Numer. Math. Theor. Meth. Appl. Vol. **4**, No. 1, pp. 53-67
doi: 10.4208/nmtma.2011.m99009 February 2011

On Newton's Method for Solving Nonlinear Equations and Function Splitting

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Received 25 February 2009; Accepted (in revised version) 18 October 2009

Available online 29 October 2010

Abstract. We provided in [14] and [15] a semilocal convergence analysis for Newton's method on a Banach space setting, by splitting the given operator. In this study, we improve the error bounds, order of convergence, and simplify the sufficient convergence conditions. Our results compare favorably with the Newton-Kantorovich theorem for solving equations.

AMS subject classifications: 65H99, 65H10, 65G99, 49M15, 47J20, 47H04, 90C30, 90C33

Key words: Newton's method, Banach space, majorizing sequence, semilocal convergence, splitting of an operator.

1. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, let $\mathscr{D}$ be an open convex subset of $\mathscr{X}$, and let $F: \mathscr{D} \longmapsto \mathscr{Y}$ be a Fréchet differentiable function, fixed all throughout this paper. In the sequel, we will assume that the function F has a splitting

$$F = f + g, \tag{1.1}$$

where $f, g: \mathscr{D} \longmapsto \mathscr{Y}$ are Fréchet differentiable functions satisfying the condition

$$F'(x) = F'(u_0),\ \forall x \in \mathscr{D} \implies f'(x) = f'(u_0),\ \forall x \in \mathscr{D}. \tag{1.2}$$

What this means is that the splitting functions f and g should only be nonlinear if the initial function F is nonlinear. Given any $u_0 \in \mathscr{D}$ and $r > 0$, $\overline{U}(u_0, r)$ will designate the set $\{x \in \mathscr{X} : \|x - u_0\| \leq r\}$, and $U(u_0, r)$ the corresponding interior ball.

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We are interested in the solvability of the equation

$$F(u) = 0. \tag{1.3}$$

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$ (for some suitable operator Q), where x is the state. Then the equilibrium states are determined by solving equation (1.3). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative — when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

In previous papers co-authored by the first author, see [14, 15], we have provided (see also Theorems 2.2, 2.3, 3.1 and 3.2) an analysis under weaker sufficient convergent conditions than the celebrated Newton-Kantorovich theorem for solving equations (see Theorem 2.1).

Here, we improve the bounds on the distances $\| x_{k+1} - x_k \|$, $\| x_k - x^\star \|$, $(k \geq 0)$, and also simplify the sufficient convergence conditions for Newton method (2.1).

2. Preliminaries

In using the Newton's method

$$u_{m+1} = u_m - F'(u_m)^{-1} F(u_m), \tag{2.1}$$

one of the most important theorems in nonlinear analysis is the following result due — essentially — to Kantorovich [7].

Theorem 2.1 (The Kantorovich theorem). *Suppose that $F'(u_0)^{-1}$ exists for some $u_0 \in \mathscr{D}$, and that there exists $K \geq 0$ and $\eta \geq 0$ such that*

$$\| F'(u_0)^{-1} F(u_0) \| \leq \eta, \tag{2.2}$$

$$\| F'(u_0)^{-1} (F'(x) - F'(y)) \| \leq K \ \| x - y \|, \quad \forall x, y \in \mathscr{D}, \tag{2.3}$$

$$2 K \eta \leq 1. \tag{2.4}$$

Let

$$t_\star = \frac{2 \eta}{1 + \sqrt{1 - 2 K \eta}}, \qquad T_\star = \frac{2}{K} - t_\star,$$

and suppose that $\overline{U}(u_0, t_\star) \subset \mathscr{D}$. *Then Eq.* (1.3) *has a unique solution u in the set* $\overline{U}(u_0, t_\star) \cup (\mathscr{D} \cap U(u_0, T_\star))$, *and the Newton iterations* (2.1) *generate a sequence that converges to u. The sequence defined iteratively as*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{m+1} = t_m + \frac{K\,(t_m - t_{m-1})^2}{2\,(1 - K\,t_m)}, \quad m = 1, \cdots$$

converges monotonically to $t_\star$ *and the following majorant error bounds hold:*

$$\| u_m - u_{m-1} \| \leq t_m - t_{m-1}, \tag{2.5a}$$
$$\| u_m - u_0 \| \leq t_m, \tag{2.5b}$$
$$\| u - u_m \| \leq t_\star - t_m. \tag{2.5c}$$

Several versions of this theorem can be found in a large number of works including [4–6, 9]. Extensions of the theorem with improved *a-priori* and/or *a-posteriori* error bounds have been derived by several authors including Ostrowski [9], Gragg & Tapia [6], Potra [10], Potra & Pták [11] and Yamamoto [16]. One of these results is the following fundamental generalization of the Kantorovich theorem proved recently by Argyros [1].

