

Extrapolation of Mixed Finite Element Approximations for the Maxwell Eigenvalue Problem

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Abstract. In this paper, a general method to derive asymptotic error expansion formulas for the mixed finite element approximations of the Maxwell eigenvalue problem is established. Abstract lemmas for the error of the eigenvalue approximations are obtained. Based on the asymptotic error expansion formulas, the Richardson extrapolation method is employed to improve the accuracy of the approximations for the eigenvalues of the Maxwell system from $\mathcal{O}(h^2)$ to $\mathcal{O}(h^4)$ when applying the lowest order Nédélec mixed finite element and a nonconforming mixed finite element. To our best knowledge, this is the first superconvergence result of the Maxwell eigenvalue problem by the extrapolation of the mixed finite element approximation. Numerical experiments are provided to demonstrate the theoretical results.

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Key words: Maxwell eigenvalue problem, mixed finite element, asymptotic error expansion, Richardson extrapolation.

1. Introduction

The Maxwell eigenvalue problems are of basic importance in designing the microwave resonators, microwave ovens, and communication equipments [2,3,16]. Assuming that the electromagnetic fields are time-harmonic, after elimination of the magnetic field intensity, the Maxwell eigenvalue problem can be formulated by [5, 11]

$$\nabla \times \nabla \times p = \lambda p, \quad \text{in } \Omega, \quad (1.1)$$

$$\nabla \cdot p = 0, \quad \text{in } \Omega, \quad (1.2)$$

$$p \times n = 0, \quad \text{on } \partial\Omega, \quad (1.3)$$

where Ω is a bounded cubic domain in \mathcal{R}^3 .

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There are many results for the Maxwell eigenvalue problems based on the finite element modeling and analysis, see, e.g., [1, 3, 5, 8, 15]. It is known, see [19], that the modeling of electromagnetic resonances is delicate. The early attempts to calculate FEMs approximation may lead to the occurrence of non-physical, so-called spurious eigenmodes [4, 16]. In order to overcome this difficulty, many methods have been proposed. In general, there are two possibilities: either one imposes the constraint of divergence-freeness on the problem, or one looks for an easy identification of the eigenvectors from the kernel of the curl operator. In order to impose the divergence-free constraint on the problem, one may try to incorporate this property in the definition of the discrete function spaces [7, 11, 30, 31, 37]. Several researchers prefer to impose this constraint implicitly, using mixed formulations [2, 3, 15, 20]. The finite element approximation of the Maxwell equation has also been studied to solve the electromagnetic field in [11, 21, 28].

When approximating the eigenvalue of Maxwell equations by the finite element method, it has been proved that it is not easy to distinguish the spurious values [4] using the nodal elements, especially for the lower order nodal elements. Such spurious values can be removed by the weighted regularization method [10] or by the least-square method [6]. For the non-smoothing solution not in $H^1(\Omega)$, a element-local L_2 projection technique was presented in [14] to deal with the nonconvex Lipschitz polyhedron with reentrant corners and edges. However, it has been shown that edge elements for the eigenvalue Maxwell equations can have a good approximation on the affine mesh [2–4], though can not achieve optimal approximation on non-affine mesh for the lower order edge elements [5]. In [23, 24], one of the variational form EQ_1^{rot} of Rannacher-Turek nonconforming element was proposed and numerical examples were shown that it can produce better approximations for the eigenvalue of elliptic problem. In three dimensions, this nonconforming finite element is a face-element. And it has been applied to solve the time-harmonic Maxwell's equations with the absorbing boundary condition by Douglağ etc. in [12].

To enhance the finite element eigenvalues approximation, one of the most important techniques is the polynomial preserving recovery (PPR) [29, 36]. Remarkable fourth order convergence is observed for linear elements under structured meshes as well as unstructured initial meshes with the conventional refinement. The extrapolation method is another important technique to enhance the finite element eigenvalue approximation. For many eigenvalue problems, the extrapolation method can be applied to improve the accuracy of eigenvalue approximation on the structured mesh. Based on asymptotic expansions of error approximation, one or two orders of convergence rate can be improved by a simple extrapolation postprocessing, see, e.g., [9, 17, 18, 22, 25, 35]. These accelerators are all based on an extremely important assumption that the solution should be higher smoothing. Most of these researches were presented for conforming finite elements or conforming mixed finite elements except for [25, 35]. For mixed finite element, one of the difficulties is that it is hard to derive asymptotic error expansion formulations. So far, we have not found any superconvergence results for the Maxwell eigenvalue problem by the extrapolation of the mixed finite element approximation.

In this paper, we assume that the exact solution is sufficiently smooth in order to employ the extrapolation technique. For the lowest order Nédélec mixed finite element and

a nonconforming mixed finite element introduced in [12, 32], we derive some necessary asymptotic error expansion formulas in discrete $H(\text{curl})$ space automatically and formulate abstract theorems for the eigenvalue and its approximation on the uniform mesh. In the end, a posterior extrapolation method is used to improve the accuracy from $\mathcal{O}(h^2)$ to $\mathcal{O}(h^4)$. The extrapolation method has also been applied to get the superconvergence solution of the electromagnetic field for the Maxwell equation by mixed finite element approximations in [32, 33].

This paper is organized as follows: In Section 2, the lowest order Nédélec mixed finite space and a nonconforming mixed finite element space are introduced. In Section 3, abstract theorems of asymptotic eigenvalues of the Maxwell system by the lowest order Nédélec mixed finite element and the nonconforming mixed finite element are derived. Some helpful asymptotic error expansion formulas are also presented. In Section 4, the Richardson extrapolation method is used to enhance the accuracy of eigenvalue approximations. A remarkably convergence rate $\mathcal{O}(h^4)$ for the eigenvalue approximation is obtained. In Section 5, numerical experiments are given to show the efficiency of our method. The paper ends with some conclusions in Section 6.

