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# A-Posteriori Error Estimation for the Legendre Spectral Galerkin Method in One-Dimension

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**Abstract.** In this paper, a-posteriori error estimators are proposed for the Legendre spectral Galerkin method for two-point boundary value problems. The key idea is to postprocess the Galerkin approximation, and the analysis shows that the postprocess improves the order of convergence. Consequently, we obtain asymptotically exact a-posteriori error estimators based on the postprocessing results. Numerical examples are included to illustrate the theoretical analysis.

**AMS subject classifications**: 65N35, 65N30

**Key words**: Legendre spectral Galerkin method, two-point boundary value problem, superconvergence, a-posteriori error estimation.

## 1. Introduction

A-posteriori error estimation has now become an accepted and even expected tool in modern scientific computing. Many a-posteriori error estimators have been developed for low order finite element methods (FEM) (see, e.g., [1, 4, 27], and references therein), which are mainly based on the residual method [2, 3, 5, 8] or on the recovery method [35, 36].

In contrast to the low order FEM ($h$-FEM), a-posteriori error estimation for high order methods such as the spectral methods, the $p$-version FEM and the $hp$-version FEM is much less developed and lacks of substantial progress in the past two decades. There are only few papers on this topic in the current literature, see, e.g., [9, 12–16, 22–24].

In the present paper, we develop a-posteriori error estimation for the Legendre spectral Galerkin method [10, 13, 25, 29] for a certain class of two-point boundary value problems. We first construct a semi-$H^1$ projection which plays an important role in the analysis of

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high order methods in one space dimension, and investigate it's approximation property. Following the classical superconvergence analysis of the $h$-FEM [4,11,21,28,32,34], some superconvergence results are obtained. Then, we propose a postprocessing technique to enhance the Legendre spectral Galerkin method. It is proved that the postprocess improves the order of convergence of the Galerkin approximation. Actually, the postprocessing technique is essentially a correction for the Galerkin approximation, and the correction scheme (4.5) in section 4 shows that the correction quantities to the numerical solution can be expressed by a sum of some higher order polynomials, and the overcost of the postprocessing procedure is nearly negligible. Finally, it is possible to define recovery-based a-posteriori error estimators that are asymptotically exact by using the postprocessing results.

This paper is organized as follows: in the next section we present the model problem and construct the Legendre spectral Galerkin approximation scheme. In Section 3, we investigate the approximation properties of the semi-$H^1$ projection. In Section 4, we propose a postprocessing technique for the Galerkin approximation and the asymptotically exact a-posteriori error estimators are analyzed. The analytical results are illustrated by numerical examples in Section 5. We summarize the work and also discuss some possible future works in the last section.

Let $I \subset \mathbb{R}$ be an open and bounded interval. In this paper, we adopt the standard notation $W^{m,q}(I)$ for Sobolev spaces on $I$ with the norm $\|\cdot\|_{m,q}$ and the seminorm $|\cdot|_{m,q}$. In addition, We denote $W^{m,2}(I)$, $W_0^{m,2}(I)$ by $H^m(I)$, $H_0^m(I)$, respectively. Hereafter, we denote by $C$ a generic positive constant independent of any function and $N$, the order of the Galerkin approximation.

## 2. Legendre spectral Galerkin method

We consider the following two-point boundary value problem

$$\begin{cases} -u''(x)+b(x)u(x)=f(x), & \text{in } I=(-1,1), \\ u(-1)=u(1)=0, \end{cases} \tag{2.1}$$

with $b(x)\geq 0$, and we assume that $b$ and $f$ are sufficiently smooth for our analysis.

