Numer. Math. Theor. Meth. Appl. Vol. **3**, No. 1, pp. 97-118
doi: 10.4208/nmtma.2009.m9004 February 2010

Bivariate Simplex Spline Quasi-Interpolants

D. Sbibih, A. Serghini and A. Tijini*

University Mohammed I, ESTO, MATSI Laboratory, Oujda, Morocco.

Received 3 February 2009; Accepted (in revised version) 29 June 2009

Available online 30 November 2009

Abstract. In this paper we use the simplex B-spline representation of polynomials or piecewise polynomials in terms of their polar forms to construct several differential or discrete bivariate quasi interpolants which have an optimal approximation order. This method provides an efficient tool for describing many approximation schemes involving values and (or) derivatives of a given function.

AMS subject classifications: 41A05, 41A25, 41A50, 65D05

Key words: Polar form, quasi-interpolation, simplex B-spline.

1. Introduction

The construction of most classical approximants of a given data set or a function usually requires the solving of linear systems. Spline quasi-interpolants are local approximants avoiding this problem, so they are very convenient in practice. In general, a quasi-interpolant of a given function f is obtained as a linear combination of some elements of a suitable set of basis functions. In order to achieve local control, these functions are required to be positive, to ensure the stability and to have small local supports. The coefficients of the linear combination are the values of linear functionals depending on f and (or) its derivatives or integrals.

Various methods for building univariate and bivariate quasi-interpolants have been developed in the literature see for examples [1,5,11,16,22], and references therein. Usually, the appropriate basis functions are B-splines. In the univariate case, these B-splines are well-known and satisfy the above required properties. So the study of the corresponding quasi-interpolants is well developed and they are used for solving several problems in different fields. However, in the bivariate setting, the construction of B-splines which are positive and form a stable basis of spline space is difficult, even impossible except on very special cases, such as the Powell-Sabin basis introduced by P. Dierckx in [13] and the simplex B-spline basis. Several authors have presented collections of simplex B-splines which guarantee such properties. The first collection has been introduced by de

*Corresponding author. *Email addresses:* `sbibih@yahoo.fr` (D. Sbibih), `sergabdel@yahoo.fr` (A. Serghini), `tijiniahmed@yahoo.fr` (A. Tijini)

Boor in [10], which proposes to consider the cylinder or "Slab" $\mathbb{R}^s \times \Omega$, where $\Omega \subset \mathbb{R}^n$ is a suitable convex hull polytope of unit volume, and subdivide this Slab into non trivial simplices. Since then, the above theory has been explored extensively by Dahmen and Micchelli in [7, 8] and by Höllig in [15]. From the point of view of blossoming (i.e., using the polar form), two collections are proposed. The first one, called triangular B-splines (or DMS B-splines), introduced by Dahmen, Micchelli and Seidel in [9]. The second one, studied by Neamtu [20], is the natural generalization of univariate B-splines.

Simplex B-splines are a powerful and flexible class of geometric objects defined over arbitrary, non-rectangular domains. Despite their great potential advantages and their interesting theoretical properties, practical techniques and computational tools with simplex B-splines are less-developed. The first works which use theoretical simplex B-splines in approximation were introduced by W. Dahmen and C. A. Micchelli in [7, 8]. They proposed an approximation process to construct a differential simplex spline quasi-interpolant which generalize the quasi-interpolant introduced by C. de Boor and G. J. Fix [12]. In this paper, we propose a method for constructing discrete and differential simplex spline quasi-interpolants which reproduce the space of polynomials of degree at most n or the whole space of simplex splines associated with a given set of knots. This method (see [23], for instance) is based on the polar form of a chosen local polynomial approximant like local interpolants or other operators having the optimal approximation order. It also requires a collection of simplex B-splines which allows to express any bivariate polynomial p or simplex spline s as a combination of normalized simplex B-splines; and coefficients in the combinations should be given in terms of the polar forms of p or s. This latter condition is only satisfied by the DMS B-splines and Neamtu B-splines. In our work we use DMS B-splines to describe different schemes for differential and discrete quasi-interpolants.

The paper is organized as follows. In Section 2 we give some definitions and properties of bivariate simplex B-splines. In Section 3 we introduce the B-spline representation of all bivariate polynomials or DMS splines over a triangulation Δ of a bounded domain $D \subset \mathbb{R}^2$, in terms of their polar forms. In Section 4 we apply the approach introduced in [23] to DMS B-splines. Then we describe some differential and discrete simplex spline quasi interpolants which reproduce bivariate polynomials and provide the full approximation order in the space of bivariate simplex splines. In Section 5 we give some upper bounds of the infinity norms of some families of discrete quasi-interpolants. Finally, some numerical examples are proposed in Section 6.

2. Bivariate simplex B-splines

For any ordered set of affinely independent points $W = \{w_0, w_1, w_2\} \subset \mathbb{R}^2$ and any point $x \in \mathbb{R}^2$ we define

$$d(W) = det \begin{pmatrix} 1 & 1 & 1 \\ w_0 & w_1 & w_2 \end{pmatrix},$$

$$d_0(W|x) = det\begin{pmatrix} 1 & 1 & 1 \\ x & w_1 & w_2 \end{pmatrix},$$
$$d_1(W|x) = det\begin{pmatrix} 1 & 1 & 1 \\ w_0 & x & w_2 \end{pmatrix},$$
$$d_2(W|x) = det\begin{pmatrix} 1 & 1 & 1 \\ w_0 & w_1 & x \end{pmatrix},$$

where det denotes the determinant operator. The barycentric coordinates of x with respect to the ordered set W are given as follows

$$\lambda_j(x) = \frac{d_j(W|x)}{d(W)}, \quad j = 0,1,2.$$

It is well-known that the scalars $\lambda_j(x)$, $j = 0,1,2$, satisfy the following properties

$$x = \sum_{j=0}^{2} \lambda_j(x) w_j, \quad \text{with} \quad \sum_{j=0}^{2} \lambda_j(x) = 1.$$

Let $V = \{v_0, \cdots, v_m\}$ be an arbitrary set in $\mathbb{R}^2$. We denote by $[V] = [v_0, \cdots, v_m]$ the convex hull of the set V. The simplex B-splines $M(x|V) = M(x|v_0, \cdots, v_m)$ are defined recursively as follows: for $V = \{v_0, v_1, v_2\}$ we set

$$M(x|V) = \frac{\mathscr{X}_{[v_0,v_1,v_2)}(x)}{|d(V)|}, \tag{2.1}$$

where $\mathscr{X}_{[v_0,v_1,v_2)}(x)$ is the characteristic function of the half-open convex hull $[v_0, v_1, v_2)$. For $V = \{v_0, \cdots, v_m\}$, $m > 2$, we set

$$M(x|V) = \sum_{j=0}^{2} \frac{d_j(W|x)}{d(W)} M(x|V \setminus \{v_{i_j}\}), \tag{2.2}$$

where $W = \{v_{i_0}, v_{i_1}, v_{i_2}\}$ is any subset of affinely independent points in V (for more information see [14]).

The function $M(x|V)$ is a positive piecewise polynomial of degree $n = m-2$, supported on the convex hull $[V]$, and is C^{n-1} continuous everywhere. Simplex B-splines and their univariate counterparts share many useful properties (see, e.g., [7, 19]). For examples, the simplex B-splines can be evaluated by the recurrence relation (2.2), which is similar to the well-known recursion for univariate B-splines. Of course, while individual simplex B-splines are mathematically appealing functions in their own right, it is the linear combinations of such splines that are of main interest in modeling and data applications. Thus it is very important to choose an appropriate collection of knots to construct meaningful spaces spanned by bivariate B-splines.

3. Polar form and bivariate simplex spline space

3.1. Polar form

Definition 3.1 ([2]). *Given a bivariate polynomial $p \in \mathbb{P}_n(\mathbb{R}^2)$, the polar form $\hat{p}$ of p is defined as the unique function of n vector variables $u_1, \cdots, u_n \in \mathbb{R}^2$ satisfying the following properties:*

- *multiaffine: for any index $i = 1, \cdots, n$ and any real number λ*

$$\begin{aligned}&\hat{p}(u_1, \cdots, u_{i-1}, \lambda u + \overline{\lambda} v, u_{i+1}, \cdots, u_n)\\ &= \lambda \hat{p}(u_1, \cdots, u_{i-1}, u, u_{i+1}, \cdots, u_n) + \overline{\lambda}\hat{p}(u_1, \cdots, u_{i-1}, v, u_{i+1}, \cdots, u_n),\end{aligned}$$

  where $\overline{\lambda} = 1 - \lambda$.

- *symmetry: for any permutation π of $\{1, 2, \cdots, n\}$*

$$\hat{p}(u_1, \cdots, u_n) = \hat{p}(u_{\pi(1)}, \cdots, u_{\pi(n)}),$$

- *diagonal: $\hat{p}$ reduces to p when evaluated on its diagonal, i.e.,*

$$\hat{p}(u, \cdots, u) = p(u).$$

The following result gives a method for computing the polar form of bivariate polynomials in a special case.

Proposition 3.1. *Let $A_1, \cdots, A_n$ be n points of $\mathbb{R}^2$ and $R_1, \cdots, R_n$ be n polynomials in $\mathbb{P}_1(\mathbb{R}^2)$. If*

$$p(x, y) = \prod_{i=1}^{n} R_i(x, y),$$

then we have

$$\hat{p}(A_1, \cdots, A_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^{n} R_i(A_{\pi(i)}), \tag{3.1}$$

where $\mathfrak{S}_n$ is the symmetric group of all permutations of the set $\{1, \cdots, n\}$.