Theorem 2.2. *Suppose that* $F'(u_0)^{-1}$ *exists for some* $u_0 \in \mathscr{D}$, *and there exist constants* $K \geq K_0 \geq 0$ *and* $\eta \geq 0$, *such that* (2.2)-(2.3) *hold, and*

$$\| F'(u_0)^{-1}\,(F'(x) - F'(u_0)) \| \leq K_0\ \| x - u_0 \|, \quad \forall x \in \mathscr{D}. \tag{2.6}$$

Suppose further that there exists $0 \leq \theta < 1$, *such that*

$$(K + K_0\,\theta)\,\eta \leq \theta. \tag{2.7}$$

Then the sequence defined iteratively by

$$r_0 = 0, \quad r_1 = \eta, \quad r_{m+1} = r_m + \frac{K\,(r_m - r_{m-1})^2}{2\,(1 - K_0\,r_m)}, \quad m = 1, \cdots \tag{2.8}$$

converges monotonically to a real number $r_\star$. *If* $\overline{U}(u_0, r_\star) \subset \mathscr{D}$, *then Eq.* (1.3) *has a unique solution u in* $\overline{U}(u_0, r_\star)$, *and the Newton iterations generated from* (2.1) *converge to u and satisfy the majorant error bounds:*

$$\|u_m - u_{m-1}\| \leq r_m - r_{m-1},$$
$$\|u_m - u_0\| \leq r_m,$$
$$\|u - u_m\| \leq r_\star - r_m.$$

If there exists $T > r_\star$ *such that* $U(u_0, T) \subseteq \mathscr{D}$, *and* $K_0\,(r_\star + T) \leq 2$, *then the solution u is unique in* $U(u_0, T)$.

One sees that the basic hypothesis (2.4) of the Kantorovich theorem can be replaced with the weaker Argyros condition $(K_0 + K)\,\eta \leq 1$, and that the case $K_0 = K$ corresponds to the Kantorovich theorem.

In the present paper we will generalize Theorem 2.2 even further by replacing the Newton iterations in (2.1) with the generalized Newton scheme

$$f'(u_m)\,u_{m+1} + g(u_{m+1}) = f'(u_m)\,u_m - f(u_m), \quad m = 0, 1, \cdots \tag{2.9}$$

based on the function splitting (1.1). The generalized Newton scheme (2.9) is a sequence of partial linearizations based on the splitting (1.1). It has been studied in Uko and Adeyeye [12] as a possible alternative — in appropriate contexts — to the Newton iterations (2.1) for the numerical solution of equation (1.3). Our aim in the present paper is to use the method of majorant sequences and the generalized Newton scheme (2.9) to obtain an significant generalization of the Kantorovich theorem.

Theorem 2.3. *Suppose that $F'(u_0)^{-1}$ exists for some $u_0 \in \mathscr{D}$, and that there exists $K \geq K_0 \geq 0$, and $\eta \geq 0$, such that* (2.2), (2.3), (2.6) *hold, and*

$$(4\,K_0 + K + \sqrt{K^2 + 8\,K\,K_0})\,\eta \leq 4, \text{ with strict inequality if } K_0 = 0. \tag{2.10}$$

Then the sequence given by (2.8) *converges monotonically to a limit $t_\star$. Let*

$$t_{\star\star} = \frac{2\,\eta}{1 + \sqrt{1 - 2\,K_0\,\eta}}, \qquad T_{\star\star} = \frac{2}{K_0} - t_{\star\star},$$

and suppose that $\overline{U}(u_0, t_{\star\star}) \subset \mathscr{D}$. Then Eq. (1.3) *has a unique solution u in the set $\overline{U}(u_0, t_{\star\star}) \cup (\mathscr{D} \cap U(u_0, T_{\star\star}))$, and Newton's method* (2.1) *generate a sequence that converges to u. Moreover the following estimates hold for all $m \geq 1$:*

$$\| u_m - u_{m-1} \| \leq r_m - r_{m-1}, \tag{2.11a}$$

$$\| u_m - u_0 \| \leq r_m, \tag{2.11b}$$

$$\| u - u_m \| \leq t_\star - r_m. \tag{2.11c}$$

The solution u is also unique in the set $\overline{U}(u_0, t_{\star\star})$, and $\overline{U}(u_0, T) \cap \mathscr{D}$, whenever T is any non-negative number such that $K_0\,(t_{\star\star} + T) < 2$.

3. Semilocal convergence analysis of Newton's method

We need two results on majorizing sequences for the Newton's method (2.1), whose proofs can be founds [14], and [15] respectively.

Lemma 3.1. *Let $\eta \geq 0$, $L \geq L_0 \geq 0$, $0 \leq \theta_0 < 1$, and suppose:*

$$(L + \theta_0\,L_0)\,\eta \leq \theta_0. \tag{3.1}$$

Then, scalar sequence $\{t_k\}$ $(k \geq 1)$ *given by:*

$$t_0 = 0,\ t_1 = \eta,\ t_{k+1} = t_k + \frac{L\,(t_k - t_{k-1})^2}{2\,(1 - L_0\, t_k)} \tag{3.2}$$

is well defined, nondecreasing, bounded above by $t^{\star\star}$*, and converges to its unique least upper bound* $t^\star \in [0, t^{\star\star}]$*, where*

$$t^{\star\star} = \frac{2\,\eta}{2 - \theta_0}.$$

Moreover, the following estimates hold for all $k \geq 0$*:*

$$0 \leq t_{k+1} - t_k \leq \left(\frac{\theta_0}{2}\right)(t_k - t_{k-1}) \leq \left(\frac{\theta_0}{2}\right)^k \eta, \tag{3.3a}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\theta_0}{2}\right)^k t^{\star\star}. \tag{3.3b}$$