The weak form of (2.1) is to find $u\in H_0^1(I)$ such that

$$a(u,v)=(f,v), \quad \forall v\in H_0^1(I),$$

where

$$a(u,v)=\int_I (u'v'+buv)dx, \quad (f,v)=\int_I f v dx.$$

Let

$$L_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n], \quad n\geq 0$$

be the Legendre polynomials which form an orthogonal basis of $L^2[-1,1]$. For the properties of Legendre polynomials see [26]. We define

$$\begin{aligned}\phi_0(x) &= \frac{1-x}{2} = \frac{L_0(x)-L_1(x)}{2},\\ \phi_1(x) &= \frac{1+x}{2} = \frac{L_0(x)+L_1(x)}{2},\\ \phi_{k+1}(x) &= \sqrt{\frac{2k+1}{2}}\int_{-1}^{x} L_k(\xi)d\xi, \quad k\geq 1.\end{aligned} \tag{2.2}$$

The following properties are valid

$$\phi_{k+1}(x) = \sqrt{\frac{1}{2(2k+1)}}(L_{k+1}(x)-L_{k-1}(x)), \quad k\geq 1, \tag{2.3}$$

$$\phi_{k+1}(x) = -\sqrt{\frac{2k+1}{2}}\frac{1}{k(k+1)}(1-x^2)L_k'(x), \quad k\geq 1. \tag{2.4}$$

Let $V_N = span\{\phi_0,\phi_1\cdots\phi_N\}$. The Legendre spectral Galerkin method is to find $u_N \in V_N^0 = V_N \cap H_0^1(I)$, such that

$$a(u_N,v) = (f,v), \quad \forall v\in V_N^0.$$

We end this section by listing some properties of $L_j$ ad $\phi_j$ which will be used later:

$$\|L_j\|^2_{L^2(I)} = \frac{2}{2j+1}, \quad j\geq 0, \tag{2.5}$$

$$\|\phi_j\|^2_{L^2(I)} = \frac{2}{(2j+1)(2j-3)}, \quad j\geq 2, \tag{2.6}$$

$$\phi_j(\pm 1) = 0, \quad \phi_j'(x) = \sqrt{\frac{2j-1}{2}}L_{j-1}(x), \quad j\geq 2. \tag{2.7}$$

The proof of the above results is straightforward and we only verify the second one. In fact, by (2.3) and (2.5),

$$\|\phi_j\|^2_{L^2(I)} = \frac{1}{2(2j-1)}\Big(\frac{2}{2j+1}+\frac{2}{2j-3}\Big) = \frac{2}{(2j+1)(2j-3)}.$$

In the following section, we shall investigate the approximation properties of a semi-$H^1$ projection.

## 3. Semi-$H^1$ projection and super-approximation

Suppose that $u\in H^1(I)$. Then $u'\in L^2(I)$, and consequently we have Legendre expansion

$$u'(x) = \sum_{k=0}^{\infty}\alpha_k L_k(x)$$

with $\alpha_k = (k+\frac{1}{2})(u', L_k)_I$, $k \geq 0$. It follows that

$$u(x) = \sum_{j=0}^{\infty} \beta_j \phi_j(x), \quad \forall x \in I,$$

where $\phi_j$ is defined by (2.2) and

$$\begin{cases} \beta_0 = u(-1), \quad \beta_1 = u(1), \\ \beta_{j+1} = \sqrt{\dfrac{2j+1}{2}}(u', L_j), \quad j \geq 1. \end{cases} \tag{3.1}$$

Let $\Pi_N^0 u$ be the Legendre projection which satisfies

$$(u - \Pi_N^0 u, v) = 0, \quad \forall v \in P_N,$$

where $P_N$ stands for the set of all polynomials of degree at most $N$. We define the operator

$$\Pi_N : H^1(I) \to V_N(I), \quad \Pi_N u(x) = \sum_{j=0}^{N} \beta_j \phi_j(x), \quad \forall x \in I. \tag{3.2}$$

Obviously, there always holds

$$(\Pi_N u)' = \Pi_{N-1}^0 u', \tag{3.3}$$

which implies that $\Pi_N$ is a semi-$H^1$ projection operator.