Proof. Let us consider the function q defined by:

$$\begin{aligned} q: \quad & (\mathbb{R}^2)^n \to \mathbb{R} \\ & (A_1, \cdots, A_n) \mapsto \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \prod_{i=1}^{n} R_i(A_{\pi(i)}) = q(A_1, \cdots, A_n). \end{aligned}$$

It is easy to see that the function q satisfies the properties given in Definition 3.1. Hence q is the polar form of p. ∎

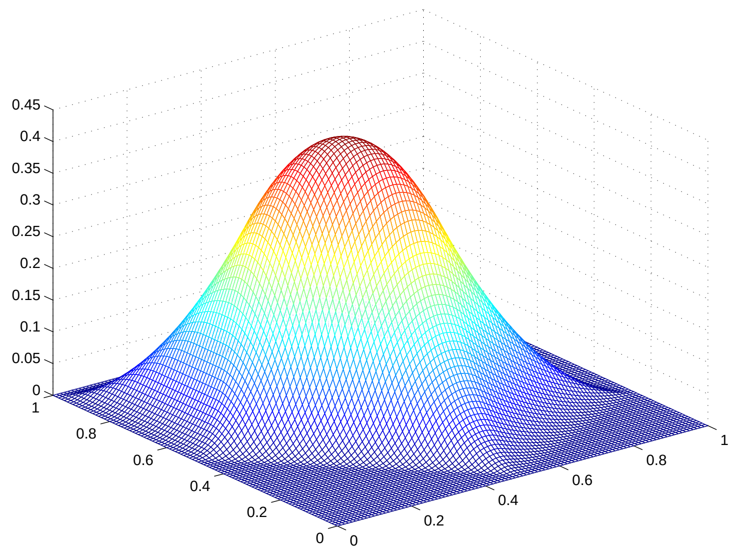

Figure 1: A perspective plot of a bivariate quadratic simplex spline with knots $X = \{(0,1),(0,0.5),(0.5,0),(1,0.25),(0.75,1)\}$.

3.2. Bivariate simplex spline space

It is well-known that univariate B-splines form a basis of a spline space, which has optimal approximation properties in the sense that continuous functions can be approximated by linear combinations of B-splines with the approximation order equal to the order of the splines. This is a consequence of the two following facts:

(1) B-splines provide a local basis.

(2) The linear space spanned by B-splines contains all polynomials of degree less than the order of the splines.

In order to use simplex B-splines in applications (for examples data fitting, computer-aided design, image analysis), it is necessary to construct an appropriate simplex B-spline space $\mathscr{S}_n$, which has the optimal approximation properties like the univariate one. There exist several constructions of the space $\mathscr{S}_n$ (see, e.g., [8–10, 15, 20]), but in practice the most well-known are the ones introduced by W. Dahmen et al. in [9] and by M. Neamtu in [20]. These constructions are obtained by solving the following problem: for any integer $n \geq 0$, and any generic set of knots (i.e., without multiplicities) K, find a spline space $\mathscr{S}_n$ such that the following properties hold:

(i) Each spline in $\mathscr{S}_n$ is a piecewise polynomial of degree n, associated with a rectilinear partition determined by K;

(ii) Each spline in $\mathscr{S}_n$ has optimal smoothness, i.e., $\mathscr{S}_n \in C^{n-1}(\mathbb{R}^2)$;

(iii) The space of bivariate polynomials of degree less than or equal to n is a subset of $\mathscr{S}_n$, i.e., $\mathbb{P}_n(\mathbb{R}^2) \subset \mathscr{S}_n$;

(iv) $\mathscr{S}_n$ is locally finite dimensional, i.e., dim $(\mathscr{S}_n/\Omega) < \infty$, for any compact $\Omega \subset \mathbb{R}^2$;

(v) There exists a countable collection $\mathscr{B}_n$ of compactly-supported functions which forms a basis for $\mathscr{S}_n$, in the sense that for every $x \in \mathbb{R}^2$, all but a finite number of functions in $\mathscr{B}_n$ vanish at x, and every spline in $\mathscr{S}_n$ can be uniquely represented as a linear combination of the form $\Sigma_{B\in\mathscr{B}_n} c_B B$, where $c_B \in \mathbb{R}$, $B \in \mathscr{B}_n$.

In our study we are interested in property (v), that illustrates how each spline s in $\mathscr{S}_n$ and each polynomial p in $\mathbb{P}_n(\mathbb{R}^2)$ can be written as a linear combination of the form $\Sigma_{B\in\mathscr{B}_n} c_B B$. Since $c_B, B \in \mathscr{B}_n$, are given in the DMS-construction and Neamtu's construction by the same expression (with the polar form of p or the polar form of the polynomial restriction of s in the support of B), we will use in this work the DMS-space for constructing different quasi-interpolating schemes.

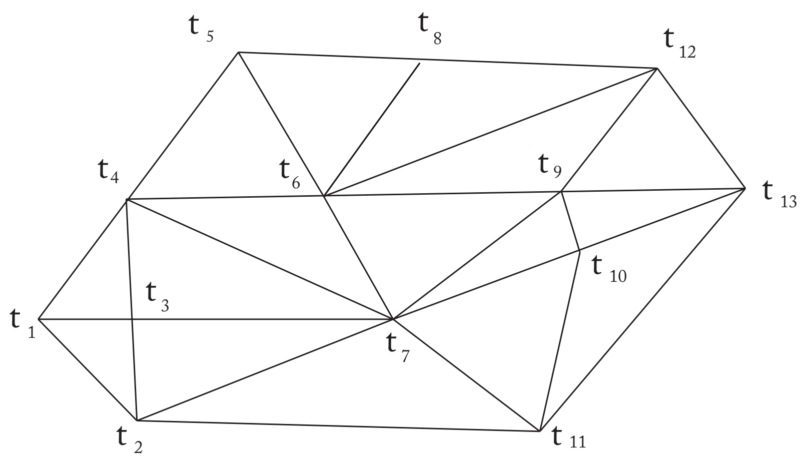

Figure 2: Triangulation of a bounded domain $D \subset \mathbb{R}^2$.

Let $T = \{\Delta(I) = [t_{i_0}, t_{i_1}, t_{i_2}], I = (i_0, i_1, i_2) \in \mathscr{I} \subset \mathbb{Z}^3_+\}$ be a triangulation of a bounded domain $D \subset \mathbb{R}^2$. This means that $D = \cup_{I\in\mathscr{I}} \Delta(I)$, where for any pair $(I,J) \in \mathscr{I} \times \mathscr{I}$, $\Delta(I) \cap \Delta(J)$ is empty or is a common point or edge of $\Delta(I)$ and $\Delta(J)$ (see Fig. 2). In order to construct simplex B-splines of degree n over the triangulation T we first assign a sequence of auxiliary knots $t_{i,1}, \cdots, t_{i,n}$ to every vertex $t_i = t_{i,0}$ of the triangulation in such a way that any set of three knots of $\{t_{i,0}, \cdots, t_{i,n}\}$ is affinely independent, i.e., forms a proper triangle (see, e.g., [24]). Consider the regions

$$\Omega^I_\beta = \cap_{\gamma\leq\beta} \Delta^I_\gamma \quad \text{and} \quad \Omega^I_n = int(\bigcap_{\beta\in\Gamma_n} \Omega^I_\beta),$$

where $int(A)$ denotes the interior of the set A,

$$\beta, \gamma \in \mathbb{Z}^3_+, \quad \Gamma_n = \{\beta = (\beta_0, \beta_1, \beta_2) \in \mathbb{Z}^3_+ : |\beta| = \beta_0 + \beta_1 + \beta_2 = n\},$$

and

$$\Delta^I_\gamma = \Delta^I_{(\gamma_0,\gamma_1,\gamma_2)} = [t_{i_0,\gamma_0}, t_{i_1,\gamma_1}, t_{i_2,\gamma_2}].$$

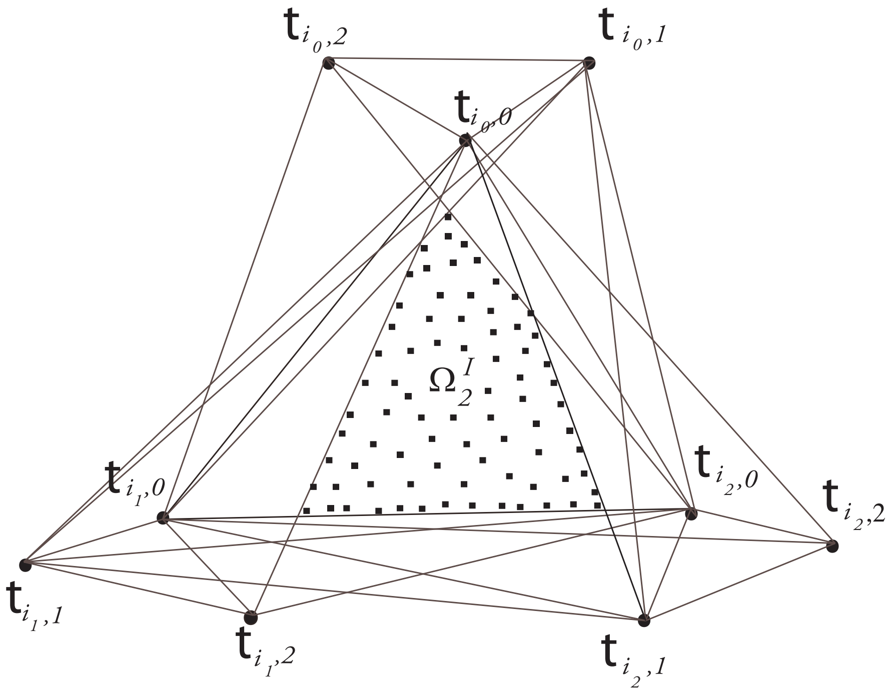

Figure 3: The region Ω_2^I.