2. Mixed finite element formulation

Denote the Hilbert space

$$\begin{aligned} Q &= \mathbf{H}_0(\text{curl}, \Omega) = \{p \in (L^2(\Omega))^3; \nabla \times p \in (L^2(\Omega))^3, p \times n = 0 \text{ on } \partial\Omega\}, \\ V &= (L^2(\Omega))^3, \end{aligned}$$

and the standard variational formulation for problem (1.1)-(1.3) is: finding $0 \neq \lambda \in R$ such that $p \in Q, (p, p) = 1$ and

$$(\nabla \times p, \nabla \times q) = \lambda(p, q), \quad \forall q \in Q. \tag{2.1}$$

An equivalent mixed formulation [3] is: finding $(\lambda, u, p) \in R \times V \times Q$ such that $\lambda \neq 0, (p, p) = 1$ and

$$a(u, v) + b(v, p) = 0, \quad \forall v \in V, \tag{2.2}$$

$$b(u, q) = -\lambda(p, q), \quad \forall q \in Q, \tag{2.3}$$

where

$$a(u, v) = \int_{\Omega} uv dx dy dz, \quad b(v, p) = \int_{\Omega} v \cdot \nabla \times p dx dy dz.$$

2.1. Nédélec mixed finite element approximation

Here, we consider the lowest order Nédélec mixed finite element to approximate (2.2)-(2.3). In [26, 31], the lowest Nédélec mixed finite element space is

$$Q_h = \{\phi \in (L^2(\Omega))^3; \phi|_K \in Q_{0,1,1} \times Q_{1,0,1} \times Q_{1,1,0}\}, \tag{2.4}$$

$$U_h = \{\psi \in (L^2(\Omega))^3; \psi|_K \in Q_{1,0,0} \times Q_{0,1,0} \times Q_{0,0,1}\}. \tag{2.5}$$

Obviously $Q_h \subset Q$. The corresponding interpolation operator is defined by π_h, I_h , respectively.

The approximation version of (2.2)-(2.3) is: finding $(\lambda_h, u_h, p_h) \in R \times U_h \times Q_h$ such that $\lambda_h \neq 0, (p_h, p_h) = 1$ and

$$a(u_h, v) + b(v, p_h) = 0, \quad \forall v \in U_h, \tag{2.6}$$

$$b(u_h, q) = -\lambda_h(p_h, q), \quad \forall q \in Q_h. \tag{2.7}$$

2.2. Nonconforming mixed finite element approximation

Now we introduce a nonconforming mixed finite element. Let \mathcal{T}_h be a uniform partition of $\Omega \subset R^3$, and denote each element $K = [x_K - h_x, x_K + h_x] \times [y_K - h_y, y_K + h_y] \times [z_K - h_z, z_K + h_z]$ with size $h = \max\{h_x, h_y, h_z\}$.

Let $\hat{K} = [-1, 1]^3$ be a reference cube element, and denote $\hat{Q} = \hat{Q}_x \times \hat{Q}_y \times \hat{Q}_z$, where

$$\hat{Q}_x = \text{Span}\{1, \hat{y}, \hat{z}, \hat{y}^2, \hat{z}^2\},$$

$$\hat{Q}_y = \text{Span}\{1, \hat{z}, \hat{x}, \hat{z}^2, \hat{x}^2\},$$

$$\hat{Q}_z = \text{Span}\{1, \hat{x}, \hat{y}, \hat{x}^2, \hat{y}^2\}.$$

Denote $\hat{g}_i, i = 1, \dots, 6$, be the front, back, left, right, upper and lower face of the reference. For $\hat{u} \in \mathbf{H}_0(\mathbf{curl}, \hat{K})$, the local interpolation operator is defined as follows: $\hat{\pi} : \mathbf{H}_0(\mathbf{curl}, \hat{K}) \rightarrow \hat{Q}(\hat{K})$

$$\frac{1}{|\hat{g}_i|} \int_{\hat{g}_i} (\hat{\pi}_{\hat{K}} \hat{\phi} - \hat{\phi}) d\hat{s} = 0, \quad i = 1, \dots, 6, \quad \frac{1}{|\hat{K}|} \int_{\hat{K}} (\hat{\pi}_{\hat{K}} \hat{\phi} - \hat{\phi}) d\hat{x} d\hat{y} d\hat{z} = 0. \tag{2.8}$$

In every element K , the local interpolation operator is defined by

$$\pi_K \phi = (\hat{\pi}_{\hat{K}} \hat{\phi}) \circ F_K^{-1},$$

where affine mapping $F_K : \hat{K} \rightarrow K$ is

$$x = h_x \hat{x} + x_K, \quad y = h_y \hat{y} + y_K, \quad z = h_z \hat{z} + z_K.$$

So the interpolation operator π_h in the domain Ω is defined by

$$\pi_h \phi|_K = \pi_K \phi.$$

Note that (2.8) provides fifteen degrees of freedom needed to determine an element in $\hat{Q}(\hat{K})$. Next let $\hat{S} = \hat{S}_x \times \hat{S}_y \times \hat{S}_z$, where

$$\hat{S}_x = \text{Span}\{1, \hat{y}, \hat{z}\},$$

$$\hat{S}_y = \text{Span}\{1, \hat{z}, \hat{x}\},$$

$$\hat{S}_z = \text{Span}\{1, \hat{x}, \hat{y}\}.$$

and define a local interpolation $\hat{I} : (L^2(\hat{K}))^3 \rightarrow \hat{S}(\hat{K})$ as follows. For $\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3) \in \hat{S}(\hat{K})$

$$\int_{\hat{K}} (\hat{I}_{\hat{K}} \hat{\psi}_l - \hat{\psi}_l) d\hat{x} d\hat{y} d\hat{z} = 0, \quad \int_{\hat{K}} \text{curl}(\hat{I}_{\hat{K}} \hat{\psi}_l - \hat{\psi}_l) d\hat{x} d\hat{y} d\hat{z} = 0, \quad l = 1, 2, 3, \quad (2.9)$$

where curl is defined as

$$\text{curl} \hat{\psi}_1 = \left(\frac{\partial \hat{\psi}_1}{\partial \hat{z}}, -\frac{\partial \hat{\psi}_1}{\partial \hat{y}} \right), \quad \text{curl} \hat{\psi}_2 = \left(\frac{\partial \hat{\psi}_2}{\partial \hat{x}}, -\frac{\partial \hat{\psi}_2}{\partial \hat{z}} \right), \quad \text{curl} \hat{\psi}_3 = \left(\frac{\partial \hat{\psi}_3}{\partial \hat{y}}, -\frac{\partial \hat{\psi}_3}{\partial \hat{x}} \right).$$