**Lemma 3.1.** *The operator $\Pi_N$ has the following properties:*

1) $\Pi_N u = u, \ \forall u \in P_N, \quad \Pi_N u(\pm 1) = u(\pm 1).$
2) $(u - \Pi_N u, v)_I = 0, \ \forall v \in P_{N-2}, \quad ((u - \Pi_N u)', v)_I = 0, \ \forall v \in P_{N-1}.$

*Proof.* From the definition of $\Pi_N$ in (3.2), we can verify these properties directly. □

To derive approximation results, we introduce the Legendre-weighted Sobolev space $H^{k,0}(I)$ furnished with the norm

$$||u||_{H^{k,0}(I)}^2 = \sum_{l=0}^{k} \int_I (1-x^2)^l |u^{(l)}(x)|^2 dx.$$

Moreover, $|\cdot|_{H^{k,0}(I)}$ is a semi-norm involving the $k$-th derivative only. Actually, $H^{k,0}(I)$ is a special case of the Jacobi-weighted Sobolev space $H^{k,\beta}(I)$ with $\beta = 0$ (cf. [6, 7, 17–20]).

Note that it is always true that for $k \geq 0$

$$||u||_{H^{k,0}(I)} \leq ||u||_{H^k(I)}. \tag{3.4}$$

**Lemma 3.2.** *Let $u \in H^{k+1}(I)$ with $k \geq 0$. Then we have the following estimates*

$$\|u-\Pi_N u\|_m \leq CN^{-(k+1-m)}\|u'\|_{H^{k,0}(I)} \leq CN^{-(k+1-m)}\|u\|_{k+1}, \quad m=0,1. \tag{3.5}$$

*Proof.* Let $\Pi^0_{N-1}u' = \sum_{k=0}^{N-1}\alpha_k L_k(x)$ be the Legendre projection of $u'(x)$ on $P_{N-1}$. It can be verified [17] that for $l \leq k$

$$\|u'-\Pi^0_{N-1}u'\|_{H^{l,0}(I)} \leq CN^{-(k-l)}\|u'\|_{H^{k,0}(I)}. \tag{3.6}$$

Recalling the fact that for any integers $n \geq 0$ and $k \geq 0$ there holds [10]

$$\int_{-1}^{1}(1-x^2)L_n'(x)L_k'(x)dx = n(n+1)\int_{-1}^{1}L_n(x)L_k(x)dx,$$

which, together with (2.4) and (3.1), gives

$$\begin{aligned}
\|u-\Pi_N u\|_0^2 &= \|\sum_{k=N+1}^{\infty}\beta_k\phi_k\|_0^2 = \|\sum_{k=N+1}^{\infty}\sqrt{\frac{2}{2k-1}}\alpha_{k-1}\phi_k\|_0^2 \\
&\leq \sum_{k=N+1}^{\infty}\frac{2}{(k-1)k(2k-1)}|\alpha_{k-1}|^2 = \sum_{k=N}^{\infty}\frac{2}{k(k+1)(2k+1)}|\alpha_k|^2 \\
&\leq \sum_{k=N}^{\infty}\frac{1}{k^2}\frac{2}{(2k+1)}|\alpha_k|^2 \leq N^{-2}\|u'-\Pi^0_{N-1}u'\|_0^2 \\
&\leq CN^{-2(k+1)}\|u'\|^2_{H^{k,0}(I)}.
\end{aligned}$$

The above results, together with (3.3) and (3.6), implies (3.5). □

An important weak error estimate will be given as follows.

**Theorem 3.1.** *Let $N \geq 2$. Then the following basic estimates hold for $m = 1,2$*

$$|a(u-\Pi_N u, v)| \leq CN^{-(k+1+m)}\|u\|_{k+1}\|v\|_m, \quad \forall v \in V_N^0. \tag{3.7}$$