If $int(\Omega_n^I)$ is not empty, then

$$\mathscr{K} = \{t_{i_0,0}, \cdots, t_{i_0,n}, t_{i_1,0}, \cdots, t_{i_1,n}, t_{i_2,0}, \cdots, t_{i_2,n}\}$$

is called a knot net associated with the triangle $\Delta(I) = [t_{i_0,0}, t_{i_1,0}, t_{i_2,0}]$. The set $C_i = \{t_{i,0}, \cdots, t_{i,n}\}$ is called the cloud of knots associated with the vertex $t_i = t_{i,0}$. This condition states that the clouds of knots associated with different vertices of a triangle are all separated from each other. In particular, all triangles $\Delta_\beta^I = [t_{i_0,\beta_0}, t_{i_1,\beta_1}, t_{i_2,\beta_2}]$ have the same orientation [24] (see Fig. 3).

If we put

$$V_\beta^I = \{t_{i_0,0}, \cdots, t_{i_0,\beta_0}, t_{i_1,0}, \cdots, t_{i_1,\beta_1}, t_{i_2,0}, \cdots, t_{i_2,\beta_2}\}, \quad \beta \in \Gamma_n,$$

then the functions

$$N_\beta^I(x) = \mid d(t_{i_0,\beta_0}, t_{i_1,\beta_1}, t_{i_2,\beta_2}) \mid \cdot M(x \mid V_\beta^I), \quad I \in \mathscr{I}, \quad \beta \in \Gamma_n,$$

called DMS-splines (or Triangular B-splines) over $\mathscr{K}$, are linearly independent and locally linearly independent over Ω_n^I. Furthermore, the space

$$\mathscr{S}_n := \text{span}\left\{N_\beta^I(x), I \in \mathscr{I}, \beta \in \Gamma_n\right\}$$

satisfies the properties (i), (ii), (iii), (iv) and (v) (see [9,14,24]). More precisely we have the following results:

Theorem 3.1. *Let s be any piecewise polynomial of degree n over the triangulation T that is C^{n-1} continuous everywhere, then for each x in D we have*

$$s(x)=\sum_{I\in\mathscr{I}}\sum_{\beta\in\Gamma_n}\hat{s}_I(t_{i_0,0},\cdots,t_{i_0,\beta_0-1},t_{i_1,0},\cdots,t_{i_1,\beta_1-1},t_{i_2,0},\cdots,t_{i_2,\beta_2-1})N_\beta^I(x), \tag{3.2}$$

where s_I is the restriction of s to the triangle $\Delta(I)$.

A full proof of Theorem 3.1 is given in [25]. We note that β_0,β_1,β_2 are strictly positive.

Corollary 3.1. *For each polynomial $p\in\mathbb{P}_n(\mathbb{R}^2)$ and $x\in D$ we have*

$$p(x)=\sum_{I\in\mathscr{I}}\sum_{\beta\in\Gamma_n}\hat{p}(t_{i_0,0},\cdots,t_{i_0,\beta_0-1},t_{i_1,0},\cdots,t_{i_1,\beta_1-1},t_{i_2,0},\cdots,t_{i_2,\beta_2-1})N_\beta^I(x). \tag{3.3}$$

For $p\equiv 1$, we deduce that

$$\sum_{I\in\mathscr{I}}\sum_{\beta\in\Gamma_n}N_\beta^I=1,$$

i.e., the normalized simplex B-splines N_β^I form a partition of unity. The following theorem shows how each spline s in $\mathscr{S}_n$ can be explicitly represented as a linear combination of the normalized B-splines N_β^I, $\beta\in\Gamma_n$, $I\in\mathscr{I}$.

Theorem 3.2. *For each spline $s\in\mathscr{S}_n$ and $x\in D$ we have*

$$s(x)=\sum_{I\in\mathscr{I}}\sum_{\beta\in\Gamma_n}\hat{s}_I(t_{i_0,0},\cdots,t_{i_0,\beta_0-1},t_{i_1,0},\cdots,t_{i_1,\beta_1-1},t_{i_2,0},\cdots,t_{i_2,\beta_2-1})N_\beta^I(x), \tag{3.4}$$

where s_I is the restriction of s to the region Ω_n^I.

Proof. Let $s\in\mathscr{S}_n$. Then we have

$$s=\sum_{I\in\mathscr{I}}\sum_{\beta\in\Gamma_n}c_\beta^I N_\beta^I.$$

If we denote by s_I the restriction of s to the region Ω_n^I, then we have

$$s_I(x)=\sum_{\beta\in\Gamma_n}c_\beta^I N_\beta^I(x),\quad \forall x\in\Omega_n^I.$$

Furthermore, s_I is a polynomial of degree n. By using Corollary 3.1, we get for each $x\in\Omega_n^I$

$$s_I(x)=\sum_{\beta\in\Gamma_n}\hat{s}_I(t_{i_0,0},\cdots,t_{i_0,\beta_0-1},t_{i_1,0},\cdots,t_{i_1,\beta_1-1},t_{i_2,0},\cdots,t_{i_2,\beta_2-1})N_\beta^I(x).$$

Since N_β^I, $\beta\in\Gamma_n$ are linearly independent over Ω_n^I, it is easy to see that

$$c_\beta^I=\hat{s}_I(t_{i_0,0},\cdots,t_{i_0,\beta_0-1},t_{i_1,0},\cdots,t_{i_1,\beta_1-1},t_{i_2,0},\cdots,t_{i_2,\beta_2-1}).$$

This completes the proof of the theorem. ∎

4. Bivariate simplex spline quasi-interpolants

Since the quasi-interpolants are good approximation tools, we propose in this section a method for constructing some quasi-interpolants using the bivariate simplex B-splines in order to approximate given values or derivatives of a function f. We are interested in quasi-interpolants of the form:

$$\mathcal{Q}f = \sum_{I \in \mathscr{I}} \sum_{\beta \in \Gamma_n} \mu_\beta^I(f) N_\beta^I, \tag{4.1}$$

where μ_β^I, $I \in \mathscr{I}$, $\beta \in \Gamma_n$ are linear functionals. We first propose a method for constructing the linear functionals μ_β^I, $I \in \mathscr{I}$, $\beta \in \Gamma_n$. In order to ensure the locality property of the quasi-interpolant, the support of μ_β^I must be a subset of the support of N_β^I and contains some data sites which are used for determining $\mu_\beta^I(f)$. Assume that the scattered data given at these data sites allow to construct a local linear polynomial operator $\mathscr{J}_\beta^I$ reproducing the space of polynomials of degree n, i.e., $\mathscr{J}_\beta^I(f) = f$ for all $f \in \mathbb{P}_n(\mathbb{R}^2)$. Then we have the following results.

Theorem 4.1. *Let f be a function defined from $\mathbb{R}^2$ to $\mathbb{R}$ such that the values and (or) the derivatives of f are given at some discrete points in the support of N_β^I, $I \in \mathscr{I}$, $\beta \in \Gamma_n$. Set $p_\beta^I = \mathscr{J}_\beta^I(f)$. Then, the quasi-interpolant defined by* (4.1) *with*

$$\mu_\beta^I(f) = \hat{p}_\beta^I(t_{i_0,0}, \cdots, t_{i_0,\beta_0-1}, t_{i_1,0}, \cdots, t_{i_1,\beta_1-1}, t_{i_2,0}, \cdots, t_{i_2,\beta_2-1}), \tag{4.2}$$

satisfies

$$\mathcal{Q}p = p, \quad \forall p \in \mathbb{P}_n(\mathbb{R}^2).$$

Proof. For $f = p \in \mathbb{P}_n(\mathbb{R}^2)$ we have $p_\beta^I = \mathscr{J}_\beta^I(f) = p$ for any $I \in \mathscr{I}$, and $\beta \in \Gamma_n$. Then by using Corollary 3.1, it is easy, to see that

$$\mathcal{Q}p = p, \quad \forall p \in \mathbb{P}_n(\mathbb{R}^2),$$

which is the desired result. □

Theorem 4.2. *If the data sites corresponding to p_β^I, $\beta \in \Gamma_n$ belong to the same region Ω_n^I, for all $I \in \mathscr{I}$, then the quasi-interpolant $\mathcal{Q}f$ defined by Theorem* 4.1 *satisfies*

$$\mathcal{Q}s = s, \quad \forall s \in \mathscr{S}_n.$$

Proof. Let $s \in \mathscr{S}_n$ and let s_I be its restriction to the region Ω_n^I. We know that Ω_n^I contains the interpolation points which determine the polynomials p_β^I, $\beta \in \Gamma_n$. Then we have

$$p_\beta^I = s_I, \quad \text{for } I \in \mathscr{I}.$$

By using Theorem 3.2, it is easy to see that $\mathcal{Q}s = s$, $\forall s \in \mathscr{S}_n$. □

If s is a piecewise polynomial of degree n over the triangulation T that is of class C^{n-1} everywhere, then by using Theorem 3.1 we can obtain the following result.

