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Convergence Analysis of the Legendre Spectral Collocation Methods for Second Order Volterra Integro-Differential Equations

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Abstract. A class of numerical methods is developed for second order Volterra integro-differential equations by using a Legendre spectral approach. We provide a rigorous error analysis for the proposed methods, which shows that the numerical errors decay exponentially in the L^∞-norm and L^2-norm. Numerical examples illustrate the convergence and effectiveness of the numerical methods.

AMS subject classifications: 45L10, 65R20, 65D15

Key words: Second order Volterra integro-differential equations, Gauss quadrature formula, Legendre-collocation methods, convergence analysis.

1. Introduction

Second order Volterra integro-differential equations (VIDEs) arise in the mathematical model of physical and biological phenomena. This fact has led researchers to develop the theoretical and numerical analysis for such equations. For a survey of early results we refer the reader to [12, 19–21, 25]. More recently, polynomial spline collocation methods were investigated in [9, 23]. Bologna [22] found an asymptotic solution for first and second order VIDEs containing an arbitrary kernel. In [24], Sinc-collocation method was developed to approximate the second order VIDEs with boundary conditions.

So far, very few works have touched the spectral approximations to second order VIDEs. Spectral methods have been used in applied mathematics and scientific computing to numerically solve certain partial differential equations (PDEs) [2, 7, 10, 17]. In practice, spectral methods have excellent convergence properties with the so-called "exponential convergence" being the fastest possible. Recently, several authors have developed

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the spectral methods for the solutions of Volterra integral equations (VIEs) of the second kind [27,29,30], pantograph-type delay differential equations [15,16] and singularly perturbed problems [28]. The main purpose of this work is to apply the Legendre spectral-collocation methods for second order VIDEs. We will provide a rigorous error analysis which theoretically justifies the spectral rate of convergence.

For simplicity, denote $y^{(j)}(t) = \left(\partial^j/\partial t^j\right) y(t),\ j = 0,1,2$. In order to discuss the numerical solution of the second order VIDEs we consider the following linear integro-differential equation:

$$y^{(2)}(t) = q(t) + \sum_{j=0}^{1} p_j(t) y^{(j)}(t) + \sum_{j=0}^{1} \int_0^t K_j(t,s) y^{(j)}(s) ds, \qquad t \in \tilde{I} := [0,T], \tag{1.1}$$

with

$$y(0) = y_0, \quad y^{(1)}(0) = y_1, \tag{1.2}$$

where $q : \tilde{I} \to R$, $p_j : \tilde{I} \to R$ and $K_j : D \to R$ ($j = 0,1$) (with $D := \{(t,s) : 0 \le s \le t \le T\}$) are given functions and are assumed to be sufficiently smooth in the respective domains. The above equation is usually known as basic test equation and is suggested by Brunner and Lambert [14]. It has been widely used for analyzing the solution and stability properties of various methods.

For ease of analysis, we will describe the spectral methods on the standard interval $\hat{I} := [-1,1]$. Hence, we employ the transformation

$$t = \frac{T}{2}(1+x), \qquad x = \frac{2}{T}t - 1.$$

Then the above problem becomes

$$\begin{aligned} u^{(2)}(x) = \left(\frac{T}{2}\right)^2 \sum_{j=0}^{1} \int_0^{\frac{T}{2}(1+x)} K_j\left(\frac{T}{2}(1+x), s\right) y^{(j)}(s) ds \\ + b(x) + \sum_{j=0}^{1} a_j(x) u^{(j)}(x), \qquad x \in \hat{I} := [-1,1], \end{aligned} \tag{1.3}$$

with

$$u(-1) = u_{-1}, \qquad u^{(1)}(-1) = u'_{-1}, \tag{1.4}$$

where

$$\begin{aligned} &u(x) = y\left(\frac{T}{2}(1+x)\right), && b(x) = \left(\frac{T}{2}\right)^2 q\left(\frac{T}{2}(1+x)\right), \\ &a_0(x) = \left(\frac{T}{2}\right)^2 p_0\left(\frac{T}{2}(1+x)\right), && a_1(x) = \frac{T}{2} p_1\left(\frac{T}{2}(1+x)\right), \\ &u_{-1} = y_0, && u'_{-1} = \frac{T}{2} y_1. \end{aligned}$$