In every element K , the local interpolation operator I_K is defined by

$$I_K \psi = (\hat{I}_{\hat{K}} \hat{\psi}) \circ F_K^{-1},$$

so the interpolation operator I_h in the domain Ω is defined by

$$I_h \psi|_K = I_K \psi.$$

Note that (2.9) provides nine degrees of freedom needed to determine an element in $\hat{S}(\hat{K})$ and that

$$\nabla \times \hat{Q} = \hat{S}.$$

The nonconforming mixed finite element space will be defined as

$$W_h = \left\{ \phi \in (L^2(\Omega))^3 : \phi|_K \circ F_K \in \hat{Q}, K \in \mathcal{T}_h, \int_g [\phi] ds = 0, \phi \times n|_{\partial\Omega} = 0 \right\}, \quad (2.10)$$

$$V_h = \{ \psi \in (L^2(\Omega))^3 : \psi|_K \circ F_K \in \hat{S}, K \in \mathcal{T}_h \}, \quad (2.11)$$

with the respective norm

$$\|\phi\|_h = \sum_{K \in \mathcal{T}_h} (\|\phi\|_0^2 + \|\nabla \times \phi\|_0^2)^{1/2}, \quad \|\psi\|_0 = \left(\sum_{K \in \mathcal{T}_h} \int_K \phi^2 dx dy dz \right)^{1/2}.$$

The constructed finite element spaces are divergence-free by

$$\nabla \cdot \phi_h = \frac{\partial \phi_{1h}}{\partial x} + \frac{\partial \phi_{2h}}{\partial y} + \frac{\partial \phi_{3h}}{\partial z} = 0.$$

For each ϕ_{1h} , it is the integration continuous on the adjoint face of two element in y -direction and z -direction, and it is continuous on the face in x -direction because of E_{1h} not containing the variant x . For ϕ_{2h}, ϕ_{3h} , we have the similar conclusion. On the other hand, the constructed mixed finite element spaces satisfy inf-sup condition

$$\begin{aligned} \sup_{\phi \in W_h} \frac{b_h(\phi, \psi)}{\|\phi\|_h} &\geq \sup_{q \in H_0(\text{curl}, \Omega)} \frac{b_h(\pi_h q, \psi)}{\|\pi_h q\|_h} \\ &= \sup_{q \in H_0(\text{curl}, \Omega)} \frac{b_h(q, \psi)}{\|\pi_h q\|_h} \geq C \sup_{q \in H_0(\text{curl}, \Omega)} \frac{b_h(q, \psi)}{\|q\|_{H_0(\text{curl}, \Omega)}}, \end{aligned}$$

which can impose the divergence-free condition on the eigenvalue problem implicitly. We can overcome the spurious modes [15] when using this new mixed finite element space. From the above analysis, this new element method will satisfy Gaussian’s laws, and it can not exist spurious modes.

Therefore, the mixed finite element discrete version of Eqs. (2.2)- (2.3) is defined by: find $(\lambda_h, u_h, p_h) \in R \times V_h \times W_h$ such that $\lambda_h \neq 0, (p_h, p_h) = 1$ and

$$a_h(u_h, v) + b_h(v, p_h) = 0, \quad \forall v \in V_h, \tag{2.12}$$

$$b_h(u_h, q) = -\lambda_h(p_h, q), \quad \forall q \in W_h, \tag{2.13}$$

where

$$a_h(u_h, v) = \sum_K \int_K u_h v dx dy dz, \quad b_h(v, p_h) = \sum_K \int_K v \cdot \nabla \times p_h dx dy dz.$$

Using the method in [9,17,27,34], we can will get a general error estimation as follows:

Theorem 2.1. *Suppose that (λ, u, p) and (λ_h, u_h, p_h) is the eigenpair of problem (2.2)-(2.3) and (2.6)-(2.7) or (2.12)-(2.13), respectively, then the following error estimations for the eigenvalue and eigenfunction hold:*

$$|\lambda - \lambda_h| \leq ch^2, \quad \|p - p_h\|_0 + \|u - u_h\|_0 \leq ch^2(\|p\|_2 + \|u\|_2). \tag{2.14}$$

In the following two sections, we can see that the accuracy of the approximations of the Maxwell eigenvalues can be improved remarkably from $\mathcal{O}(h^2)$ to $\mathcal{O}(h^4)$ by the Richardson extrapolation post process.

3. Asymptotic expansion of eigenvalue error

In order to employ the Richardson extrapolation post process, in this section, we will establish the asymptotic expansion formulations of the eigenvalue error firstly.

3.1. Asymptotic expansion formulation

When

$$h_x \equiv h_1, \quad h_y \equiv h_2, \quad h_z \equiv h_3, \quad \forall K \in \mathcal{T}_h, \tag{3.1}$$

the mesh \mathcal{T}_h is called uniform [23]. The following lemmas give the important asymptotic expansion formulations.