Suppose further that $M \geq 0$*,* $0 < K_0 \leq M + L$*,* $0 \leq \theta < 2$*, and*

$$(4M + L\theta^2 + 4K_0\theta)t^\star < 4\theta, \tag{3.4a}$$

$$4M\theta + L\theta^3 + 4K_0\theta^2 \leq 8M + 2L\theta^2. \tag{3.4b}$$

Define sequence $\{v_k\}$ $(k \geq 0)$ *by*

$$v_0 = 0, \quad v_1 = t^\star, \quad v_{k+1} = v_k + \frac{M\,(v_k - v_{k-1})^2}{1 - K_0\, v_k + \sqrt{(1 - K_0\, v_k)^2 - L\, M\,(v_k - v_{k-1})^2}} \tag{3.5}$$

is well defined, nondecreasing, bounded above by $v^{\star\star}$*, and converges to its unique least upper bound* $v^\star \in [0, v^{\star\star}]$*, where*

$$v^{\star\star} = \frac{2\,\eta}{2 - \theta}.$$

Moreover, the following estimates hold for all $k \geq 0$*:*

$$0 \leq v_{k+1} - v_k \leq \left(\frac{\theta}{2}\right)(v_k - v_{k-1}) \leq \left(\frac{\theta}{2}\right)^k \eta, \tag{3.6a}$$

$$0 \leq v^\star - v_k \leq \left(\frac{\theta}{2}\right)^k v^{\star\star}. \tag{3.6b}$$

Lemma 3.2. *Suppose that there exist constants* $M \geq 0$*,* $L \geq L_0 \geq 0$*,* $\eta \geq 0$*, and* $0 \leq K_0 \leq M + L_0$*, such that:*

$$(L + 4\,L_0 + \sqrt{L^2 + 8\,L_0\,L})\,\eta \leq 4. \tag{3.7}$$

The inequality (3.7) *is strict if* $L_0 = 0$*.*

Let $\theta_1 = \theta^\star(M, K_0, L)$ *be defined as the largest real zero of the cubic function*

$$\rho(\theta) = L\,\theta^3 + 2\,(2\,K_0 - L)\,\theta^2 + 4\,M\,\theta - 8\,M \tag{3.8}$$

on the interval $[0,2]$. *If*

$$4\ M\ w_1 \leq (2 - L\ w_1)\ \theta_1^2 \tag{3.9}$$

($\textit{this inequality is strict if } K_0 = 0$), *then the sequence* $\{w_k\}$ $(k \geq 0)$ *given by*

$$w_0 = 0, \qquad w_1 = \frac{2\ \eta}{1 + \sqrt{1 - 2\ L_0\ \eta}}, \tag{3.10a}$$

$$w_{k+1} = w_k + \frac{M\ (w_k - w_{k-1})^2}{1 - K_0\ w_k + \sqrt{(1 - K_0\ w_k)^2 - L\ M\ (w_k - w_{k-1})^2}} \tag{3.10b}$$

is well defined, nondecreasing, bounded above by $w^{\star\star}$, *and converges to its unique least upper bound* $w^\star \in [0, w^{\star\star}]$, *where*

$$w^{\star\star} = \frac{2\ \eta}{2 - \theta_1}.$$

Moreover, the following estimates hold for all $k \geq 0$*:*

$$0 \leq w_{k+1} - w_k \leq \left(\frac{\theta_1}{2}\right)(w_k - w_{k-1}) \leq \left(\frac{\theta_1}{2}\right)^k \eta, \tag{3.11a}$$

$$0 \leq w^\star - w_k \leq \left(\frac{\theta_1}{2}\right)^k w^{\star\star}. \tag{3.11b}$$

In the next result, we show that under the same sufficient convergence condition (3.7) (which always weaker than (3.1) unless $L_0 = L$) for Newton's method (2.1), we can improve upon the linear error estimates (3.3) and (3.11), and show instead the quadratic convergence of the majorizing sequence $\{t_n\}$.

Lemma 3.3. *Assume there exist constants* $L_0 \geq 0$, $L \geq 0$, *and* $\eta \geq 0$, *such that*

$$q_0 = \overline{L}\ \eta \leq \frac{1}{2}, \tag{3.12}$$

where

$$\overline{L} = \frac{1}{8}\left(L + 4\ L_0 + \sqrt{L^2 + 8\ L_0\ L}\right). \tag{3.13}$$

The inequality in (3.12) *is strict if* $L_0 = 0$. *Then, the sequence* $\{t_k\}$ $(k \geq 0)$ *given by*

$$t_0 = 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L_1\ (t_k - t_{k-1})^2}{2\ (1 - L_0\ t_k)}, \qquad (k \geq 1), \tag{3.14}$$

is well defined, nondecreasing, bounded above by $t^{\star\star}$, *and converges to its unique least upper bound* $t^\star \in [0, t^{\star\star}]$, *where*

$$L_1 = \begin{cases} L_0, & \textit{if}\ \ k = 1, \\ L, & \textit{if}\ \ k > 1, \end{cases} \qquad t^{\star\star} = \frac{2\ \eta}{2 - \delta}, \tag{3.15a}$$

$$1 \leq \delta = \frac{4\ L}{L + \sqrt{L^2 + 8\ L_0\ L}} < 2 \quad \text{for } L_0 \neq 0. \tag{3.15b}$$