Furthermore, to transfer the integral interval $[0,\ T(1+x)/2]$ to the interval $[-1,x]$, we make a linear transformation: $s=T(1+\tau)/2,\ \ \tau\in[-1,x]$. Then Eq. (1.3) becomes

$$u^{(2)}(x)=b(x)+\sum_{j=0}^{1}a_j(x)u^{(j)}(x)+\sum_{j=0}^{1}\int_{-1}^{x}\tilde{K}_j(x,\tau)u^{(j)}(\tau)d\tau,\quad x\in\hat{I},\tag{1.5}$$

where

$$\tilde{K}_0(x,\tau)=\left(\frac{T}{2}\right)^3K_0\left(\frac{T}{2}(1+x),\frac{T}{2}(1+\tau)\right),$$
$$\tilde{K}_1(x,\tau)=\left(\frac{T}{2}\right)^2K_1\left(\frac{T}{2}(1+x),\frac{T}{2}(1+\tau)\right).$$

This paper is organized as follows. In Section 2, we introduce the spectral approaches for second order Volterra integro-differential equations. Some useful lemmas are provided in Section 3. These lemmas will play a key role in the derivation of the convergence analysis. The convergence analysis is provided in Section 4. In Section 5 we present some numerical examples in order to illustrate the convergence of the methods. These results show that the new methods are quite effective. Section 6 compares the solutions produced by both the polynomial spline collocation methods and the spectral methods. Finally, in Section 7, we end with conclusion and future work.

Throughout the paper C will denote a generic positive constant that is independent of N but which will depend on T and on the bounds for the given functions $p_j(t)$ and $K_j(t,s),\ \ j=0,1$.

2. Legendre-collocation methods

As demonstrated in the last section, we can assume that the solution domain is [-1,1]. The second order Volterra integro-differential equations in one-dimension are of the form (1.5), namely,

$$u^{(2)}(x)=b(x)+\sum_{j=0}^{1}a_j(x)u^{(j)}(x)+\sum_{j=0}^{1}\int_{-1}^{x}\tilde{K}_j(x,\tau)u^{(j)}(\tau)d\tau,\quad x\in\hat{I},\tag{2.1}$$

with

$$u(-1)=u_{-1},\qquad u^{(1)}(-1)=u'_{-1}.\tag{2.2}$$

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $(N+1)$ Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto points, and by $\{\omega_i\}_{i=0}^N$ the corresponding weights. Let $\mathscr{P}_N$ denote the space of all polynomials of degree not exceeding N. For any $v\in\mathscr{C}[-1,1]$ (see, e.g., [3,5,18]), we can define the Lagrange interpolating polynomial $I_Nv\in\mathscr{P}_N$, satisfying

$$I_Nv(x_i)=v(x_i),\qquad 0\le i\le N.$$

The Lagrange interpolating polynomial can be written in the form

$$I_N v(x)=\sum_{i=0}^{N} v(x_i)F_i(x),$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with the Legendre collocation points $\{x_i\}_{i=0}^N$.

In order that the spectral collocation methods are carried out naturally, integrating Eq. (2.1) and using (2.2), we get

$$u^{(1)}(x)=u'_{-1}+\sum_{j=0}^{1}\int_{-1}^{x} a_j(s)u^{(j)}(s)ds+\int_{-1}^{x} v(s)ds, \tag{2.3}$$

$$u(x)=u_{-1}+\int_{-1}^{x} u^{(1)}(s)ds, \tag{2.4}$$

$$v(x)=b(x)+\sum_{j=0}^{1}\int_{-1}^{x}\tilde{K}_j(x,s)u^{(j)}(s)ds. \tag{2.5}$$

Firstly, Eqs. (2.3)-(2.5) hold at the collocation points $\{x_i\}_{i=0}^N$ on [-1,1], namely,

$$u^{(1)}(x_i)=u'_{-1}+\sum_{j=0}^{1}\int_{-1}^{x_i} a_j(s)u^{(j)}(s)ds+\int_{-1}^{x_i} v(s)ds, \tag{2.6}$$

$$u(x_i)=u_{-1}+\int_{-1}^{x_i} u^{(1)}(s)ds, \tag{2.7}$$

$$v(x_i)=b(x_i)+\sum_{j=0}^{1}\int_{-1}^{x_i}\tilde{K}_j(x_i,s)u^{(j)}(s)ds, \tag{2.8}$$