Theorem 4.3. *If the data sites corresponding to p^I_β, $\beta \in \Gamma_n$ belong to the same triangle $\Delta(I)$, for all $I \in \mathscr{I}$, then the quasi-interpolant $\mathscr{Q}f$ defined by Theorem* 4.1 *satisfies $\mathscr{Q}s = s$ for each piecewise polynomial s of degree n over the triangulation T and of class C^{n-1} everywhere.*

4.1. Differential bivariate simplex spline quasi interpolants

For constructing a differential simplex spline quasi-interpolant of degree n, we choose the local operator $\mathscr{J}^I_\beta$ as the Hermite interpolant at some fixed points $Z^I_\beta = (z^I_{\beta,1}, z^I_{\beta,2}) \in supp(N^I_\beta)$. For this, it suffices to have the values and derivatives of f at these points which give rise to a unisolvent scheme in $\mathbb{P}_n(\mathbb{R}^n)$. In this subsection we describe how the approach given in the Theorems 4.1-4.3 can be used to construct some particular differential bivariate simplex spline quasi-interpolants. Let f be a function of class C^n on D. For $I \in \mathscr{I}$ and $\beta \in \Gamma_n$, we denote by p^I_β the Taylor expansion of order n of f at a point Z^I_β in the support of N^I_β, i.e., for $V = (x, y) \in \mathbb{R}^2$ we have

$$p^I_\beta(V) = \sum_{0 \le i,j \le n} \frac{(x - z^I_{\beta,1})^i (y - z^I_{\beta,2})^j}{i!j!} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(Z^I_\beta). \tag{4.3}$$

Theorem 4.4. *Let $\mathscr{Q}f$ be the quasi-interpolant defined by* (4.1), (4.2) *and* (4.3). *Then*

$$\mathscr{Q}p = p, \quad \forall p \in \mathbb{P}_n(\mathbb{R}^2).$$

Moreover, if each Z^I_β, $\beta \in \Gamma_n$, belongs to the region Ω^I_n (resp. the triangle $\Delta(I)$), then $\mathscr{Q}s = s$ for each simplex spline s (resp. piecewise polynomial of degree n over the triangulation T and of class C^{n-1} everywhere).

Proof. As

$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(V) = \frac{\partial^{i+j} f}{\partial y^j \partial x^i}(V), \quad 1 \le i, j \le n,$$

the polynomial p^I_β is the unique interpolant of f which satisfies:

$$p^I_\beta(Z^I_\beta) = f(Z^I_\beta) \text{ and } \frac{\partial^{i+j} p^I_\beta}{\partial x^i \partial y^j}(Z^I_\beta) = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(Z^I_\beta), \quad 1 \le i, j \le n. \tag{4.4}$$

Since $\mathscr{J}^I_\beta(f) = f$ for all $f \in \mathbb{P}_n(\mathbb{R}^2)$, the claim follows from the Theorems 4.1-4.3. $\square$

Now we give some differential quasi-interpolants constructed with specific values of n. Consider

$$\mathscr{W}^I_\beta = \{t_{i_0,0}, \cdots, t_{i_0,\beta_0-1}, t_{i_1,0}, \cdots, t_{i_1,\beta_1-1}, t_{i_2,0}, \cdots, t_{i_2,\beta_2-1}\}.$$

In the case $n = 2$, it is easy to see that for each $\beta \in \Gamma_2$ there exist two points $A^I_\beta = (a^I_{\beta,1}, a^I_{\beta,2})$ and $B^I_\beta = (b^I_{\beta,1}, b^I_{\beta,2})$ such that $\mathscr{W}^I_\beta = \{A^I_\beta, B^I_\beta\}$. If we put $Q^I_\beta = (x^I_\beta, y^I_\beta)$ the midpoint of $[A^I_\beta B^I_\beta]$, then by using the definition of the polar form and Corollary 3.1, we get the following results:

- If $Z^I_\beta = A^I_\beta$ for $I \in \mathscr{I}$, $\beta \in \Gamma_2$, then

$$\hat{p}^I_\beta(A^I_\beta, B^I_\beta) = f(A^I_\beta) + \frac{\partial f}{\partial x}(A^I_\beta)(x^I_\beta - a^I_{\beta,1}) + \frac{\partial f}{\partial y}(A^I_\beta)(y^I_\beta - a^I_{\beta,2}). \tag{4.5}$$

- If $Z^I_\beta = B^I_\beta$ for $I \in \mathscr{I}$, $\beta \in \Gamma_2$, then

$$\hat{p}^I_\beta(A^I_\beta, B^I_\beta) = f(B^I_\beta) - \frac{\partial f}{\partial x}(B^I_\beta)(x^I_\beta - a^I_{\beta,1}) - \frac{\partial f}{\partial y}(B^I_\beta)(y^I_\beta - a^I_{\beta,2}). \tag{4.6}$$

- If $Z^I_\beta = Q^I_\beta$ for $I \in \mathscr{I}$, $\beta \in \Gamma_2$, then

$$\mu^I_\beta(f) = \hat{p}^I_\beta(A^I_\beta, B^I_\beta) = f(Q^I_\beta) - \frac{1}{2}(A^I_\beta - Q^I_\beta)\nabla^2 f(Q^I_\beta)(A^I_\beta - Q^I_\beta)^T. \tag{4.7}$$

In the case $n = 3$, we have for each $\beta \in \Gamma_3$ there exist three points

$$A^I_\beta = (a^I_{\beta,1}, a^I_{\beta,2}), \quad B^I_\beta = (b^I_{\beta,1}, b^I_{\beta,2}), \quad C^I_\beta = (c^I_{\beta,1}, c^I_{\beta,2})$$

such that $\mathscr{W}^I_\beta = \{A^I_\beta, B^I_\beta, C^I_\beta\}$. If we put

$$Q^I_\beta = (x^I_\beta, y^I_\beta) = \frac{A^I_\beta + B^I_\beta + C^I_\beta}{3},$$

then we obtain

- If $Z^I_\beta = A^I_\beta$ for $I \in \mathscr{I}$, $\beta \in \Gamma_3$, then

$$\begin{aligned}\hat{p}^I_\beta(A^I_\beta, B^I_\beta, C^I_\beta) = f(A^I_\beta) &+ \frac{1}{3}\nabla f(A^I_\beta)(Q^I_\beta - A^I_\beta)^T \\ &+ \frac{1}{6}(B^I_\beta - A^I_\beta)\nabla^2 f(A^I_\beta)(C^I_\beta - A^I_\beta)^T.\end{aligned} \tag{4.8}$$

- If $Z^I_\beta = B^I_\beta$ for $I \in \mathscr{I}$, $\beta \in \Gamma_3$, then

$$\begin{aligned}\hat{p}^I_\beta(A^I_\beta, B^I_\beta, C^I_\beta) = f(B^I_\beta) &+ \frac{1}{3}\nabla f(B^I_\beta)(Q^I_\beta - B^I_\beta)^T \\ &- \frac{1}{6}(B^I_\beta - A^I_\beta)\nabla^2 f(B^I_\beta)(C^I_\beta - B^I_\beta)^T.\end{aligned} \tag{4.9}$$

- If $Z^I_\beta = C^I_\beta$ for $I \in \mathscr{I}$, $\beta \in \Gamma_3$, then

$$\begin{aligned}\hat{p}^I_\beta(A^I_\beta, B^I_\beta, C^I_\beta) = f(C^I_\beta) &+ \frac{1}{3}\nabla f(C^I_\beta)(Q^I_\beta - C^I_\beta)^T \\ &- \frac{1}{6}(B^I_\beta - C^I_\beta)\nabla^2 f(C^I_\beta)(C^I_\beta - A^I_\beta)^T.\end{aligned} \tag{4.10}$$

Remark 4.1. This method allows to construct several other differential simplex spline quasi-interpolants. In particular the quasi-interpolants proposed in this subsection. To construct other differential quasi-interpolants, it suffices to have local data values and derivatives of f providing a unisolvent interpolation scheme in $\mathbb{P}_n(\mathbb{R}^2)$.

4.2. Discrete bivariate simplex spline quasi interpolants

To construct a discrete bivariate spline quasi-interpolant, it suffices to take data points in the support of N^I_β, for $I \in \mathscr{I}$ and $\beta \in \Gamma_n$, which allow to interpolate f, in unique way, in $\mathbb{P}_n(\mathbb{R}^2)$. In order to successfully interpolate with a unique function in the space of bivariate polynomials of degree less or equal to n at a set of points $X \subset \mathbb{R}^2$ with cardinal

$$|X| = \frac{(n+1)(n+2)}{2} = \alpha,$$

Liang [17] proved that it is necessary and sufficient that X is not a subset of any algebraic curve of degree n. To simplify the solution of the interpolation problem, Chung et al. [6] showed that if a set of nodes X satisfies the geometric characterization (GC condition), i.e., if for each $x \in X$, there exist n lines $R_{1,x}, \cdots, R_{n,x}$ such that

$$X \backslash \{x\} \subset \bigcup_{i=1}^{n} R_{i,x} \quad \text{and} \quad x \notin \bigcup_{i=1}^{n} R_{i,x},$$

then the Lagrange interpolation problem in $\mathbb{P}_n(\mathbb{R}^2)$ is uniquely solvable, and each element L_x of the Lagrange basis is given by the following expression

$$L_x = \frac{R_{1,x} \times \cdots \times R_{n,x}}{R_{1,x}(x) \times \cdots \times R_{n,x}(x)}.$$

Let $Z^I_{\beta,k}$, $1 \le k \le \alpha$, be distinct points in the support of N^I_β satisfying the GC condition. Then there exists a Lagrange basis $\{L^I_{\beta,k},\ k = 1, \cdots, \alpha\}$ such that $L^I_{\beta,r}(Z^I_{\beta,s}) = \delta_{r,s}$, $r,s = 1, \cdots, \alpha$ and the polynomial

$$\mathscr{I}^I_\beta(f) = p^I_\beta = \sum_{k=1}^{\alpha} f(Z^I_{\beta,k}) L^I_{\beta,k}$$

interpolates f at the points $Z^I_{\beta,k}$, $k = 1, \cdots, \alpha$. Therefore we have the following theorem.