*Proof.* With the aid of Lemma 3.1, we have

$$a(u-\Pi_N u, v) = (u-\Pi_N u, bv), \quad \forall v \in V_N^0(I). \tag{3.8}$$

Let $w = bv$. It follows from Lemmas 3.1 and 3.2 that for $N \geq 2$

$$\begin{aligned}
|(u-\Pi_N u, w)| &= |(u-\Pi_N u, w-\Pi_{N-2}w)| \\
&\leq \|u-\Pi_N u\|_0\|w-\Pi_{N-2}w\|_0 \\
&\leq CN^{-(k+1)}\|u\|_{k+1}N^{-m}\|w'\|_{H^{m-1,0}(I)} \\
&\leq CN^{-(k+1)}N^{-m}\|u\|_{k+1}\|w'\|_{m-1,I} \\
&\leq CN^{-(k+1+m)}\|u\|_{k+1}\|v\|_m.
\end{aligned} \tag{3.9}$$

Consequently, the desired result follows from (3.8). □

**Corollary 3.1.** *Let the assumptions of Theorem* 3.1 *hold. Then,*

$$|a(u-\Pi_N u,\Pi_N\phi)|\le CN^{-(k+3)}\|u\|_{k+1}\|\phi\|_2,\quad \forall\phi\in H^2(I)\cap H_0^1(I). \tag{3.10}$$

*Proof.* By inspecting the proof of Theorem 3.1, it suffices to prove

$$\|(b\Pi_N\phi)'\|_{H^{1,0}(I)}\le C\|\phi\|_2. \tag{3.11}$$

It follows from (3.4)-(3.6) that

$$\begin{aligned}
&\|(b\Pi_N\phi)'\|_{H^{1,0}(I)}=\|b'\Pi_N\phi+b\Pi^0_{N-1}\phi'\|_{H^{1,0}(I)}\\
\le{}& C\Big(\|\Pi_N\phi\|_{H^{1,0}(I)}+\|\Pi^0_{N-1}\phi'\|_{H^{1,0}(I)}\Big)\\
\le{}& C\Big(\|\Pi_N\phi-\phi\|_{H^{1,0}(I)}+\|\phi\|_{H^{1,0}(I)}+\|\Pi^0_{N-1}\phi'-\phi'\|_{H^{1,0}(I)}+\|\phi'\|_{H^{1,0}(I)}\Big)\\
\le{}& C\Big(\|\phi\|_{H^{1,0}(I)}+\|\phi'\|_{H^{1,0}(I)}\Big)\le C\|\phi\|_2,
\end{aligned}$$

which completes the proof of (3.11). □

**Corollary 3.2.** *Let the assumptions of Theorem* 3.1 *hold. If* $b(x)=0$*, then,*

$$a(u-\Pi_N u,v)=0,\quad \forall v\in V_N^0,\quad N\ge 1.$$

*Consequently,* $\Pi_N u=u_N$*, where* $u_N$ *is the corresponding Galerkin approximation.*

The main result in this section is presented in the following theorem. It is shown that the Galerkin approximation $u_N$ is a better approximation to the semi-$H^1$ projection than to the solution $u$ itself.

**Theorem 3.2.** *Let* $u_N$ *be the Galerkin approximation to Eq.* (2.1)*, and let* $N\ge 2$*. Then we have the super-approximation estimates*

$$\|u_N-\Pi_N u\|_m\le CN^{-(k+3-m)}\|u\|_{k+1},\quad m=0,1. \tag{3.12}$$

*Proof.* Using Theorem 3.1 gives

$$\begin{aligned}
\|u_N-\Pi_N u\|_1^2&\le Ca(u_N-\Pi_N u,u_N-\Pi_N u)\\
&=Ca(u-\Pi_N u,u_N-\Pi_N u)\\
&\le CN^{-(k+2)}\|u\|_{k+1}\|u_N-\Pi_N u\|_1
\end{aligned}$$

which implies that (3.12) holds for $m=1$. For $m=0$, we consider the auxiliary problem:

$$\begin{cases}-\phi''(x)+b(x)\phi(x)=\varphi(x), & \text{in } I=(-1,1),\\ \phi(-1)=\phi(1)=0,\end{cases}$$

with $\varphi\in L^2(I)$. Then the equation has a unique solution $\phi\in H^2(I)\cap H_0^1(I)$, and