Lemma 3.1. *Suppose that mesh \mathcal{T}_h is uniform, π_h is the interpolation operator in Q_h . Let $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3), \forall p \in (H^4(\Omega))^3 \cap \mathbf{H}_0(\mathbf{curl}, \Omega), \forall q \in Q_h$, we have*

$$\begin{aligned} (p - \pi_h p, q) = & -\frac{h_1^2}{3} \int_{\Omega} (p_{2xx}q_2 + p_{3xx}q_3) - \frac{h_2^2}{3} \int_{\Omega} (p_{1yy}q_1 + p_{3yy}q_3) \\ & - \frac{h_3^2}{3} \int_{\Omega} (p_{1zz}q_1 + p_{2zz}q_2) + \mathcal{O}(h^4)|p|_4\|q\|_0. \end{aligned} \tag{3.2}$$

Proof. The asymptotic error expansion formula in two-dimension can be found Lemma 3.1 in [33] and we can extend this result to three-dimension with the same technique. \square

Lemma 3.2. *Suppose that mesh \mathcal{J}_h is uniform, π_h is the interpolation operator in Q_h . Let $p = (p_1, p_2, p_3), v = (v_1, v_2, v_3), \forall p \in (H^4(\Omega))^3 \cap \mathbf{H}_0(\mathbf{curl}, \Omega), \forall v \in U_h$, we have*

$$\begin{aligned}
 b(v, \pi_h p - p) &= \frac{h_1^2}{3} \int_{\Omega} p_{3xy} v_1 - \frac{h_1^2}{3} \int_{\Omega} p_{2xz} v_1 - \frac{h_2^2}{3} \int_{\Omega} p_{3xy} v_2 \\
 &+ \frac{h_2^2}{3} \int_{\Omega} p_{1yz} v_2 + \frac{h_3^2}{3} \int_{\Omega} p_{2xz} v_3 - \frac{h_3^2}{3} \int_{\Omega} p_{1yz} v_3 + \mathcal{O}(h^4) |p|_4 \|v\|_0. \tag{3.3}
 \end{aligned}$$

Proof. For different $\hat{p}_i, i = 1, 2, 3$ on the reference element \hat{K} , their interpolations $\hat{\pi}_{\hat{K}} \hat{p}_i$ are shown in the following table

$\hat{p}_i,$	1	\hat{x}	\hat{y}	\hat{z}	$\hat{x}\hat{y}$	$\hat{x}\hat{z}$	$\hat{y}\hat{z}$	\hat{x}^2	\hat{y}^2	\hat{z}^2
$\hat{\pi}_{\hat{K}} \hat{p}_3$	1	\hat{x}	\hat{y}	0	$\hat{x}\hat{y}$	0	0	1	1	$\frac{1}{3}$
$\hat{\pi}_{\hat{K}} \hat{p}_2$	1	\hat{x}	0	\hat{z}	0	$\hat{x}\hat{z}$	0	1	$\frac{1}{3}$	1
$\hat{\pi}_{\hat{K}} \hat{p}_1$	1	0	\hat{y}	\hat{z}	0	0	$\hat{y}\hat{z}$	$\frac{1}{3}$	1	1
\hat{p}_i	\hat{x}^3	\hat{y}^3	\hat{z}^3	$\hat{x}^2\hat{y}$	$\hat{x}\hat{y}^2$	$\hat{x}^2\hat{z}$	$\hat{x}\hat{z}^2$	$\hat{y}^2\hat{z}$	$\hat{y}\hat{z}^2$	$\hat{x}\hat{y}\hat{z}$
$\hat{\pi}_{\hat{K}} \hat{p}_3$	\hat{x}	\hat{y}	0	\hat{y}	\hat{x}	0	$\frac{1}{3}\hat{x}$	0	$\frac{1}{3}\hat{y}$	0
$\hat{\pi}_{\hat{K}} \hat{p}_2$	\hat{x}	0	\hat{z}	0	$\frac{1}{3}\hat{x}$	\hat{z}	\hat{x}	$\frac{1}{3}\hat{z}$	0	0
$\hat{\pi}_{\hat{K}} \hat{p}_1$	0	\hat{y}	\hat{z}	$\frac{1}{3}\hat{y}$	0	$\frac{1}{3}\hat{z}$	0	\hat{z}	\hat{y}	0

Using the result

$$\begin{aligned}
 b(v, \pi_h p - p) &= \sum_K \int_K \left[((\pi_K p - p)_{3y} - (\pi_K p - p)_{2z}) v_1 \right. \\
 &\quad \left. - ((\pi_K p - p)_{3x} - (\pi_K p - p)_{1z}) v_2 + ((\pi_K p - p)_{2x} - (\pi_K p - p)_{1y}) v_3 \right],
 \end{aligned}$$

we can define a bilinear form

$$B(\hat{p}, \hat{v}_1) = \int_{\hat{K}} (\hat{\pi}_{\hat{K}} \hat{p} - \hat{p})_{3y} \hat{v}_1.$$

Consequently, we have

$$|B(\hat{p}, \hat{v}_1)| \leq C \|\hat{p}\|_{4\hat{K}} \|\hat{v}_1\|_{0\hat{K}}.$$

Here and after let $\hat{v}_1 = (1, \hat{x})$. By calculating,

$$\begin{aligned}
 B(\hat{x}^2\hat{y}, \hat{v}_1) &= \int_{\hat{K}} (\hat{y} - \hat{x}^2\hat{y})_{\hat{y}}(1, \hat{x}) = \frac{16}{3}(1, 0) = \frac{1}{3} \int_{\hat{K}} p_{3\hat{x}\hat{y}} \hat{v}_1, \\
 \hat{p}_3 = \hat{x}^2\hat{y}, \quad B(\hat{p}_3/\hat{x}^2\hat{y}, \hat{v}_1) &= 0.
 \end{aligned}$$

Let

$$H(\hat{p}_3, \hat{v}_1) = B(\hat{p}_3, \hat{v}_1) - \frac{1}{3} \int_{\hat{K}} p_{3\hat{x}\hat{x}\hat{y}} \hat{v}_1.$$

Then

$$H(\hat{P}_3, \hat{v}_1) = 0, \quad |H(\hat{P}_3, \hat{v}_1)| \leq C |\hat{p}_3|_{4\hat{K}} \|\hat{v}_1\|_{0\hat{K}}.$$