Moreover the following estimates hold:

$$L_0\, t^\star < 1, \tag{3.16a}$$

$$0 \le t_{k+1} - t_k \le \frac{\delta}{2}\,(t_k - t_{k-1}) \le \cdots \le \left(\frac{\delta}{2}\right)^k \eta, \qquad (k \ge 1), \tag{3.16b}$$

$$t_{k+1} - t_k \le \left(\frac{\delta}{2}\right)^k (2\, q_0)^{2^k - 1}\, \eta, \qquad (k \ge 0), \tag{3.16c}$$

$$0 \le t^\star - t_k \le \left(\frac{\delta}{2}\right)^k \frac{(2\, q_0)^{2^k - 1}\, \eta}{1 - (2\, q_0)^{2^k}}, \qquad (2\, q_0 < 1), \qquad (k \ge 0). \tag{3.16d}$$

Proof. If $L_0 = 0$, then (3.16a) holds trivially. In this case, for $L > 0$, an induction argument shows that

$$t_{k+1} - t_k = \frac{2}{L}\,(2\, q_0)^{2^k}, \qquad (k \ge 0),$$

and therefore

$$t_{k+1} = t_1 + (t_2 - t_1) + \cdots + (t_{k+1} - t_k) = \frac{2}{L}\,\sum_{m=0}^{k}(2\, q_0)^{2^m},$$

$$t^\star = \lim_{k\to\infty} t_k = \frac{2}{L}\,\sum_{k=0}^{\infty}(2\, q_0)^{2^k}.$$

Clearly, this series converges, (as $k \le 2^k$ and $2\, q_0 < 1$), and is bounded above by the number

$$\frac{2}{L}\,\sum_{k=0}^{\infty}(2\, q_0)^{k} = \frac{4}{L\,(2 - L\,\eta)}.$$

If $L = 0$, then in view of (3.14), $0 \le L_0 \le L$, we deduce that $L_0 = 0$ and $t^\star = t_k = \eta$ $(k \ge 1)$.

In the rest of the proof, we assume that $L_0 > 0$. The result until estimate (3.16b) follows from Lemma 1 in [2] (see also [1, 3]).

Note that in particular Newton-Kantorovich-type convergence condition (3.12) is given in [2, page 387, Case 3 for δ given by (3.15b). The factor η is missing from the left hand side of the inequality three lines before the end of page 387].

In order for us to show (3.16c) we need the estimate:

$$\frac{1 - (\delta/2)^{k+1}}{1 - \delta/2}\,\eta \le \frac{1}{L_0}\left(1 - \left(\frac{\delta}{2}\right)^{k-1} \frac{L}{4\,\overline{L}}\right), \qquad (k \ge 1). \tag{3.17}$$

For $k = 1$, (3.17) becomes

$$\left(1 + \frac{\delta}{2}\right)\eta \le \frac{4\,\overline{L} - L}{4\,\overline{L}\,L_0}$$

or using (3.15b) gives

$$\left(1+\frac{2\,L}{L+\sqrt{L^2+8\,L_0\,L}}\right)\eta \leq \frac{4\,L_0-L+\sqrt{L^2+8\,L_0\,L}}{L_0\,(4\,L_0+L+\sqrt{L^2+8\,L_0\,L})}.$$

In view of (3.12), it suffices to show that

$$\frac{L_0\,(4\,L_0+L+\sqrt{L^2+8\,L_0\,L})\,(3\,L+\sqrt{L^2+8\,L_0\,L})}{(L+\sqrt{L^2+8\,L_0\,L})\,(4\,L_0-L+\sqrt{L^2+8\,L_0\,L})} \leq 2\,\overline{L},$$

which is true as equality.

Let us now assume estimate (3.17) is true for all integers smaller or equal to k. We must show (3.17) holds for k being $k+1$ that

$$\frac{1-(\delta/2)^{k+2}}{1-\delta/2}\,\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^k\frac{L}{4\,\overline{L}}\right), \qquad (k\geq 1). \tag{3.18a}$$

or

$$\left(1+\frac{\delta}{2}+\left(\frac{\delta}{2}\right)^2+\cdots+\left(\frac{\delta}{2}\right)^{k+1}\right)\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^k\frac{L}{4\,\overline{L}}\right). \tag{3.18b}$$

We use the induction hypothesis to show (3.20c). It suffices

$$\frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^{k-1}\frac{L}{4\,\overline{L}}\right)+\left(\frac{\delta}{2}\right)^{k+1}\eta \leq \frac{1}{L_0}\left(1-\left(\frac{\delta}{2}\right)^k\frac{L}{4\,\overline{L}}\right), \tag{3.19a}$$

$$\text{or}\qquad \left(\frac{\delta}{2}\right)^{k+1}\eta \leq \frac{1}{L_0}\left(\left(\frac{\delta}{2}\right)^{k-1}-\left(\frac{\delta}{2}\right)^k\right)\frac{L}{4\,\overline{L}}, \tag{3.19b}$$

$$\text{or}\qquad \delta^2\,\eta \leq \frac{L\,(2-\delta)}{2\,\overline{L}\,L_0}. \tag{3.19c}$$

In view of (3.12) it now suffices to show that

$$\frac{2\,\overline{L}\,L_0\,\delta^2}{L\,(2-\delta)} \leq 2\,\overline{L},$$

which holds as equality by the choice of δ given by (3.15b). This completes the induction for the estimate (3.17).