$$\|\phi\|_2\le C\|\varphi\|_0. \tag{3.13}$$

Note that $\phi$ satisfies the variational equation

$$a(\phi,v)=(\varphi,v),\quad \forall v\in H_0^1(I). \tag{3.14}$$

Let $\phi_N \in V_N^0$ be the Galerkin approximation to (3.14). Applying Theorem 3.1 and (3.12) with $m=1$, we obtain

$$\begin{aligned}
&a(\phi-\Pi_N\phi,u_N-\Pi_Nu)\\
&=a(\phi_N-\Pi_N\phi,u_N-\Pi_Nu)\\
&=a(\phi_N-\Pi_N\phi,u-\Pi_Nu)\\
&\le CN^{-(k+2)}\|u\|_{k+1}\|\Pi_N\phi-\phi_N\|_1\\
&\le CN^{-(k+5)}\|u\|_{k+1}\|\phi\|_2.
\end{aligned}\tag{3.15}$$

Let $\varphi=|u_N-\Pi_Nu|sgn(u_N-\Pi_Nu)\in L^2(I)$. Combining (3.13), (3.15) and Corollary 3.1, we obtain

$$\begin{aligned}
\|u_N-\Pi_Nu\|_0^2&=(\varphi,u_N-\Pi_Nu)=a(\phi,u_N-\Pi_Nu)\\
&=a(\phi-\Pi_N\phi,u_N-\Pi_Nu)+a(\Pi_N\phi,u_N-\Pi_Nu)\\
&=a(\phi-\Pi_N\phi,u_N-\Pi_Nu)+a(u-\Pi_Nu,\Pi_N\phi)\\
&\le CN^{-(k+5)}\|u\|_{k+1}\|\phi\|_2+CN^{-(k+3)}\|u\|_{k+1}\|\phi\|_2\\
&\le CN^{-(k+3)}\|u\|_{k+1}\|\varphi\|_0,
\end{aligned}\tag{3.16}$$

which is (3.12) for $m=0$. This completes the proof of the theorem. □

## 4. The postprocessed method and a-posteriori error estimation

Under certain regularity assumptions on the exact solution, [30,31] considered spectral collocation methods and the $p$-version FEM for two-point boundary value problems, and obtained some natural superconvergent points. In this section, we shall propose a postprocessed method to enhance the spectral Galerkin approximation. A similar postprocessing technique was first developed for the $h$-version FEM in [33], where global superconvergence results are obtained. Based on the postprocessing results we can define a-posteriori error estimators that are asymptotically exact.

Let $u\in H^{k+1}(I)$ be the exact solution of Eq. (2.1). Then

$$u''=bu-f. \tag{4.1}$$

Note that for $l\ge 2$

$$\begin{aligned}
\beta_l&=\sqrt{l-\frac{1}{2}}\,(u',L_{l-1})\\
&=-(u'',\phi_l)=(f-bu,\phi_l)\\
&=(f-bu_N,\phi_l)+(b(u_N-u),\phi_l).
\end{aligned}\tag{4.2}$$

Let

$$\beta_l^* = (f - bu_N, \phi_l). \tag{4.3}$$

Thanks to the well-known estimate $\|u - u_N\|_0 \le CN^{-(k+1)}\|u\|_{k+1}$, it follows that

$$|\beta_l - \beta_l^*| = |(b(u_N - u), \phi_l)| \le CN^{-(k+1)}\|u\|_{k+1}\|\phi_l\|_0. \tag{4.4}$$

Let

$$u_N^* = u_N + \sum_{l=N+1}^{M} \beta_l^* \phi_l, \quad M \ge N+1. \tag{4.5}$$

Then $u_N^*$ is a corrected value of $u_N$ which can be easily calculated, and it is a high accurate approximation to $u$.