Therefore, by Bramble–Hilbert Lemma [see Lemma 2.6, [23]], we have

$$B(\hat{p}_3, \hat{v}_1) = \frac{1}{3} \int_{\hat{K}} p_{3\hat{x}\hat{x}\hat{y}} \hat{v}_1 + \mathcal{O}(1) |\hat{p}_3|_{4\hat{K}} \|\hat{v}_1\|_{0\hat{K}}.$$

Hence,

$$\int_K (p - \pi_K p)_{3y} v_1 dx dy dz = \frac{h_1^2}{3} \int_K p_{3xx} v_1 + \mathcal{O}(h^4) |p_3|_{4K} \|v_1\|_{0K}. \tag{3.4}$$

Similarly,

$$-\int_K (p - \pi_K p)_{2z} v_1 dx dy dz = -\frac{h_1^2}{3} \int_K p_{2xxz} v_1 + \mathcal{O}(h^4) |p_2|_{4K} \|v_1\|_{0K}, \tag{3.5}$$

$$-\int_K (p - \pi_K p)_{3x} v_2 dx dy dz = -\frac{h_2^2}{3} \int_K p_{3xy} v_2 + \mathcal{O}(h^4) |p_3|_{4K} \|v_2\|_{0K}, \tag{3.6}$$

$$\int_K (p - \pi_K p)_{1z} v_2 dx dy dz = \frac{h_2^2}{3} \int_K p_{1yz} v_2 + \mathcal{O}(h^4) |p_1|_{4K} \|v_2\|_{0K}, \tag{3.7}$$

$$\int_K (p - \pi_K p)_{2x} v_3 dx dy dz = \frac{h_3^2}{3} \int_K p_{2xzz} v_3 + \mathcal{O}(h^4) |p_2|_{4K} \|v_3\|_{0K}, \tag{3.8}$$

$$-\int_K (p - \pi_K p)_{1y} v_3 dx dy dz = -\frac{h_3^2}{3} \int_K p_{1yzz} v_3 + \mathcal{O}(h^4) |p_1|_{4K} \|v_3\|_{0K}. \tag{3.9}$$

If the mesh is uniform, combining (3.4)-(3.9) on each element, we can finish the proof. \square

By the definition of interpolation, from [26, 33], we know

Lemma 3.3. I_h is the interpolation operator in Q_h , $\forall q \in Q_h, v \in U_h$, we have

$$a(u - I_h u, v) = 0, \tag{3.10}$$

$$b(u - I_h u, q) = 0. \tag{3.11}$$

From [32], we have the following lemma:

Lemma 3.4. Suppose that mesh \mathcal{T}_h is uniform, $\forall q \in W_h$, π_h is the interpolation operator in W_h , $\forall p \in (H^6(\Omega))^3 \cap \mathbf{H}_0(\mathbf{curl}, \Omega)$, $u \in (H^5(\Omega))^3$ and $v \in V_h$, I_h is the interpolation operator

in V_h , we have

$$(p - \pi_h p, q) = \mathcal{O}(h^4) \|p\|_4 \|q\|_0, \tag{3.12}$$

$$a_h(u - I_h u, v) = \mathcal{O}(h^4) \|u\|_4 \|v\|_{1h}, \tag{3.13}$$

$$b_h(u - I_h u, q) = \mathcal{O}(h^4) \|u\|_4 \|q\|_{2h}, \tag{3.14}$$

$$b_h(v, \pi_h p - p) = 0, \tag{3.15}$$

$$\begin{aligned} \sum_K \int_{\partial K} u \times n q ds &= \frac{h_1^2}{3} \int_{\Omega} (-u_{1xxy} q_3 + u_{1xxz} q_2) dx dy dz \\ &+ \frac{h_2^2}{3} \int_{\Omega} (-u_{2yyz} q_1 + u_{2xyy} q_3) dx dy dz \\ &+ \frac{h_3^2}{3} \int_{\Omega} (-u_{3xzz} q_2 + u_{3yzz} q_1) dx dy dz + \mathcal{O}(h^4) \|u\|_5 \|q\|_{2h}. \end{aligned} \tag{3.16}$$

3.2. Abstract asymptotic expansion formulation

Using the lowest order Nédélec mixed finite element (2.4)-(2.5) to approximate the eigenvalue problem of (2.2)-(2.3), we have the following abstract asymptotic expansion formulation:

Theorem 3.1. *Suppose that $(\lambda, u, p) \in R \times V \times Q$ and $(\lambda_h, u_h, p_h) \in R \times U_h \times Q_h$ are the eigenpairs of problem (2.2)-(2.3) and (2.6)-(2.7), respectively, then the following expansion for the eigenvalue error holds:*

$$\begin{aligned} \lambda_h - \lambda &= \lambda(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) \\ &+ a(u - I_h u, \bar{u}_h) - b(I_h u - u, \bar{p}_h) + \mathcal{O}(h^4), \end{aligned} \tag{3.17}$$

where $\bar{u}_h = \frac{u_h}{(p, p_h)}$, $\bar{p}_h = \frac{p_h}{(p, p_h)}$.