Theorem 4.5. *Let $\mathscr{Q}f$ be the quasi-interpolant defined by* (4.1), (4.2)*, with*

$$p^I_\beta = \sum_{k=1}^{\alpha} f(Z^I_{\beta,k}) L^I_{\beta,k}.$$

Then

$$\mathscr{Q}p = p, \quad \forall p \in \mathbb{P}_n(\mathbb{R}^2).$$

Moreover, if the points $Z^I_{\beta,k}$, $k = 1, \cdots, \alpha$, $\beta \in \Gamma_n$, belong to the region Ω^I_n (*resp. the triangle $\Delta(I)$*)*, then $\mathscr{Q}s = s$ for each simplex spline s* (*resp. piecewise polynomial functions of degree n over the triangulation T and of class C^{n-1} everywhere*)*.*

In the case $n = 2$, the coefficients $\mu^I_\beta(f)$ are in general expressed by six scattered data at points in the support of N^I_β. However we can construct $\mu^I_\beta(f)$ by using only three scattered data. More precisely, if we set $\mathscr{W}^I_\beta = \{A^I_\beta, B^I_\beta\}$, then we have the following results.

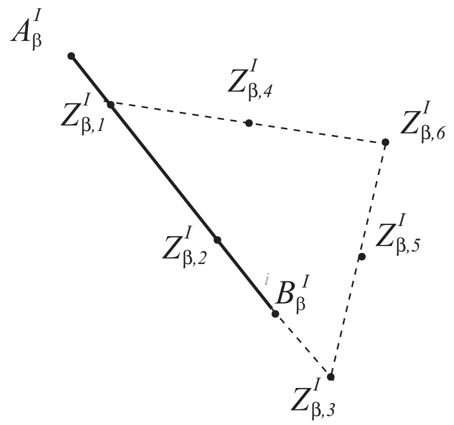

Figure 4: Position of auxiliary interpolation points for the simplex B-spline N^I_β.

Theorem 4.6. *Let $I \in \mathscr{I}$, $\beta \in \Gamma_2$ and $Z^I_{\beta,k}$, $k = 1,2,3$, be three distinct points in the support of N^I_β. The points $Z^I_{\beta,k}$, $k = 1,2,3$, are collinear with A^I_β and B^I_β if and only if there exist $q^I_{\beta,k} \in \mathbb{R}\backslash\{0\}$, $k = 1,2,3$ such that the quasi-interpolant*

$$\mathscr{Q}f = \sum_{I\in\mathscr{I}} \sum_{\beta\in\Gamma_2} \mu^I_\beta(f) N^I_\beta$$

with

$$\mu^I_\beta(f) = \sum_{k=1}^{3} q^I_{\beta,k} f(Z^I_{\beta,k}), \quad I \in \mathscr{I}, \quad \beta \in \Gamma_2$$

satisfies

$$\mathscr{Q}p = p, \quad \forall p \in \mathbb{P}_2(\mathbb{R}^2).$$

Proof. We first prove the necessary condition. Indeed, let

$$Z^I_{\beta,1} = (z^I_{\beta,1,1}, z^I_{\beta,1,2}), \quad Z^I_{\beta,2} = (z^I_{\beta,2,1}, z^I_{\beta,2,2}), \quad Z^I_{\beta,3} = (z^I_{\beta,3,1}, z^I_{\beta,3,2})$$

be three points collinear with A^I_β and B^I_β. Assume that $Z^I_{\beta,2} \in [Z^I_{\beta,1} Z^I_{\beta,3}]$. Let $Z^I_{\beta,4}$, $Z^I_{\beta,5}$ and $Z^I_{\beta,6}$ be three auxiliary points in the support of N^I_β (see Fig. 4) such that

$$Z^I_{i,4} \in [Z^I_{i,1} Z^I_{i,6}] \text{ and } Z^I_{i,5} \in [Z^I_{i,3} Z^I_{i,6}].$$

We note that $Z^I_{\beta,k}$, $k = 1, \cdots, 6$ satisfy the *GC* condition. Let $L^I_{i,k}$, $k = 1, \cdots, 6$ be the Lagrange basis corresponding respectively to $Z^I_{\beta,k}$, $k = 1 \cdots, 6$. The *GC* condition implies that

$$L^I_{\beta,k}(x,y) = \frac{R_{1,k}(x,y) R_{2,k}(x,y)}{R_{1,k}(Z^I_{\beta,k}) R_{2,k}(Z^I_{\beta,k})},$$

where $R_{1,k}$ and $R_{2,k}$ are two lines containing the nodes $Z^I_{\beta,j}$, $j=1,\cdots,6$, $j\neq k$. Since the points $Z^I_{\beta,1}$, $Z^I_{\beta,2}$ and $Z^I_{\beta,3}$ are collinear with A^I_β and B^I_β, we deduce that for $k=4,5,6$, $R_{1,k}$ or $R_{2,k}$ is the line $(A^I_\beta B^I_\beta)$. Hence, by using Corollary 3.1, it is easy to see that

$$\hat{L}^I_{\beta;k}(A^I_\beta,B^I_\beta)=0,\quad k=4,5,6.$$

We set

$$\hat{L}^I_{\beta,k}(A^I_\beta,B^I_\beta)=q^I_{\beta,k},\quad k=1,2,3,$$

then the quasi-interpolant $\mathscr{Q}$ such that

$$\mathscr{Q}f=\sum_{I\in\mathscr{I}}\sum_{\beta\in\Gamma_2}\mu^I_\beta(f)N^I_\beta$$

with

$$\mu^I_\beta(f)=\sum_{k=1}^{3}q^I_{\beta,k}f(Z^I_{\beta,k}),\quad I\in\mathscr{I},\quad \beta\in\Gamma_2 \tag{4.11}$$

satisfies

$$\mathscr{Q}p=p,\quad \forall p\in\mathbb{P}_2(\mathbb{R}^2).$$

Now, let us prove the sufficient condition. Set $A^I_\beta=(0,0)$. If we write each monomial of $\mathbb{P}_2(\mathbb{R}^2)$ in terms of the simplex B-spline of $\mathscr{S}_2$, firstly by using Corollary 3.1 and secondly by using the relation (4.11), we obtain the following system

$$q^I_{\beta,1}+q^I_{\beta,2}+q^I_{\beta,3}=1, \tag{4.12a}$$
$$q^I_{\beta,1}z^I_{\beta,1,1}+q^I_{\beta,2}z^I_{\beta,2,1}+q^I_{\beta,3}z^I_{\beta,3,1}=x^I_\beta, \tag{4.12b}$$
$$q^I_{\beta,1}z^I_{\beta,1,2}+q^I_{\beta,2}z^I_{\beta,2,2}+q^I_{\beta,3}z^I_{\beta,3,2}=y^I_\beta, \tag{4.12c}$$
$$q^I_{\beta,1}(z^I_{\beta,1,1})^2+q^I_{\beta,2}(z^I_{\beta,2,1})^2+q^I_{\beta,3}(z^I_{\beta,3,1})^2=0, \tag{4.12d}$$
$$q^I_{\beta,1}(z^I_{\beta,1,2})^2+q^I_{\beta,2}(z^I_{\beta,2,2})^2+q^I_{\beta,3}(z^I_{\beta,3,2})^2=0, \tag{4.12e}$$
$$q^I_{\beta,1}z^I_{\beta,1,1}z^I_{\beta,1,2}+q^I_{\beta,2}z^I_{\beta,2,1}z^I_{\beta,2,2}+q^I_{\beta,3}z^I_{\beta,3,1}z^I_{\beta,3,2}=0. \tag{4.12f}$$

System (4.12) admits a solution only if the points $Z^I_{\beta,1}$, $Z^I_{\beta,2}$ and $Z^I_{\beta,3}$ are collinear with A^I_β and Q^I_β (for a complete proof see [18] Theorem 3, p. 8). □

We can also compute the coefficients $q^I_{\beta,k}$, $k=1,2,3$. Indeed, let us consider three distinct points $Z^I_{\beta,1}$, $Z^I_{\beta,2}$ and $Z^I_{\beta,3}$ on the line $(A^I_\beta B^I_\beta)$. Then if we put

$$Z^I_{\beta,1}=\mu^I_\beta A^I_\beta+(1-\mu^I_\beta)B^I_\beta,\quad Z^I_{\beta,2}=\varrho^I_\beta A^I_\beta+(1-\varrho^I_\beta)B^I_\beta,\quad Z^I_{\beta,3}=\varepsilon^I_\beta A^I_\beta+(1-\varepsilon^I_\beta)B^I_\beta,$$

we obtain the following theorem.