We shall show (3.16c) using induction on $k\geq 0$: estimate (3.16c) is true for $k=0$ by (3.12), (3.14), and (3.15b). In order to show estimate (3.16c) for $k=1$, by noting that

$$t_2-t_1=\frac{L\,(t_1-t_0)^2}{2\,(1-L_0\,t_1)},$$

it suffices to observe

$$\frac{L\,\eta^2}{2\,(1-L_0\,\eta)} \le \delta\,\overline{L}\,\eta^2, \tag{3.20a}$$

$$\text{or} \qquad \frac{L}{1-L_0\,\eta} \le \frac{16\,\overline{L}\,L}{L+\sqrt{L^2+8\,L_0\,L}}, \qquad (\eta \neq 0), \tag{3.20b}$$

$$\text{or} \qquad \eta \le \frac{1}{L_0}\left(1-\frac{L+\sqrt{L^2+8\,L_0\,L}}{16\,\overline{L}}\right), \qquad (L_0\neq 0,\ L\neq 0). \tag{3.20c}$$

It follows from (3.12) that

$$\eta \le \frac{4}{L+4\,L_0+\sqrt{L^2+8\,L_0\,L}}. \tag{3.21}$$

It then suffices to show that

$$\frac{4}{L+4\,L_0+\sqrt{L^2+8\,L_0\,L}} \le \frac{1}{L_0}\left(1-\frac{L+\sqrt{L^2+8\,L_0\,L}}{8\,\overline{L}}\right), \tag{3.22a}$$

$$\text{or} \qquad \frac{L+\sqrt{L^2+8\,L_0\,L}}{8\,\overline{L}} \le 1-\frac{4\,L_0}{L+4\,L_0+\sqrt{L^2+8\,L_0\,L}}, \tag{3.22b}$$

$$\text{or} \qquad \frac{L+\sqrt{L^2+8\,L_0\,L}}{8\,\overline{L}} \le \frac{L+\sqrt{L^2+8\,L_0\,L}}{L+4\,L_0+\sqrt{L^2+8\,L_0\,L}}, \tag{3.22c}$$

which is true as equality.

Let us assume (3.16c) holds for all integers smaller or equal to k. We shall show (3.16c) holds for k being replaced by $k+1$.

Using (3.14), and the induction hypothesis, we have in turn

$$\begin{aligned} t_{k+2}-t_{k+1} &= \frac{L}{2\,(1-L_0\,t_{k+1})}\,(t_{k+1}-t_k)^2 \\ &\le \frac{L}{2\,(1-L_0\,t_{k+1})}\,\left((\delta/2)^k\,(2\,q_0)^{2^k-1}\,\eta\right)^2 \\ &\le \frac{L}{2\,(1-L_0\,t_{k+1})}\,\left((\delta/2)^{k-1}\,(2\,q_0)^{-1}\,\eta\right)\,\left((\delta/2)^{k+1}\,(2\,q_0)^{2^{k+1}-1}\,\eta\right) \\ &\le (\delta/2)^{k+1}\,(2\,q_0)^{2^{k+1}-1}\,\eta, \end{aligned} \tag{3.23}$$

where we have used the fact

$$\frac{L}{2\,(1-L_0\,t_{k+1})}\,\left((\delta/2)^{k-1}\,(2\,q_0)^{-1}\,\eta\right) \le 1, \qquad (k\ge 1). \tag{3.24}$$

Indeed, we can show instead of (3.24):

$$t_{k+1} \leq \frac{1}{L_0} \left(1 - (\delta/2)^{k-1} \frac{L}{4 \overline{L}} \right),$$

is true. First, by (3.16b), and the induction hypothesis:

$$\begin{aligned} t_{k+1} &\leq t_k + \frac{\delta}{2} (t_k - t_{k-1}) \\ &\leq t_1 + \frac{\delta}{2} (t_1 - t_0) + \cdots + \frac{\delta}{2} (t_k - t_{k-1}) \\ &\leq \eta + (\delta/2) \ \eta + \cdots + (\delta/2)^k \ \eta \\ &= \frac{1 - (\delta/2)^{k+1}}{1 - \delta/2} \ \eta \\ &\leq \frac{1}{L_0} \left(1 - (\delta/2)^{k-1} \frac{L}{4 \overline{L}} \right). \end{aligned} \tag{3.25}$$