**Theorem 4.1.** *Suppose that $u$ and $u_N$ be the exact solution and the Galerkin approximation to Problem* (2.1)*, respectively. Let $u_N^*$ be the correction value determined by* (4.5)*, and let $M$ be the minimum integer not less than $N^{1+\sigma}$ with $0<\sigma<1$. Then we have the superconvergence estimates*

$$\|u - u_N^*\|_m \le CN^{-(k+1-m)-\alpha_m}\|u\|_{k+1}, \quad m = 0, 1, \tag{4.6}$$

*where $\alpha_0 = \min\{\sigma(k+1), 1\}$ and $\alpha_1 = \min\{\sigma k, 1 - \frac{\sigma}{2}\}$.*

*Proof.* Note that

$$u - u_N^* = (u - \Pi_M u) + (\Pi_N u - u_N) + \Big(\Pi_M u - \Pi_N u - \sum_{l=N+1}^{M} \beta_l^* \phi_l\Big).$$

Using Lemma 3.2, we get

$$\|u - \Pi_M u\|_m \le CM^{-(k+1-m)}\|u\|_{k+1}, \quad m = 0, 1. \tag{4.7}$$

It follows from (2.6) and (4.4) that

$$\begin{aligned} \Big\|\Pi_M u - \Pi_N u - \sum_{l=N+1}^{M} \beta_l^* \phi_l\Big\|_0 &= \Big\|\sum_{l=N+1}^{M} (\beta_l - \beta_l^*)\phi_l\Big\|_0 \\ &\le CN^{-(k+1)}\|u\|_{k+1} \sum_{l=N+1}^{M} \|\phi_l\|_0^2 \\ &\le CN^{-(k+2)}\|u\|_{k+1}. \end{aligned} \tag{4.8}$$

Here we have used the fact that

$$\sum_{l=N+1}^{M} \|\phi_l\|_0^2 \le \sum_{l=N+1}^{M} \frac{1}{l^2} \le \sum_{l=N+1}^{M} \frac{1}{l(l-1)} \le \frac{1}{N}.$$

Combine (4.7), (4.8) and (3.12) to conclude that

$$\begin{aligned}\|u-u_N^*\|_0 &\le \|u-\Pi_M u\|_0 + \|\Pi_N u - u_N\|_0 + \Big\|\sum_{l=N+1}^{M}(\beta_l-\beta_l^*)\phi_l\Big\|_0 \\ &\le CN^{-(k+1)}\Big(N^{-\sigma(k+1)}+N^{-2}+N^{-1}\Big)\|u\|_{k+1} \\ &\le CN^{-(k+1)-\alpha_0}\|u\|_{k+1},\end{aligned}$$

with $\alpha_0=\min\{\sigma(k+1),1\}$, which is $m=0$ for (4.6).

Analogously, by Cauchy-Schwarz inequality we have

$$\begin{aligned}&\Big\|\Pi_M u-\Pi_N u-\sum_{l=N+1}^{M}\beta_l^*\phi_l\Big\|_1=\Big\|\sum_{l=N+1}^{M}(\beta_l-\beta_l^*)\phi_l\Big\|_1 \\ \le\,& CN^{-(k+1)}\|u\|_{k+1}\sum_{l=N+1}^{M}\|\phi_l\|_0\|\phi_l\|_1 \\ \le\,& CN^{-(k+1)}N^{-\frac{1}{2}}\|u\|_{k+1}\Big(\sum_{l=N+1}^{M}\|\phi_l\|_1^2\Big)^{\frac{1}{2}} \\ \le\,& CN^{-(k+\frac{3}{2})}(M-N)^{\frac{1}{2}}\|u\|_{k+1} \\ \le\,& CN^{-(k+1-\frac{\sigma}{2})}\|u\|_{k+1}.\end{aligned} \tag{4.9}$$

Combine (4.7), (4.9) and (3.12) to conclude that

$$\begin{aligned}\|u-u_N^*\|_1 &\le \|u-\Pi_M u\|_1 + \|\Pi_N u - u_N\|_1 + \Big\|\sum_{l=N+1}^{M}(\beta_l-\beta_l^*)\phi_l\Big\|_1 \\ &\le CN^{-k}(N^{-\sigma k}+N^{-2}+N^{\frac{\sigma}{2}-1})\|u\|_{k+1} \\ &\le CN^{-k-\alpha_1}\|u\|_{k+1}\end{aligned}$$

with $\alpha_1=\min\{\sigma k, 1-\frac{\sigma}{2}\}$, which is $m=1$ for (4.6). □