Theorem 4.7. *The coefficients* $q^I_{\beta,k}$, $k=1,2,3$, *defined in Theorem* 4.6, *are given by:*

$$q^I_{\beta,1}=\frac{\varepsilon^I_\beta(1-\varrho^I_\beta)+\varrho^I_\beta(1-\varepsilon^I_\beta)}{2(\varrho^I_\beta-\mu^I_\beta)(\mu^I_\beta-\varepsilon^I_\beta)}, \tag{4.13a}$$

$$q^I_{\beta,2}=\frac{\varepsilon^I_\beta(1-\mu^I_\beta)+\mu^I_\beta(1-\varepsilon^I_\beta)}{2(\mu^I_\beta-\varrho^I_\beta)(\varrho^I_\beta-\varepsilon^I_\beta)}, \tag{4.13b}$$

$$q^I_{\beta,3}=\frac{\mu^I_\beta(1-\varrho^I_\beta)+\varrho^I_\beta(1-\mu^I_\beta)}{2(\varrho^I_\beta-\varepsilon^I_\beta)(\varepsilon^I_\beta-\mu^I_\beta)}. \tag{4.13c}$$

Proof. Assume that $Z^I_{\beta,2}\in[Z^I_{\beta,1}Z^I_{\beta,3}]$. We consider three auxiliary points $Z^I_{\beta,4}$, $Z^I_{\beta,5}$ and $Z^I_{\beta,6}$ such that $Z^I_{\beta,4}$ (resp. $Z^I_{\beta,5}$) is the midpoint of $[Z^i_{\beta,1}Z^I_{\beta,6}]$ (resp. $[Z^I_{\beta,3}Z^I_{\beta,6}]$) (see Fig. 4). If we put $A^I_\beta=(0,0)$ and $Z^I_{\beta,6}=(1,0)$, then we obtain

$$Z^I_{\beta,1}=((1-\mu^I_\beta)b^I_{\beta,1},(1-\mu^I_\beta)b^I_{\beta,2}), \tag{4.14a}$$

$$Z^I_{\beta,2}=((1-\varrho^I_\beta)b^I_{\beta,1},(1-\varrho^I_\beta)b^I_{\beta,2}), \tag{4.14b}$$

$$Z^I_{\beta,3}=((1-\varepsilon^I_\beta)b^I_{\beta,1},(1-\varepsilon^I_\beta)b^I_{\beta,2}), \tag{4.14c}$$

$$Z^I_{\beta,4}=\Big(\frac{(1-\mu^I_\beta)}{2}b^I_{\beta,1}+\frac{1}{2},\frac{(1-\mu^I_\beta)}{2}b^I_{\beta,2}\Big), \tag{4.14d}$$

$$Z^I_{\beta,5}=\Big(\frac{(1-\varepsilon^I_\beta)}{2}b^I_{\beta,1}+\frac{1}{2},\frac{(1-\varepsilon^I_\beta)}{2}b^I_{\beta,2}\Big). \tag{4.14e}$$

Hence, the lines $(Z^j_{i,2}Z^j_{i,4})$, $(Z^j_{i,2}Z^j_{i,5})$, $(Z^j_{i,1}Z^j_{i,6})$ and $(Z^j_{i,3}Z^j_{i,6})$ have respectively the following equations:

$$\begin{aligned} R_{2,4}(x,y)=&-\left(\frac{(1-\mu^I_\beta)}{2}-(1-\varrho^I_\beta)\right)b^I_{\beta,2}x+\left(\Big(\frac{(1-\mu^I_\beta)}{2}\right.\\ &\left.-(1-\varrho^I_\beta)\Big)b^I_{\beta,1}+\frac{1}{2}\right)y-\frac{(1-\varrho^I_\beta)}{2}b^I_{\beta,2}=0, \end{aligned} \tag{4.15a}$$

$$\begin{aligned} R_{2,5}(x,y)=&-\left(\frac{(1-\varepsilon^I_\beta)}{2}-(1-\varrho^I_\beta)\right)b^I_{\beta,2}x+\left(\Big(\frac{(1-\varepsilon^I_\beta)}{2}\right.\\ &\left.-(1-\varrho^I_\beta)\Big)b^I_{\beta,1}+\frac{1}{2}\right)y-\frac{(1-\varrho^I_\beta)}{2}b^I_{\beta,2}=0, \end{aligned} \tag{4.15b}$$

$$R_{1,6}(x,y)=-(1-\mu^I_\beta)b^I_{\beta,2}x+((1-\mu^I_\beta)b^I_{\beta,1}-1)y+(1-\mu^I_\beta)b^I_{\beta,2}=0, \tag{4.15c}$$

$$R_{3,6}(x,y)=-(1-\varepsilon^I_\beta)b^I_{\beta,2}x+((1-\varepsilon^I_\beta)b^I_{\beta,1}-1)y+(1-\varepsilon^I_\beta)b^I_{\beta,2}=0. \tag{4.15d}$$

Now, it is easy to see that

$$
\begin{array}{ll}
R_{2,4}(Z^I_{\beta,1}) = (\varrho^I_\beta - \mu^I_\beta)b^I_{\beta,2}/2, & R_{2,4}(Z^I_{\beta,3}) = (\varrho^I_\beta - \varepsilon^I_\beta)b^I_{\beta,2}/2,\\
R_{2,4}(A^I_\beta) = -(1-\varrho^I_\beta)b^I_{\beta,2}/2, & R_{2,4}(B^I_\beta) = \varrho^I_\beta b^I_{\beta,2}/2,\\
R_{2,5}(Z^I_{\beta,1}) = (\varrho^I_\beta - \mu^I_\beta)b^I_{\beta,2}/2, & R_{2,5}(Z^I_{\beta,3}) = (\varrho^I_\beta - \varepsilon^I_\beta)b^I_{\beta,2}/2,\\
R_{2,5}(A^I_\beta) = -(1-\varrho^I_\beta)b^I_{\beta,2}/2, & R_{2,5}(B^I_\beta) = \varrho^I_\beta b^I_{\beta,2}/2,\\
R_{1,6}(Z^I_{\beta,2}) = (\varrho^I_\beta - \mu^I_\beta)b^I_{\beta,2}, & R_{1,6}(Z^I_{\beta,3}) = (\varepsilon^I_\beta - \mu^I_\beta)b^I_{\beta,2},\\
R_{1,6}(A^I_\beta) = (1-\mu^I_\beta)b^I_{\beta,2}, & R_{1,6}(B^I_\beta) = -\mu^I_\beta b^I_{\beta,2},\\
R_{3,6}(Z^I_{\beta,2}) = (\varrho^I_\beta - \varepsilon^I_\beta)b^I_{\beta,2}, & R_{3,6}(Z^I_{\beta,3}) = (\mu^I_\beta - \varepsilon^I_\beta)b^I_{\beta,2},\\
R_{3,6}(A^I_\beta) = (1-\varepsilon^I_\beta)b^I_{\beta,2}, & R_{3,6}(B^I_\beta) = -\varepsilon^I_\beta b^I_{\beta,2}.
\end{array}
$$

Corollary 3.1 and the GC condition imply that

$$
\begin{aligned}
q^I_{\beta,1} &= \frac{R_{2,4}(A^I_\beta)R_{3,6}(B^I_\beta) + R_{3,6}(A^I_\beta)R_{2,4}(B^I_\beta)}{2R_{2,4}(Z^I_{\beta,1})R_{3,6}(Z^I_{\beta,1})} = \frac{\varepsilon^I_\beta(1-\varrho^I_\beta) + \varrho^I_\beta(1-\varepsilon^I_\beta)}{2(\varrho^I_\beta - \mu^I_\beta)(\mu^I_\beta - \varepsilon^I_\beta)},\\
q^I_{\beta,2} &= \frac{R_{1,6}(A^I_\beta)R_{3,6}(B^I_\beta) + R_{3,6}(A^I_\beta)R_{1,6}(B^I_\beta)}{2R_{1,6}(Z^I_{\beta,2})R_{3,6}(Z^I_{\beta,2})} = \frac{\varepsilon^I_\beta(1-\mu^I_\beta) + \mu^I_\beta(1-\varepsilon^I_\beta)}{2(\mu^I_\beta - \varrho^I_\beta)(\varrho^I_\beta - \varepsilon^I_\beta)},\\
q^I_{\beta,3} &= \frac{R_{1,6}(A^I_\beta)R_{2,5}(B^I_\beta) + R_{2,5}(A^I_\beta)R_{1,6}(B^I_\beta)}{2R_{1,6}(Z^I_{\beta,3})R_{2,5}(Z^I_{\beta,3})} = \frac{\mu^I_\beta(1-\varrho^I_\beta) + \varrho^I_\beta(1-\mu^I_\beta)}{2(\varrho^I_\beta - \varepsilon^I_\beta)(\varepsilon^I_\beta - \mu^I_\beta)}.
\end{aligned}
$$

This completes the proof of the theorem. □

Remark 4.2. For a particular choice of the parameters μ^I_β, ε^I_β and ϱ^I_β, the coefficients $\mu^I_\beta(f)$ can be expressed using only values of f on two data points which are collinear with A^I_β and B^I_β. For example, if we set

$$\mu^I_\beta = \frac{\varepsilon^I_\beta}{2\varepsilon^I_\beta - 1} \quad \text{where} \quad \varepsilon^I_\beta \neq 0, \frac{1}{2}, 1, \tag{4.16}$$

then it is easy to see that

$$q^I_{\beta,1} = -\frac{(2\varepsilon^I_\beta - 1)^2}{4\varepsilon^I_\beta(1-\varepsilon^I_\beta)}, \quad q^I_{\beta,2} = 0 \quad \text{and} \quad q^I_{\beta,3} = \frac{1}{4\varepsilon^I_\beta(1-\varepsilon^I_\beta)}. \tag{4.17}$$