That completes the induction for estimate (3.16c). Using estimate (3.20c) for $j \geq k$, we obtain in turn for $2 \, q_0 < 1$:

$$\begin{aligned} t_{j+1} - t_k &= (t_{j+1} - t_j) + (t_j - t_{j-1}) + \cdots + (t_{k+1} - t_k) \\ &\leq \left((\delta/2)^j \ (2 q_0)^{2^j - 1} + (\delta/2)^{j-1} \ (2 q_0)^{2^{j-1} - 1} + \cdots + (\delta/2)^k \ (2 q_0)^{2^k - 1} \right) \ \eta \\ &\leq \left(1 + (2 q_0)^{2^k} + \left((2 q_0)^{2^k} \right)^2 + \cdots \right) \left(\frac{\delta}{2} \right)^k (2 q_0)^{2^k - 1} \ \eta \\ &= \left(\frac{\delta}{2} \right)^k \frac{(2 q_0)^{2^k - 1} \ \eta}{1 - (2 q_0)^{2^k}}. \end{aligned} \tag{3.26}$$

Estimate (3.16d) follows from (3.26) by letting $j \longrightarrow \infty$. This completes the proof of Lemma 3.3. □

Remark 3.1. The sufficient convergence conditions (3.4a), (3.4b), (3.8), and (3.9) for sequences $\{v_k\}$, $\{w_k\}$, respectively are not so easy to verify in general, and their convergence is only linear. In view of Lemma 3.3, we can recover the quadratic convergence.

Let us define sequences $\{\overline{v}_k\}$, $\{\overline{w}_k\}$, $\{z_k\}$, $\{\overline{z}_k\}$ for $k \geq 1$:

$$\overline{v}_0 = 0, \quad \overline{v}_1 = t^\star, \quad \overline{v}_{k+1} = \overline{v}_k + \frac{M \ (\overline{v}_k - \overline{v}_{k-1})^2}{1 - K_0 \ \overline{v}_k}, \tag{3.27a}$$

$$\overline{w}_0 = 0, \quad \overline{w}_1 = \frac{2 \ \eta}{1 + \sqrt{1 - 2 \ L_0 \ \eta}}, \quad \overline{w}_{k+1} = \overline{w}_k + \frac{M \ (\overline{w}_k - \overline{w}_{k-1})^2}{1 - K_0 \ \overline{w}_k}, \tag{3.27b}$$

$$z_0 = 0, \quad z_1 = t^\star, \quad z_{k+1} = z_k + \frac{\sqrt{L \ M} \ (z_k - z_{k-1})^2}{1 - K_0 \ z_k}, \tag{3.27c}$$

$$\overline{z}_0 = 0, \quad \overline{z}_1 = w_1, \quad \overline{z}_{k+1} = \overline{z}_k + \frac{\sqrt{L \ M} \ \ (\overline{z}_k - \overline{z}_{k-1})^2}{1 - K_0 \ \overline{z}_k}. \tag{3.27d}$$

Note that under hypotheses of Lemmas 3.1 and 3.2, sequences $\{\overline{v}_k\}$, $\{\overline{w}_k\}$ can replace $\{v_k\}$, $\{w_k\}$, respectively. However, an inductive argument shows:

$$v_k \le \overline{v}_k, \tag{3.28a}$$
$$0 \le v_{k+1} - v_k \le \overline{v}_{k+1} - \overline{v}_k, \tag{3.28b}$$
$$0 \le v^\star - v_k \le \overline{v^\star} - \overline{v}_k, \tag{3.28c}$$
$$v^\star \le \overline{v^\star}, \qquad w_k \le \overline{w}_k, \tag{3.28d}$$
$$0 \le w_{k+1} - w_k \le \overline{w}_{k+1} - \overline{w}_k, \tag{3.28e}$$
$$0 \le w^\star - w_k \le \overline{w^\star} - \overline{w}_k, \tag{3.28f}$$
$$w^\star \le \overline{w^\star}, \tag{3.28g}$$

where

$$\overline{v^\star} = \lim_{k\to\infty} \overline{v}_k, \qquad \overline{w^\star} = \lim_{k\to\infty} \overline{w}_k.$$

Let us define constants:

$$\alpha = L + 4L_0 + \sqrt{L^2 + 8L_0 L}, \tag{3.29a}$$
$$\beta = 2\left(K + 2K_0 + \sqrt{M^2 + 4K_0 M}\right), \tag{3.29b}$$
$$\gamma = 2\left(\sqrt{LM} + 2K_0 + \sqrt{LM + 4K_0\sqrt{LM}}\right), \quad \lambda = \frac{1}{8}\max\{\alpha, \beta, \gamma\}, \tag{3.29c}$$
$$\alpha_1 = \frac{1}{2}\left(\frac{L}{\theta_0} + L_0\right)\frac{\eta}{t^\star}, \qquad \theta_0 \ne 0,\ t^\star \ne 0, \tag{3.29d}$$
$$\beta_1 = \frac{1}{2}\left(\frac{2M}{\theta_2} + K_0\right), \qquad \theta_2 \in (0,1), \tag{3.29e}$$
$$\gamma_1 = \frac{1}{2}\left(\frac{2\sqrt{LM}}{\theta_3} + K_0\right), \qquad \theta_3 \in (0,1), \tag{3.29f}$$
$$\theta_4 = \max\{\theta_0, \theta_2, \theta_3\}, \quad \lambda_1 = \frac{1}{2}\max\{\alpha_1, \beta_1, \gamma_1\}, \tag{3.29g}$$
$$\delta_1 = \frac{M}{M + \sqrt{M^2 + 4K_0 M}}, \qquad \delta_0 = \max\{\delta, \delta_1, \delta_2\}, \tag{3.29h}$$
$$\delta_2 = \frac{\sqrt{LM}}{\sqrt{LM} + \sqrt{LM + 4K_0\sqrt{LM}}}. \tag{3.29i}$$