Proof. Define Maxwell project solution: finding $(\tilde{u}_h, \tilde{p}_h) \in U_h \times Q_h$ such that

$$a(\tilde{u}_h, v) + b(v, \tilde{p}_h) = 0, \quad \forall v \in U_h, \tag{3.18}$$

$$b(\tilde{u}_h, q) = -\lambda(p, q), \quad \forall q \in Q_h. \tag{3.19}$$

By the Maxwell's projection, we know

$$a(u - \tilde{u}_h, v) + b(v, p - \tilde{p}_h) = 0, \quad \forall v \in U_h,$$

$$b(u - \tilde{u}_h, q) = 0, \quad \forall q \in Q_h.$$

By $(p, \bar{p}_h) = 1$, we have

$$\lambda_h = \lambda_h(p, \bar{p}_h) = \lambda_h(p - \tilde{p}_h, \bar{p}_h) + \lambda_h(\tilde{p}_h, \bar{p}_h).$$

From (2.2)-(2.3), (2.6)-(2.7) and (3.18)-(3.19), we have

$$\lambda_h(\tilde{p}_h, \bar{p}_h) = -b(\bar{u}_h, \tilde{u}_h) = a(\tilde{u}_h, \bar{u}_h) = -b(\tilde{u}_h, \bar{p}_h) = \lambda(p, \bar{p}_h) = \lambda,$$

and

$$\begin{aligned}
 \lambda_h(p - \tilde{p}_h, \bar{p}_h) &= \lambda_h(p - \pi_h p, \bar{p}_h) + \lambda_h(\pi_h p - \tilde{p}_h, \bar{p}_h) \\
 &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - \tilde{p}_h) \\
 &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) - b(\bar{u}_h, p - \tilde{p}_h) \\
 &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - \tilde{u}, \bar{u}_h) \\
 &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) + a(I_h u - \tilde{u}_h, \bar{u}_h) \\
 &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) - b(I_h u - \tilde{u}_h, \bar{p}_h) \\
 &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) \\
 &\quad - b(I_h u - u, \bar{p}_h) - b(u - \tilde{u}_h, \bar{p}_h) \\
 &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) - b(I_h u - u, \bar{p}_h).
 \end{aligned}$$

By (2.14), we have $\lambda_h = \lambda + \mathcal{O}(h^2)$, which yields (3.17). Now we can finish the proof. \square

Using the nonconforming mixed finite element (2.10)–(2.11) to approximate the eigenvalues of Maxwell’s system (2.2)–(2.3), we have the following abstract asymptotic expansion formulation:

Theorem 3.2. *Suppose that $(\lambda, u, p) \in R \times V \times Q$ and $(\lambda_h, u_h, p_h) \in R \times V_h \times W_h$ are the eigenpairs of problem (2.2)–(2.3) and (2.12)–(2.13), respectively, then the following expansion for the eigenvalue error holds:*

$$\begin{aligned}
 \lambda_h - \lambda &= \lambda(p - \pi_h p, \bar{p}_h) - b_h(\bar{u}_h, \pi_h p - p) + a_h(u - I_h u, \bar{u}_h) \\
 &\quad - b_h(I_h u - u, \bar{p}_h) + \sum_K \int_{\partial K} u \times n \bar{p}_h ds + \mathcal{O}(h^4), \tag{3.20}
 \end{aligned}$$

where $\bar{u}_h = u_h/(p, p_h)$, $\bar{p}_h = p_h/(p, p_h)$.

Proof. Define Maxwell project solution: finding $(\tilde{u}_h, \tilde{p}_h) \in U_h \times Q_h$ such that

$$a_h(\tilde{u}_h, v) + b_h(v, \tilde{p}_h) = 0, \quad \forall v \in V_h, \tag{3.21}$$

$$b_h(\tilde{u}_h, q) = -\lambda(p, q), \quad \forall q \in W_h. \tag{3.22}$$

By the Maxwell’s projection, we know

$$a_h(u - \tilde{u}_h, v) + b_h(v, p - \tilde{p}_h) = 0, \quad \forall v \in U_h,$$

$$b_h(u - \tilde{u}_h, q) = -\sum_K \int_{\partial K} u \times n q ds, \quad \forall q \in Q_h.$$

By $(p, \bar{p}_h) = 1$, we have

$$\lambda_h = \lambda_h(p, \bar{p}_h) = \lambda_h(p - \tilde{p}_h, \bar{p}_h) + \lambda_h(\tilde{p}_h, \bar{p}_h).$$

From(2.2)–(2.3), (2.12)–(2.13) and (3.21)–(3.22), we have

$$\lambda_h(\tilde{p}_h, \bar{p}_h) = -b_h(\bar{u}_h, \tilde{u}_h) = a_h(\tilde{u}_h, \bar{u}_h) = -b_h(\tilde{u}_h, \bar{p}_h) = \lambda(p, \bar{p}_h) = \lambda,$$

and

$$\begin{aligned} \lambda_h(p - \tilde{p}_h, \bar{p}_h) &= \lambda_h(p - \pi_h p, \bar{p}_h) + \lambda_h(\pi_h p - \tilde{p}_h, \bar{p}_h) \\ &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - \tilde{p}_h) \\ &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) - b(\bar{u}_h, p - \tilde{p}_h) \\ &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - \tilde{u}, \bar{u}_h) \\ &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) + a(I_h u - \tilde{u}_h, \bar{u}_h) \\ &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) - b(I_h u - \tilde{u}_h, \bar{p}_h) \\ &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) \\ &\quad - b(I_h u - u, \bar{p}_h) - b(u - \tilde{u}_h, \bar{p}_h) \\ &= \lambda_h(p - \pi_h p, \bar{p}_h) - b(\bar{u}_h, \pi_h p - p) + a(u - I_h u, \bar{u}_h) - b(I_h u - u, \bar{p}_h) \\ &\quad + \sum_K \int_{\partial K} u \times n \bar{p}_h ds. \end{aligned}$$

By (2.14), we have $\lambda_h = \lambda + \mathcal{O}(h^2)$, which lead to (3.20). Now we can finish the proof. \square

4. Richardson extrapolation of the eigenvalue

Based on Theorems 2.1, 3.1–3.2 and Lemmas 3.3–3.4, we have the following theorem:

Theorem 4.1. *Suppose that $\lambda, \lambda_h \in \mathbb{R}$ is the eigenvalue of problem (2.2)–(2.3) and (2.12)–(2.13), respectively. For Nédélec mixed finite element, if the partition is uniform, $\forall p \in (H^4(\Omega))^3 \cap \mathbf{H}_0(\mathbf{curl}, \Omega), u \in (L_2(\Omega))^3$, the eigenvalues error estimation hold:*

$$\begin{aligned} \lambda - \lambda_h &= -\frac{h_1^2}{3} \int_{\Omega} (p_{2xx} p_2 + p_{3xx} p_3) - \frac{h_2^2}{3} \int_{\Omega} (p_{1yy} p_1 + p_{3yy} p_3) \\ &\quad - \frac{h_3^2}{3} \int_{\Omega} (p_{1zz} p_1 + p_{2zz} p_2) - \frac{h_1^2}{3} \int_{\Omega} p_{3xy} u_1 + \frac{h_1^2}{3} \int_{\Omega} p_{2xz} u_1 + \frac{h_2^2}{3} \int_{\Omega} p_{3xy} u_2 \\ &\quad - \frac{h_2^2}{3} \int_{\Omega} p_{1yz} u_2 - \frac{h_3^2}{3} \int_{\Omega} p_{2xz} u_3 + \frac{h_3^2}{3} \int_{\Omega} p_{1yz} u_3 + \mathcal{O}(h^4). \end{aligned} \tag{4.1}$$

Let $\lambda_h^{extra} = (4\lambda_{h/2} - \lambda_h)/3$. The Richardson extrapolation eigenvalue error is

$$\lambda - \lambda_h^{extra} = \mathcal{O}(h^4). \tag{4.2}$$

Proof. From

$$1 - (p, p_h) = \frac{1}{2}(p - p_h, p - p_h) \leq Ch^2,$$

we have $(p, p_h) \geq 1 - Ch^2$.

From Theorem 2.1 we have $\bar{u}_h = u + \mathcal{O}(h^2), \bar{p}_h = p + \mathcal{O}(h^2)$. Take $v = \bar{u}_h, q = \bar{p}_h$ in Lemmas 3.3–3.5, and from Theorem 3.1, we can derive (4.1). Using Richardson extrapolation formula, we can get (4.2). □

Theorem 4.2. *Suppose that $\lambda, \lambda_h \in R$ is the eigenvalue of problem (2.2)-(2.3) and (2.12)-(2.13), respectively. For nonconforming mixed finite element in Section 2, if the partition is uniform, $\forall p \in (H^6(\Omega))^3 \cap \mathbf{H}_0(\mathbf{curl}, \Omega), u \in (H^5(\Omega))^3$, the eigenvalues error estimation hold:*

$$\begin{aligned} \lambda - \lambda_h &= \frac{h_1^2}{3} \int_{\Omega} (-u_{1xxy}p_3 + u_{1xxz}p_2) dx dy dz + \frac{h_2^2}{3} \int_{\Omega} (-u_{2yyz}p_1 + u_{2xyy}p_3) dx dy dz \\ &\quad + \frac{h_3^2}{3} \int_{\Omega} (-u_{3xzz}p_2 + u_{3yzz}p_1) dx dy dz + \mathcal{O}(h^4). \end{aligned} \tag{4.3}$$

Let $\lambda_h^{extra} = (4\lambda_{h/2} - \lambda_h)/3$. The Richardson extrapolation eigenvalue error is

$$\lambda - \lambda_h^{extra} = \mathcal{O}(h^4). \tag{4.4}$$

Proof. From

$$1 - (p, p_h) = \frac{1}{2}(p - p_h, p - p_h) \leq Ch^2,$$

we have $(p, p_h) \geq 1 - Ch^2$. By Theorem 2.1, we have

$$\bar{u}_h = u + \mathcal{O}(h^2), \quad \bar{p}_h = p + \mathcal{O}(h^2).$$

Take $v = \bar{u}_h, q = \bar{p}_h$ in Lemma 3.6, from Theorem 3.2, we can derive (4.3). Using the Richardson extrapolation formula, we can get (4.4). □

Remark 4.1.

$$\lambda - \lambda_{h/2} = \frac{\lambda_h - \lambda_{h/2}}{3} + \mathcal{O}(h^4) \tag{4.5}$$

provides a posteriori error estimation $(\lambda_h - \lambda_{h/2})/3$ for $\lambda - \lambda_{h/2}$.

5. Numerical results

In this section, we give some numerical results to validate our theoretical analysis. Here we give some notations:

$$\begin{aligned} err_h &= \lambda - \lambda_h, & err_h^{extra} &= \lambda - \lambda_h^{extra}, \\ \alpha &= \frac{\log(err_h/err_{\frac{h}{2}})}{\log(2)}, & \alpha^{extra} &= \frac{\log(err_h^{extra}/err_{\frac{h}{2}}^{extra})}{\log(2)}. \end{aligned}$$

Here, α, α^{extra} stand for the numerical convergence order of eigenvalue from the mixed finite element approximation and the extrapolation approximation, respectively.

We choose

$$p = (\sin k\pi y \sin t\pi z, \sin k\pi z \sin t\pi x, \sin k\pi x \sin t\pi y), \quad \lambda = (k^2 + t^2)\pi^2$$

and $k, t \geq 0$ are integers. It is known that (λ, p) are eigenpairs of (1.1)–(1.3). We test the accuracy for both the Nédélec mixed finite element and the nonconforming mixed finite element on the uniform rectangular meshes.

In Tables 1–3, we list the approximations by the lowest order Nédélec mixed finite element of the Maxwell eigenvalues $2\pi^2, 5\pi^2$ and $10\pi^2$, respectively. In Tables 4–6, we list the approximations by the nonconforming mixed finite element (2.10)–(2.11) of the Maxwell eigenvalues $2\pi^2, 5\pi^2$ and $10\pi^2$, respectively. It is observed that the eigenvalues computed by the lowest order Nédélec mixed finite element approximate the Maxwell

Table 1: The approximations of eigenvalue $2\pi^2 \approx 19.73920880$ by the lowest order Nédélec mixed finite element.