5. Upper bounds of the infinity norm

In this section we provide upper bounds of the infinity norms of the discrete simplex spline quasi-interpolants of degree n. We will study the case where the interpolation points satisfy the GC condition in the support of N^I_β. Denote by $||f||_{\infty,\Upsilon} = \sup_{x\in\Upsilon} |f(x)|$, and by

$||\mathcal{Q}||_{\infty,\Upsilon}$ the corresponding induced norm. It is well-known that if $\mathcal{Q}$ reproduces bivariate polynomials of degree less or equal than n, then we have

$$||\mathcal{Q}f - f||_{\infty,D} \leq (1 + ||\mathcal{Q}||_{\infty,D}) \inf_{p\in\mathbb{P}_n} ||f - p||_{\infty,D} \tag{5.1}$$

Thus, bounding $||\mathcal{Q}||_{\infty,D}$ implies that the quasi interpolant is $(n+1)^{th}$ order accurate, that is it provides the optimal approximation order in $\mathcal{S}_n$. Let $Z^I_{\beta,k}$, $k = 1,\cdots,\alpha$, be distinct points in the support of N^I_β satisfying the GC condition. We consider the discrete simplex spline quasi interpolant defined in Theorem 4.5. It is easy to see that

$$||\mathcal{Q}||_{\infty,D} \leq \max_{I\in\mathcal{I}} \max_{\beta\in\Gamma_2} \sum_{k=1}^{\alpha} \left| \hat{L}^I_{\beta,k}(t_{i_0,0},\cdots,t_{i_0,\beta_0-1},t_{i_1,0},\cdots,t_{i_1,\beta_1-1},t_{i_2,0},\cdots,t_{i_2,\beta_2-1}) \right|. \tag{5.2}$$

We have

$$L^I_{\beta,k}(x,y) = \frac{R^I_{1,k}(x,y)\times\cdots\times R^I_{n,k}(x,y)}{R^I_{1,k}(Z^I_{\beta,k})\times\cdots\times R^I_{n,k}(Z^I_{\beta,k})}, \quad k = 1,\cdots,\alpha,$$

where

$$R^I_{s,k}(x,y) = a_s x + b_s y + c_s, \ \text{ and } \ R^I_{s,k}(Z^I_{\beta,k}) \neq 0, \quad s = 1,\cdots,n.$$

According to [4], the GC condition implies that

$$\{Z^I_{\beta,m}, \ \ m = 1,\cdots,\alpha, \ \ m \neq k\} \subset \bigcup_{i=1}^{n} R^I_{i,k}$$

and the line $R^I_{i,k}$ contains at least two points of

$$\{Z^I_{\beta,l}, \ \ l = 1,\cdots,\alpha\}.$$

Then for each $s = 1,\cdots,n$, there exists $Z^I_{\beta,m}$ in $R^I_{s,k}$ such that

$$|R^I_{s,k}(A^I_\beta)| \leq K_s ||A^I_\beta - Z^I_{\beta,m}||, \quad \forall A^I_\beta \in \mathcal{W}^I_\beta,$$

where

$$K_s = \sqrt{(a_s)^2 + (b_s)^2}.$$

Hence, if we put

$$\max_{k=1,\cdots,\alpha} \max_{A^I_\beta\in\mathcal{W}^I_\beta} ||A^I_\beta - Z^I_{\beta,k}|| = \lambda^I_\beta,$$

we obtain

$$|R^I_{s,k}(A^I_\beta)| \leq K_s \lambda^I_\beta, \quad s = 1,\cdots,n. \tag{5.3}$$

Let

$$S^I_\beta = \min_{\substack{\{area(Z^I_{\beta,p},Z^I_{\beta,q},Z^I_{\beta,r})>0,\\ p,q,r=1,\cdots,\alpha\}}} area(Z^I_{\beta,p},Z^I_{\beta,q},Z^I_{\beta,r}).$$

Since

$$|R_{s,k}(Z^I_{\beta,k})| = 2K_s \frac{area(Z^I_{\beta,p}, Z^I_{\beta,q}, Z^I_{\beta,k})}{||Z^I_{\beta,p} - Z^I_{\beta,q}||}, \quad \text{where} \quad Z^I_{\beta,p}, Z^I_{\beta,q} \in R^I_{s,k},$$

we get

$$|R^I_{s,k}(Z^I_{\beta,k})| \geq K_s S^I_\beta / \lambda^I_\beta. \tag{5.4}$$

Using relations (5.3) and (5.4) we obtain

$$|\hat{L}^I_{\beta,k}(A^I_\beta, B^I_\beta)| \leq (\lambda^I_\beta)^{2n} / (S^I_\beta)^n \quad \forall k = 1 \cdots, \alpha. \tag{5.5}$$

If we put

$$H^I_\beta = (\lambda^I_\beta)^{2n} / (S^I_\beta)^n, \tag{5.6}$$

then we get the following theorem.

Theorem 5.1. *Let $\mathcal{Q}$ be the bivariate simplex spline quasi interpolant defined in Theorem* 4.5. *Then we have*

$$||\mathcal{Q}||_\infty \leq \alpha \max_{I \in \mathcal{I}} \max_{\beta \in \Gamma_n} H^I_\beta. \tag{5.7}$$

The upper bound of the infinity norm of the discrete quasi-interpolant, defined in Theorem 4.7, satisfies the following relation:

$$||\mathcal{Q}||_\infty \leq \max_{I \in \mathcal{I}} \max_{\beta \in \Gamma_2} \sum_{k=1}^{3} |q^I_{\beta,k}|,$$

where $q^I_{\beta,j}$, $j = 1, 2, 3$ are defined by (4.13). In particular, if we choose μ^I_β given by (4.16) (see Remark 4.2), then

$$||\mathcal{Q}||_{\infty,\Omega} \leq 1 + \max_{I \in \mathcal{I}} \max_{\beta \in \Gamma_2} \frac{1}{2|\varepsilon^I_\beta(1 - \varepsilon^I_\beta)|}.$$

6. Numerical examples

In this section, we show the results of some numerical tests on the method proposed in this paper ($n = 2$, $n = 3$). To do this, we performed experiments with some differential and discrete quasi-interpolants given in Section 4. Let $\Delta^{(0)}$ be a triangulation of the bounded domain $D = [0, 1] \times [0, 1]$ (see Fig. 5 left) and let $\Delta^{(k)}$, $k \geq 1$, be the refined triangulation obtained by connecting the midpoints of the edges of each triangle in the triangulation $\Delta^{(k-1)}$ (see Fig. 5 for $k = 1$). In the case $n = 2$, we define the bivariate test function f as follows

$$f(x, y) = \exp(xy(1 - x)(1 - y)(x - y)).$$

We consider three differential quasi-interpolants $Dqi1$, $Dqi2$ and $Dqi3$, defined in Theorem 4.4. The interpolation points Z^I_β are given respectively by

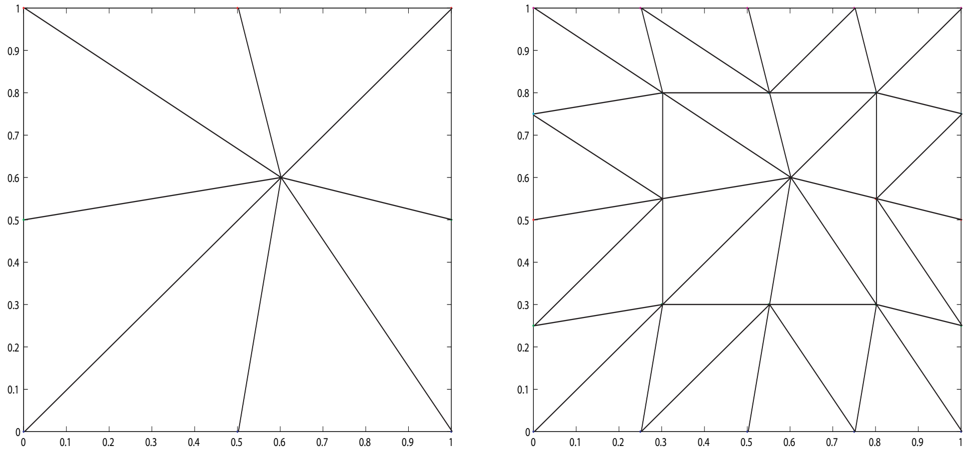

Figure 5: The triangulations $\Delta^{(0)}$ and $\Delta^{(1)}$ of the domain D.

1) $Z^I_\beta = Q^I_\beta$, and at the boundary where $Q^I_\beta \notin D$ we put $Z^I_\beta = A^I_\beta$.
2) For $j = 1,2,3$, if $\beta_j + \beta_{j+1} = 2$ with $\beta_j > 0$ and $\beta_4 = \beta_1$, then $Z^I_\beta = t_{i_j}$.
3) $Z^I_\beta \in \Omega^I_2$.

We also consider three discrete quasi-interpolants $dqi1$, $dqi2$ and $dqi3$, such that the interpolation points are given respectively as follows

1) A^I_β, Q^I_β, and B^I_β,
2) the vertices of the triangle $\Delta(I)$ and the midpoints of its edges.
3) the vertices the triangle $(Z^I_1 Z^I_2 Z^I_3)$ and the midpoints of its edges, where Z^I_1, Z^I_2 and Z^I_3 are three non collinear points in Ω^I_2.