Then simply using Lemmas 3.1 and 3.2 on sequences $\{\overline{v}_k\}$, $\{\overline{w}_k\}$ for

$$L_0 = K_0, \quad L = 2\,M, \quad \eta = t^\star,$$
$$L_0 = K_0, \quad L = 2\,\sqrt{L\,M}, \quad \eta = w_1,$$

respectively, we arrive at:

Lemma 3.4. *Suppose that*

(i) There exist nonnegative constants η, L_0, L, $t^\star$, K_0, M, $\theta_0 \in (0,1)$, $\theta_2 \in (0,1)$, $\theta_3 \in (0,1)$, *with* $t^\star \neq 0$, $0 \leq L_0 \leq L$, $K_0 \leq 2\,M$, $M \leq L$, *such that*

$$q_1 = \lambda_1 t^\star \leq \frac{1}{2}, \tag{3.30}$$

where $t^\star$ *is given in Lemma* 3.1, *and* λ_1 *is given by* (3.29g)*. The inequality in* (3.30) *is strict if* $L_0 = 0$, *or* $K_0 = 0$. *Then, the conclusions of Lemma* 3.3 *hold with* q_1, λ, $t^\star$, θ_4, $\{\overline{v}_k\}$, $\overline{v^\star}$, $\overline{v^{\star\star}}$, *replacing* q_0, $\overline{L}$, η, δ, $\{t_k\}$, $t^\star$, $t^{\star\star}$, *respectively, where,* $t^{\star\star}$ *was given in Lemma* 3.1, *and* θ_4, $\overline{v^\star}$, $\overline{v^{\star\star}}$ *in Remark* 3.1.

(ii) There exist nonnegative constants L_0, L, w_1, K_0, M, *with* $0 \leq L_0 \leq L$, $K_0 \leq 2\,M$, $M \leq L$, *such that:*

$$q_2 = \lambda\, w_1 \leq \frac{1}{2}, \tag{3.31}$$

The inequality in (3.31) *is strict if* $L_0 = 0$, *or* $K_0 = 0$. *Then, the conclusions of Lemma* 3.3 *hold with* q_2, λ, w_1, δ_0, $\{\overline{w}_k\}$, $\overline{w^\star}$, $\overline{w^{\star\star}}$, *replacing* q_0, $\overline{L}$, η, δ, $\{t_k\}$, $t^\star$, $t^{\star\star}$, *respectively, where,* $t^{\star\star}$ *was given in Lemma* 3.2, *and* $\overline{w^\star}$, $\overline{w^{\star\star}}$ *in Remark* 3.1.

Proof. Sequences $\{z_k\}$and $\{\overline{z}_k\}$ are nonnegative by (3.27c) and (3.27d), respectively. Therefore, the quantities under the radicals in (3.5), and (3.10a) are nonnegative, since

$$(1-K_0 v_k)^2 - LM(v_k - v_{k-1})^2 \geq (1-K_0 z_k)^2 - LM(z_k - z_{k-1})^2 \geq 0, \qquad (k \geq 1), \tag{3.32}$$
$$(1-K_0 w_k)^2 - LM(w_k - w_{k-1})^2 \geq (1-K_0 \overline{z}_k)^2 - LM(\overline{z}_k - \overline{z}_{k-1})^2 \geq 0, \quad (k \geq 1), \tag{3.33}$$

by the definition of sequences $\{v_k\}$, $\{w_k\}$, $\{z_k\}$, $\{\overline{z}_k\}$, $M \leq K$, and the estimates for all $k \geq 0$:

$$v_k \leq z_k, \qquad w_k \leq \overline{z}_k, \tag{3.34a}$$
$$v_{k+1} - v_k \leq z_{k+1} - z_k, \tag{3.34b}$$
$$w_{k+1} - w_k \leq \overline{z}_{k+1} - \overline{z}_k. \tag{3.34c}$$

This completes the proof of Lemma 3.4. □

Hence, we arrived at the concluding result.

Lemma 3.5. *Under hypotheses of (i) Lemmas* 3.1 *and* 3.4, *or (ii) Lemmas* 3.2 *and* 3.4, *the convergence of majorizing sequences* $\{v_k\}$, $\{w_k\}$ *is quadratic, and estimates* (3.28) *hold.*

Several applications can also be found in [3].

