end{aligned} \tag{4.26}$$

It follows from (3.10) that

$$(\tilde{f},v)_N=(P_{N-1}\partial_x[P_NF(U)-I_NF(u^*)],v).$$

Thus, by (3.6), (3.1) and (3.3), we obtain

$$\begin{aligned} &\|I_N\tilde{f}(s)\|_{1/2}\le CN^{3/2}\|P_NF(U)-I_NF(u^*)\| \\ &\le CN^{3/2}\Big\{\|P_NF(U)-F(U)\|+\|F(U)-F(u^*)\|+\|F(u^*)-I_NF(u^*)\|\Big\} \\ &\le CN^{3/2-r}. \end{aligned} \tag{4.27}$$

Let $r\ge r_1\ge 2$. A derivation analogous to the above analysis leads to

$$\begin{aligned} &\|\partial_t F(U)-\partial_t F(u^*)\| \\ &\le \left\| \int_0^1 F''(\theta U+(1-\theta)u^*)\partial_t(\theta U+(1-\theta)u^*)(U-u^*)\,\mathrm{d}\theta \right\| \\ &\quad + \left\| \int_0^1 F'(\theta U+(1-\theta)u^*)\,\mathrm{d}\theta\,\partial_t(U-P_NU) \right\| \\ &\le \|\partial_t U(s)\|_{C(I)} \max_{|z|\le\|U(s)\|_1}|\partial_z^2F(z)|CN^{-r}|U|_r+\max_{|z|\le\|U(s)\|_1}|\partial_zF(z)|CN^{-r_1}|\partial_tU|_{r_1}, \end{aligned}$$

and

$$\begin{aligned} &\|I_N\partial_t\tilde{f}(s)\|_{1/2} \\ &\le CN^{3/2}\|P_N\partial_tF(U)-I_N\partial_tF(u^*)\| \\ &\le CN^{3/2}\Big(\|P_N\partial_tF(U)-\partial_tF(U)\|+\|\partial_tF(U)-\partial_tF(u^*)\|+\|\partial_tF(u^*)-I_N\partial_tF(u^*)\|\Big) \\ &\le CN^{3/2-r_1}. \end{aligned}$$

For the initial errors, we have by (4.25) that $e(0) = 0$, and by (4.27),

$$\|\partial_t e(0)\|_{1/2} = \|I_N \tilde{f}(0)\|_{1/2} \leq CN^{3/2-r}.$$

Theorem 4.2. *Suppose* $r \geq r_1 \geq 2$, $F(z) \in C^3(\mathbb{R}) \cap C^{r+1}(\mathbb{R})$, $U \in C(0,T;H_p^r(I))$, $\partial_t U \in C(0,T;H_p^{r_1}(I))$, *and* $\partial_t^2 U \in C(\bar{I} \times [0,T])$. *Then there exists a constant* C *depending on the regularities of* U *and* F *such that for any* $0 \leq t \leq T$,

$$\|u_c(t) - U(t)\|_{1/2} \leq CN^{3/2-r},$$
$$\|\partial_t u_c(t) - \partial_t U(t)\|_{1/2} \leq CN^{3/2-r_1}.$$

5. A numerical example

We simulate the periodic soliton solution of the BO equation by the FC method (2.2). Consider the periodic BO equation:

$$\partial_t U + U\partial_x U + \mathscr{H}\partial_x^2 U = 0, \ -L < x < L, \ t > 0,$$

with a soliton solution of period $2L$ [3]:

$$U(x,t) = \frac{2c\delta^2}{1 - \sqrt{1-\delta^2}\cos(c\delta(x-ct))}, \quad \delta = \frac{\pi}{cL}.$$

The problem is computed by the FC method (2.2) with the parameters $c = 0.25$ and $L = 40$. The results are given in Table 1 with various τ and N, where L^2-error and $H^{1/2}$-error are computed approximately by using $I_N U$ instead of U. The results show that the method is of spectral accuracy in space and of second order convergence in time. Note that with N suitably large, the FC method gets the results of almost the same accuracy as the FG method [4].

Table 1: Errors at $t = 100$ of the FC method.

τ	$2N$	L^2-error	L^∞-error	$H^{1/2}$-error	L^2-order	L^∞-order	$H^{1/2}$-order
1e-1	128	9.1938e-5	2.9987e-4	7.0887e-4			
1e-2		9.1873e-7	3.0025e-6	7.0842e-6	$\tau^{2.00}$	$\tau^{2.00}$	$\tau^{2.00}$
1e-3		9.1863e-9	2.9874e-8	7.0830e-8	$\tau^{2.00}$	$\tau^{2.00}$	$\tau^{2.00}$
1e-3	32	1.0566e-2	2.2714e-2	3.0962e-2			
	64	2.9173e-5	5.5836e-5	9.0323e-5	$N^{-8.50}$	$N^{-8.67}$	$N^{-8.42}$
	128	9.1863e-9	2.9874e-8	7.0830e-8	$N^{-11.63}$	$N^{-10.87}$	$N^{-10.32}$

Acknowledgments The authors would like to thank the referees for their helpful suggestions. The work is supported by NSF of China (60874039) and Leading Academic Discipline Project of Shanghai Municipal Education Commission (J50101).

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