**Remark 4.1.** Obviously, the best choice of $\sigma\in(0,1)$ is $\sigma=\frac{1}{k+1}$ for $m=0$, and $\sigma=\frac{2}{2k+1}$ for $m=1$ such that the equality $\sigma k=1-\frac{\sigma}{2}$ holds, and then $\alpha_0=1,\alpha_1=\frac{2k}{2k+1}$. Accordingly, we have

$$\|u-u_N^*\|_0\le CN^{-(k+2)}\|u\|_{k+1}, \tag{4.10}$$

$$\|u-u_N^*\|_1\le CN^{-(k+\frac{2k}{2k+1})}\|u\|_{k+1}. \tag{4.11}$$

We now introduce the error estimators $\eta_m, m=0,1$ for the two-point boundary value problems (2.1)

$$\eta_m=\|u_N^*-u_N\|_m=\Big\|\sum_{l=N+1}^{M}\beta_l^*\phi_l\Big\|_m, \tag{4.12}$$

where $\beta_l^*$ are computed as in (4.3). Let $E = u - u_N$. The following result shows that $\eta_m$ are asymptotically exact.

**Theorem 4.2.** *Assume that the hypotheses of Theorem* 4.1 *are satisfied and that*

$$\|E\|_m = \|u - u_N\|_m \geq CN^{-(k+1-m)}, \quad m = 0, 1. \tag{4.13}$$

*Then the error estimators $\eta_m$ are asymptotically exact, i.e.,*

$$\lim_{N\to\infty} \frac{\eta_m}{\|E\|_m} = 1, \quad m = 0, 1. \tag{4.14}$$

*Proof.* Employing the splitting

$$E = (u_N^* - u_N) + (u - u_N^*)$$

and (4.6), we have

$$\|E\|_m - CN^{-(k+1-m)-\alpha_m} \leq \eta_m \ \leq \|E\|_m + CN^{-(k+1-m)-\alpha_m}. \tag{4.15}$$

The desired result follows from (4.13) and (4.15). □

## 5. Numerical experiments

In this section, we present some computational examples to illustrate the preceding discussions and we will focus on the robustness of the proposed error estimators.

The quality of the error estimators is expressed as usual in terms of the effectivity index

$$\theta_m = \frac{\eta_m}{\|E\|_m} = \frac{\|u_N^* - u_N\|_m}{\|u - u_N\|_m}, \quad m = 0, 1.$$

As the first example, we consider the problem

$$\begin{cases} -u'' + bu = f \quad \text{in } I = (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

with the exact solution $u(x) = x^{9/2} - x$ and $b(x) = e^x + 3$. Then $u \in H^{5-\varepsilon}(I) \cap H_0^1(I), \varepsilon > 0$. Obviously, the proposed method can be applied here by a simple scaling. Let $M = N^{6/5}$. Then various norm errors, estimators and effectivity indices are computed with varying $N$ and results are summarized in Table 1, where $\|E\|_0$ and $\|E\|_1$ denote the $L^2$ and $H^1$ norm errors, respectively. It is shown that the error estimators $\eta_m$ are asymptotically sharp, which coincide with the predicted theoretical results.