Theorem 3.1. *Suppose that* $F'(u_0)^{-1}$ *exists for some* $u_0 \in \mathscr{D}$, *and there exist constants* $M \geq 0$, $L \geq 0$, $0 \leq \theta < 2$, *and* $\eta \geq 0$, *such that* $0 < K_0 \leq M + L$, *and conditions* (1.1), (1.2), (2.2), (2.3), (2.6) *and* (2.7) *hold. Let the* w_m *be defined as in* (3.10) *and let* $w^\star$ *be its limit. Suppose further that condition* (3.4) *hold, and*

$$\| F'(u_0)^{-1} (f'(x) - f'(y)) \| \le M \| x - y \|, \qquad \forall x, y \in \mathscr{D}, \tag{3.35a}$$

$$\| F'(u_0)^{-1} (g'(x) - g'(y)) \| \le L \; \| x - y \|, \qquad \forall x, y \in \mathscr{D}. \tag{3.35b}$$

If $\overline{U}(u_0, w^\star) \subset \mathscr{D}$*, then Eq.* (1.3) *has a solution* u *in* $\overline{U}(u_0, w^\star)$*, and the iterates* $\{u_m\}$ *generated from the generalized Newton scheme* (2.9) *converge to* u*, and satisfy the error estimates for all* $m \ge 1$*:*

$$\| u_m - u_{m-1} \| \le w_m - w_{m-1}, \tag{3.36a}$$

$$\| u_m - u_0 \| \le w_m, \tag{3.36b}$$

$$\| u - u_m \| \le w^\star - w_m. \tag{3.36c}$$

If $R \ge w^\star$ *and* $K_0 \, (w^\star + R) \le 2$*, then the solution* u *is unique in the set* $\mathscr{D} \cap U(u_0, R)$*.*

Theorem 3.2. *Suppose that there exist constants* $M \ge 0$*,* $L \ge L_0 \ge 0$*, and* $\eta \ge 0$ *such that* $0 \le K_0 \le M + L_0$*, and* (2.2), (2.6), (3.7), (3.9) *and* (3.35) *hold. Let*

$$\overline{r_\star} = \frac{2\,\eta}{1 + \sqrt{1 - 2\,K_0\,\eta}}, \qquad \overline{R_\star} = \frac{2}{K_0} - \overline{r_\star}.$$

If $\overline{U}(u_0, w^\star) \subset \mathscr{D}$*, then Eq.* (1.3) *has a unique solution* u *in* $\overline{U}(u_0, w^\star)$*, and the iterates generated from the generalized Newton scheme* (2.9) *converge to* u *and satisfy the error estimates for all* $m \ge 1$*:*

$$\| u_m - u_{m-1} \| \le w_m - w_{m-1}, \tag{3.37a}$$

$$\| u_m - u_0 \| \le w_m, \tag{3.37b}$$

$$\| u - u_m \| \le w^\star - w_m, \tag{3.37c}$$

where $\{w_m\}$ *and* $w^\star$ *are given in Lemma* 3.2*.*

The solution u *is also unique in sets* $\mathscr{D} \cap (\overline{U}(u_0, \overline{r_\star}) \cup U(u_0, \overline{R_\star}))$*, and* $\mathscr{D} \cap U(u_0, T)$*, where* T *is any nonnegative number such that:*

$$K_0 \, (\overline{r_\star} + T) < 2. \tag{3.38}$$

Example 3.1. Let $\mathscr{X} = Y = \mathbb{R}$, $\mathscr{D} = (-1, 1)$, and $u_0 = 0$. Define F by

$$F(x) = 9\,x^4 + 10\,x - 0.95. \tag{3.39}$$

(a) The constants employed in Theorem 2.2 are $K = 10.8$, $K_0 = 3.6$, $\eta = 0.095$. An examination of the graph of the function $\theta \mapsto (K + K_0\,\theta)\,\eta - \theta$ shows that condition (2.7) does not hold for any $\theta \in [0, 2]$. Therefore the conditions of Theorem 2.2 do not hold. However, if we introduce the splitting

$$F(x) = f(x) + g(x) \tag{3.40}$$

with $f(x) = x^4$, $g(x) = 8x^4 + 10x - 0.95$. Then the constants employed in Theorem 3.1 are $M = 1.2$, $M_0 = 0.4$, $L_0 = 3.2$, $L = 9.6$. An examination of the graphs of the functions

$$\theta \mapsto (L + L_0\,\theta)\ - \theta,$$
$$\theta \mapsto (4\,M + L\,\theta^2 + 4\,K_0\,\theta)\,w_1 - 4\,\theta,$$
$$\theta \mapsto 4\,M\,\theta + L\,\theta^3 + 4\,K_0\,\theta^2 - 8\,M - 2\,L\,\theta^2$$

shows that conditions of Theorem 3.1 hold with

$$\theta = 1,\ 1.310344828 < \theta < 2, \quad w_1 = \lim_{m\to\infty} \alpha_m \approx 0.2176326776574304,$$

where

$$\alpha_0 = 0, \quad \alpha_1 = \eta, \quad \alpha_{m+1} = \alpha_m + \frac{L\,(\alpha_m - \alpha_{m-1})^2}{2\,(1 - L_0\,\alpha_m)}.$$

(b) The constants K, K_0, and η of Theorem 2.3 are the same that (a). It is easy to verify that the conditions of Theorem 2.3 do not hold and – *a priori* – that the hypotheses of the Kantorovich theorem [7] do not hold. However, if we introduce the splitting (3.40), then, the constants employed in Theorem 3.2 are $M = 1.2$, $L_0 = 3.2$, $L = 9.6$, and $\eta = 0.095$. Since $\theta^\star(M, K_0, L) = 1$, it is easy to verify that the conditions of Theorem 3.2 hold and that there exists a unique solution u of equation (1.3) such that $|u| \leq 0.116844052$.

Acknowledgments The authors would like to thank the referees for the helpful suggestions.

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