N^3	$2 \times 2 \times 2$	$4 \times 4 \times 4$	$8 \times 8 \times 8$	$16 \times 16 \times 16$	$32 \times 32 \times 32$
λ_h	24	20.77328401	19.99416131	19.80270736	19.75506824
λ_h^{extra}	-	19.69771201	19.73445375	19.73888937	19.73918853
err_h	-4.2608	-1.0341	-2.55E-01	-6.35E-02	1.59E-02
err_h^{extra}	-	-0.0415	-0.0048	-3.19E-04	-2.03E-05
α	-	2.042748629	2.02E+00	2.01E+00	2.00E+00
α^{extra}	-	-	3.112005026	3.91E+00	3.98E+00

Table 2: The approximations of eigenvalue $5\pi^2 \approx 49.34802201$ by the lowest order Nédélec mixed finite element.

N^3	$4 \times 4 \times 4$	$8 \times 8 \times 8$	$16 \times 16 \times 16$	$32 \times 32 \times 32$
λ_h	51.32899651	49.90215741	49.48970653	49.38363202
λ_h^{extra}	-	49.42654437	49.3522229	49.34827386
err_h	-1.980974503	-0.554135403	-0.141684525	-0.03561002
err_h^{extra}	-	-0.078522369	-0.004200899	-0.000251851
α	-	1.837899865	1.96755635	1.99232706
α^{extra}	-	-	4.224333788	4.060055064

Table 3: The approximations of eigenvalue $10\pi^2 \approx 98.69604401$ by the lowest order Nédélec mixed finite element.

N^3	$4 \times 4 \times 4$	$8 \times 8 \times 8$	$16 \times 16 \times 16$	$32 \times 32 \times 32$
λ_h	1.13E+02	1.03E+02	99.70825598	98.94956997
λ_h^{extra}	-	99.21853695	98.70970614	98.69667463
err_h	-14.46396713	-4.007861487	-1.012211968	-0.253525954
err_h^{extra}	-	-0.52249294	-0.013662128	-0.000630617
α	-	1.85155875	1.985321212	1.997306083
α^{extra}	-	-	5.257157384	4.437275319

eigenvalues from above with the convergence order $\mathcal{O}(h^2)$, and the eigenvalues computed by the nonconforming mixed finite element (2.10)-(2.11) approximate the Maxwell eigenvalues from below with the convergence order $\mathcal{O}(h^2)$. After a Richardson extrapolation post process, we have successfully improved the accuracy of the approximations of the Maxwell eigenvalues from $\mathcal{O}(h^2)$ to $\mathcal{O}(h^4)$.

Table 4: The approximations of eigenvalue $2\pi^2 \approx 19.73920880$ by the nonconforming mixed finite element (2.10)–(2.11).

N^3	$2 \times 2 \times 2$	$4 \times 4 \times 4$	$8 \times 8 \times 8$	$16 \times 16 \times 16$	$32 \times 32 \times 32$
λ_h	16.84801921	18.81863788	19.49188625	19.67619616	19.72337988
λ_h^{extra}	–	19.47551076	19.71630238	19.73763279	19.73910779
err_h	2.891189587	0.920570926	0.247322549	0.063012646	0.015828919
err_h^{extra}	–	0.263698038	0.022906424	0.001576012	1.01E-04
α	–	1.651062432	1.896135097	1.972680476	1.993078624
α^{extra}	–	–	3.525062684	3.86140176	3.96E+00

Table 5: The approximations of eigenvalue $5\pi^2 \approx 49.34802201$ by the nonconforming mixed finite element (2.10)–(2.11).

N^3	$4 \times 4 \times 4$	$8 \times 8 \times 8$	$16 \times 16 \times 16$	$32 \times 32 \times 32$
λ_h	46.33042078	48.40880933	49.09927197	49.28491547
λ_h^{extra}	-	49.10160551	49.32942619	49.34679663
err_h	3.017601223	0.939212677	0.248750031	0.063106536
err_h^{extra}	-	0.246416495	0.018595816	0.001225371
α	-	1.683878379	1.916755175	1.978835359
α^{extra}	-	-	3.728048877	3.923687015

Table 6: The approximations of eigenvalue $10\pi^2 \approx 98.69604401$ by the nonconforming mixed finite element (2.10)–(2.11).

N^3	$4 \times 4 \times 4$	$8 \times 8 \times 8$	$16 \times 16 \times 16$	$32 \times 32 \times 32$
λ_h	95.36880373	96.86940843	98.15588458	98.55530348
λ_h^{extra}	-	97.36961	98.58470996	98.68844311
err_h	3.327240277	1.826635578	0.540159431	0.140740532
err_h^{extra}	-	1.326434011	0.111334049	0.007600899
α	-	0.865137216	1.757731643	1.940347416
α^{extra}	-	-	3.574586125	3.872580952

6. Conclusion and future work

The main work of this paper is to derive the abstract lemma for the error of the eigenvalues of Maxwell system by the lowest order Nédélec and a nonconforming mixed finite element, which is aimed at improving the eigenvalue approximation by Richardson extrapolation formula. Some asymptotic error expansion formulas for the eigenvalue ap-

proximation are presented. As a by-product, we can apply various finite element spaces (for example, ECHL and MECHL spaces described in [13]) with high order accuracy to form a class of *a posteriori* error estimates for the eigenvalues of Maxwell system on the uniform mesh. The Richardson extrapolation method is employed to improve the accuracy of the approximations for the eigenvalues of the Maxwell system from $\mathcal{O}(h^2)$ to remarkable $\mathcal{O}(h^4)$. Numerical experiments have demonstrated the theoretical results and the efficiency of our proposed method.

In the numerical experiments, we have observed that the eigenvalues computed by the lowest order Nédélec mixed finite element approximate the Maxwell eigenvalues always from above and the eigenvalues computed by the nonconforming mixed finite element (2.10)–(2.11) approximate the Maxwell eigenvalues always from below. It is a quite interesting phenomenon, we will analyze it in the future work.

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