As the second example, we consider the problem

$$\begin{cases} -u'' + bu = f \quad \text{in } I = (-1,1), \\ u(-1) = u(1) = 0, \end{cases}$$

Table 1: Errors, estimators and effectivity indices for Example 1 with $M = N^{6/5}$.

| $N$ | $\|\|E\|\|_0$ | $\eta_0$ | $\theta_0$ | $\|\|E\|\|_1$ | $\eta_1$ | $\theta_1$ |
|---|---|---|---|---|---|---|
| 3 | 0.0133 | 0.0143 | 1.0790 | 0.2601 | 0.2722 | 1.0464 |
| 4 | 6.8284e-004 | 7.1763e-004 | 1.0510 | 0.0173 | 0.0178 | 1.0287 |
| 5 | 3.2420e-005 | 3.3784e-005 | 1.0421 | 0.0010 | 0.0010 | 1.0182 |
| 6 | 4.2590e-006 | 4.3461e-006 | 1.0204 | 1.6079e-004 | 1.6282e-004 | 1.0126 |
| 7 | 8.6355e-007 | 8.7175e-007 | 1.0095 | 3.8279e-005 | 3.8632e-005 | 1.0092 |
| 8 | 2.2800e-007 | 2.2806e-007 | 1.0043 | 1.1651e-005 | 1.1731e-005 | 1.0068 |
| 9 | 7.2287e-008 | 7.2410e-008 | 1.0017 | 4.1948e-006 | 4.2168e-006 | 1.0052 |
| 10 | 2.6263e-008 | 2.6271e-008 | 1.0003 | 1.7099e-006 | 1.7162e-006 | 1.0037 |

Table 2: Errors, estimators and effectivity indices for Example 2 with $M = N + 2$.

| $N$ | $\|\|E\|\|_0$ | $\eta_0$ | $\theta_0$ | $\|\|E\|\|_1$ | $\eta_1$ | $\theta_1$ |
|---|---|---|---|---|---|---|
| 3 | 0.0918 | 0.1097 | 1.1948 | 0.4714 | 0.5357 | 1.1363 |
| 4 | 0.0146 | 0.0165 | 1.1269 | 0.0962 | 0.1040 | 1.0805 |
| 5 | 0.0018 | 0.0019 | 1.0838 | 0.0143 | 0.0151 | 1.0538 |
| 6 | 1.7762e-004 | 1.8799e-004 | 1.0584 | 0.0017 | 0.0017 | 1.0386 |
| 7 | 1.4664e-005 | 1.5292e-005 | 1.0428 | 1.5704e-004 | 1.6159e-004 | 1.0290 |
| 8 | 1.0394e-006 | 1.0735e-006 | 1.0328 | 1.2582e-005 | 1.2866e-005 | 1.0226 |
| 9 | 6.4554e-008 | 6.6226e-008 | 1.0259 | 8.7183e-007 | 8.8757e-007 | 1.0181 |
| 10 | 3.5673e-009 | 3.6421e-009 | 1.0210 | 5.3187e-008 | 5.3974e-008 | 1.0148 |

with the exact solution $u(x) = (1 - x^2)e^x$ and $b(x) = x^2 + 3$. For this smooth solution, we let $M = N + 2$, and the results are listed in Table 2. Note that the value of $M$ slightly larger than $N$ is enough to improve the accuracy of the approximation, and the overcost of the postprocessing procedure is nearly negligible. It can be also observed from Table 2 that the effectivity indices approach 1 as $N$ increasing, and the excellent properties of the proposed estimators are confirmed again.

## 6. Conclusion

In this paper, we develop a-posteriori error estimation for Legendre spectral Galerkin method for two-point boundary value problems. Applying the postprocessed method, superconvergence properties for the Galerkin approximation are obtained, which play essential role in the analysis of recovery-based a-posteriori error estimation.

In fact, those results obtained and techniques used can be extended to one dimensional parabolic equations and the *hp*-version FEM. However, the generalization of the results to the higher-dimensional tensor product case is not straightforward. It is hoped that we can apply the proposed method to recovery the internal boundary flux in the *p*-version FEM, this in turn can be used to develop residual-based a-posteriori estimation. The work presented in previous sections is just the beginning, and a comprehensive study on a-posteriori error analysis for high order methods, such as the spectral method, the *p*-version FEM and the *hp*-version FEM in two or higher dimensional problems is needed. Nevertheless, it certainly present a new way of thinking, a new approach, and a new direction, which seem encouraging and promising.

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