If the point Q^I_β is outside the domain D, we can replace the interpolation points given for constructing $dqi1$ by those given for constructing $dqi2$. We note that $Dqi3$ and $dqi3$ reproduce the whole space $\mathscr{S}_2$, and for each piecewise polynomial s over the triangulation T, that is C^1 everywhere, the quasi-interpolants $Dqi2$ and $dqi2$ satisfy $\mathscr{Q}s = s$. We define the error between f and each quasi-interpolant defined above by

$$\max_{0 \le r,s \le 50} |f(x_r, y_s) - \mathscr{Q}f(x_r, y_s)|, \quad \text{where } x_r = \frac{r}{50} \text{ and } y_s = \frac{s}{50}.$$

Table 1 (respectively Table 2) gives the errors between f and the three differential quasi-interpolants $Dqi1$, $Dqi2$ and $Dqi3$, (respectively the three discrete quasi-interpolants $dqi1$, $dqi2$ and $dqi3$), for the triangulations $\Delta^{(l)}$, $l = 0,1,2,3$.

Table 1: Error behaviour of $Dqi1$, $Dqi2$ and $Dqi3$ for the triangulations $\Delta^{(l)}$, $0 \leq l \leq 3$.

l	knots of $\Delta^{(l)}$	$Dqi1$	$Dqi2$	$Dqi3$
0	9	6×10^{-3}	5.9×10^{-3}	1×10^{-2}
1	25	9.19×10^{-4}	1.2×10^{-3}	1.9×10^{-3}
2	81	1.34×10^{-4}	1.54×10^{-4}	2.06×10^{-4}
3	289	1.55×10^{-5}	1.20×10^{-5}	1.43×10^{-5}

Table 2: Error behaviour of $dqi1$, $dqi2$ and $dqi3$ for the triangulations $\Delta^{(l)}$, $0 \leq l \leq 3$.

l	knots of $\Delta^{(l)}$	$dqi1$	$dqi2$	$dqi3$
0	9	6×10^{-3}	6.1×10^{-3}	7.7×10^{-3}
1	25	7.93×10^{-4}	7.68×10^{-4}	1.1×10^{-3}
2	81	1.1×10^{-4}	1.08×10^{-4}	1.46×10^{-4}
3	289	1.04×10^{-5}	1.07×10^{-5}	1.96×10^{-5}

Table 3: Error behaviour of $Dqi4$, $Dqi5$, $dqi4$ and $dqi5$ for the triangulations $\Delta^{(l)}$, $0 \leq l \leq 3$.

l	knots of $\Delta^{(l)}$	$Dqi4$	$Dqi5$	$dqi4$	$dqi5$
0	9	8.95×10^{-4}	9.87×10^{-4}	8.18×10^{-4}	1.25×10^{-3}
1	25	5.56×10^{-5}	6.35×10^{-5}	3.19×10^{-5}	7.27×10^{-5}
2	81	2.15×10^{-6}	2.56×10^{-6}	1.18×10^{-6}	2.92×10^{-6}
3	289	1.37×10^{-7}	1.40×10^{-7}	1.38×10^{-7}	1.50×10^{-7}

In the case $n = 3$, we know that $\mathscr{W}_\beta^I = \{A_\beta^I, B_\beta^I, C_\beta^I\}$. Let us define the bivariate test function g as follows

$$g(x,y) = \cos\left(\frac{x^2 - y}{2}\right).$$

We consider two differential quasi-interpolants $Dqi4$ and $Dqi5$ defined in Theorem 4.4. The interpolation points Z_β^I are given respectively by 1) $Z_\beta^I = A_\beta^I$; and 2) $Z_\beta^I \in \Omega_3^I$.

We also consider two discrete quasi-interpolants $dqi4$ and $dqi5$, such that the interpolation points are given respectively as follows

1) the Bézier points associated to the Bézier polynomial of degree 3 over the triangle $A_\beta^I B_\beta^I C_\beta^I$.
2) the Bézier points associated to the Bézier polynomial of degree 3 over the triangle $\Delta(I)$.

Table 3 gives the errors between g and the differential quasi-interpolants $Dqi4$, $Dqi5$, $dqi4$ and $dqi5$ for the triangulations $\Delta^{(l)}$, $l = 0, 1, 2, 3$.

Acknowledgments The authors would like to thank the referees for the helpful suggestions.

References

[1] D. Barrera, M. Ibanez, P. Sablonnière, D. Sbibih, *Near-best quasi-interpolants associated with H-splines on a three-direction mesh*, J. Comput. Appl. Math. 183 (2005), pp. 133-152.

[2] P. J. Barry, *An introduction to Blossoming, in Knot Insertion and Deletion Algorithms for B-Spline Curves and surfaces*, R. N. Goldman and T. Lyche (eds), SIAM, Philadelphia, (1993), pp. 1-10.

[3] J. R. Busch, *A note on Lagrange interpolation in* $\mathbb{R}^2$, Rev. Un. Mat. Argentina, 36, (1990), pp. 33-10.

[4] J. M. Carnicer, and M. Gasca, *A conjecture on multivariate polynomial interpolation*, RACSAM. 95, (2001), pp. 145-153.

[5] G. Chen, C. K. Chui and M. J. Lai, *Construction of real-time spline quasi-interpolation schemes*, Approx. Theory Appl. 4 (1988), pp. 61-75.

[6] K. C. Chung, and T. H. Yao, *On lattices admitting unique Lagrange interpolation*, SIAM J. Numer. Math. Anal. 14,(1977), pp. 735-743.

[7] W. Dahmen, and C. A. Micchelli, *Recent progress in multivariate splines*, Approximation Theory IV, C. Chui, L. Shumaker, and J. Ward (eds), Academic Press, New York, (1983), pp. 27-121.

[8] W. Dahmen, and C. A. Micchelli, *On the linear independence of multivariate B-splines. I. Triangulation of simploids*, SIAMJ. Numer. Anal. 19 (1982), pp. 993-1012.

[9] W. Dahmen, C. A. Micchelli, and H. P. Seidel, *Blossoming begets B-splines built better by B-patches*, Math. Comp. 59 (1992), pp. 97-115.

[10] C. de Boor, *Splines as linear combinations of B-splines. A survey*, in Approximation Theory, II, G. G. Lorentz, C. K Chui, and L. L. Schumaker (eds), Academic Press, New York, (1976), pp. 1-47.

[11] C. de Boor, *The quasi-interpolant as a tool in elementary polynomial spline theory*, in Approximation Theory, I, H. Berens, E. W Cheney, G. G. Lorentz, and L. L. Schumaker (eds), Academic Press, New York, (1973), pp. 269-276.

[12] C. de Boor, G. J. Fix, *Spline approximation by quasi-interpolants*, J, Approx. Threory, 8 (1973), pp. 19-45.

[13] P. Dierckx, *On calculating normalized Powell-Sabin B-splines*, Comput. Aided Geom. Design 15 (1997), pp. 61-78.

[14] P. Fong, H. P. Seidel, *Control points for multivariate B-spline surfaces over arbitrary triangulations*, Computer Graphics Forum 10 (1991), pp. 309-319.

[15] K. Höllig, *Multivariate splines*, SIAM J. Numer. Anal. 19 (1982), pp. 1013-1031.

[16] P. Lamberti and C. Manni, *Tensioned quasi-interpolation via geometric continuity*, Adv. Comput. Math. 20 (2004), pp 105-117.

[17] X-Z. Liang, *On the interpolation and approximation in several variables*, Postgraduate Thesis, Jilin University, 1965.

[18] C. Manni, P. Sablonnière, *Quadratic spline quasi-interpolants on Powell-Sabin partitions*, Adv. Comput. Math. 26 (2007), pp. 283-304.

[19] C. A. Micchelli, *Mathematical Aspects of geometric Modeling*, CBMS-NSF Regional Conference Series in applied mathematics, 65, SIAM, Philadelphia, 1995.

[20] M. Neamtu, *What is the natural generalization of univariate splines to higher dimensions?*, in Mathematical Methods for Curves and Surfaces: Oslo 2000, T. Lyche and L. L. Schumaker(eds), Vanderbilt University Press, Nashville, TN, (2001), pp. 355-392.

[21] M. Neamtu, *Bivariate simplex B-splines: A new paradigm*, in Proc. of Spring Conference on Computer Graphics, IEEE Computer Society, R. Durikovic and S. Czanner (eds.), Los Alami-

tos, 2001, pp. 71-78.

[22] O. Nouisser, D. Sbibih and P. Sablonnière, *A Family of Spline Quasi-interpolants on the Sphere*, Numer. Alg. 33 (2003), pp. 399-413.

[23] D. Sbibih, A. Serghini and A. Tijini, *Polar forms and quadratic spline quasi-interpolants on Powell-Sabin partitions*, App. Num. Math. 59 (2009), pp. 938-958.

[24] H. P. Seidel, *Polar forms and triangular B-splines surfaces*, in Blossoming: the New polar Form Approach to Spline Curves and Surfaces, SIGGRAPH 91, Course Notes #26, ACM SIGGRAPH, 1991.

[25] H. P. Seidel, *Representing piecewise polynomials as linear combinations of multivariate B-splines*, In T. Lyche and L. L. Schumaker, editors, Curves and Surfaces, Academic Press, (1992), pp. 559-566.

[26] H. P. Seidel, A. H. Vermeulen, *Simplex splines support suprisingly strong symetric structures and subdivision*, In Curves and Surfaces II, P.J Laurent, A. Le Méhaute, and L. L. Schumaker (eds), AKPeters, Boston, (1991), pp. 443-455.

[27] C. R. Traas, *Practice of bivariate quadratic simplicial splines*, In Computation of Curves and Surfaces, W. Dahmen et al. (eds), NATO ASI Series, Kluwer Academic Publishers, Dordrecht, (1990), pp. 383-422.