for $0\le i\le N$. In order to obtain high order accuracy of the approximated solution, the main difficulty is to compute the integral term. In particular, for small values of x_i, there is little information available for $u(s)$ and $u^{(1)}(s)$. To overcome this difficulty, we transfer the integral interval $[-1,x_i]$ to a fixed interval [-1,1]

$$u^{(1)}(x_i)=u'_{-1}+\frac{1+x_i}{2}\sum_{j=0}^{1}\int_{-1}^{1} a_j(s(x_i,\theta))u^{(j)}(s(x_i,\theta))d\theta+\frac{1+x_i}{2}\int_{-1}^{1} v(s(x_i,\theta))d\theta, \tag{2.9}$$

$$u(x_i)=u_{-1}+\frac{1+x_i}{2}\int_{-1}^{1} u^{(1)}(s(x_i,\theta))d\theta, \tag{2.10}$$

$$v(x_i)=b(x_i)+\frac{1+x_i}{2}\sum_{j=0}^{1}\int_{-1}^{1}\tilde{K}_j(x_i,s(x_i,\theta))u^{(j)}(s(x_i,\theta))d\theta, \tag{2.11}$$

by using the following variable change

$$s=\frac{1+x_i}{2}\theta+\frac{x_i-1}{2}\triangleq s(x_i,\theta), \qquad \theta\in[-1,1]. \tag{2.12}$$

Next, using a $(N+1)$-point Gauss quadrature formula relative to the Legendre weights $\{\omega_i\}_{i=0}^N$ to approximate the integration term, we get

$$u_i^{(1)} = u'_{-1} + \frac{1+x_i}{2}\sum_{j=0}^{1}\left(\sum_{k=0}^{N} a_j(s(x_i,\theta_k))u^{(j)}(s(x_i,\theta_k))\omega_k\right) + \frac{1+x_i}{2}\sum_{k=0}^{N} v(s(x_i,\theta_k))\omega_k, \tag{2.13}$$

$$u_i = u_{-1} + \frac{1+x_i}{2}\sum_{k=0}^{N} u^{(1)}(s(x_i,\theta_k))\omega_k, \tag{2.14}$$

$$v_i = b(x_i) + \frac{1+x_i}{2}\sum_{j=0}^{1}\left(\sum_{k=0}^{N}\tilde{K}_j(x_i,s(x_i,\theta_k))u^{(j)}(s(x_i,\theta_k))\omega_k\right), \tag{2.15}$$

where $u_i^{(1)} \approx u^{(1)}(x_i), u_i \approx u(x_i)$ and $v_i \approx v(x_i), 0 \le i \le N$. The set $\{\theta_k\}_{k=0}^N$ coincides with the collocation points $\{x_i\}_{i=0}^N$.

We expand $u^{(1)}, u$ and v using Lagrange interpolation polynomials, i.e.,

$$u^{(1)}(s) \approx \sum_{p=0}^{N} u_p^{(1)} F_p(s), \quad u(s) \approx \sum_{p=0}^{N} u_p F_p(s), \quad v(s) \approx \sum_{p=0}^{N} v_p F_p(s).$$

The Legendre collocation methods are to seek $\{u_i^{(1)}\}_{i=0}^N, \{u_i\}_{i=0}^N$ and $\{v_i\}_{i=0}^N$ such that the following collocation equations hold

$$\begin{aligned} u_i^{(1)} = u'_{-1} &+ \frac{1+x_i}{2}\sum_{p=0}^{N} v_p \sum_{k=0}^{N} F_p(s(x_i,\theta_k))\omega_k \\ &+ \frac{1+x_i}{2}\sum_{j=0}^{1}\left(\sum_{p=0}^{N} u_p^{(j)} \sum_{k=0}^{N} a_j(s(x_i,\theta_k))F_p(s(x_i,\theta_k))\omega_k\right), \end{aligned} \tag{2.16}$$

$$u_i = u_{-1} + \frac{1+x_i}{2}\sum_{p=0}^{N} u_p^{(1)} \sum_{k=0}^{N} F_p(s(x_i,\theta_k))\omega_k, \tag{2.17}$$

$$v_i = b(x_i) + \frac{1+x_i}{2}\sum_{j=0}^{1}\left(\sum_{p=0}^{N} u_p^{(j)} \sum_{k=0}^{N}\tilde{K}_j(x_i,s(x_i,\theta_k))F_p(s(x_i,\theta_k))\omega_k\right). \tag{2.18}$$

Remark 2.1. Integrating both sides of Eq. (2.1) over $[-1,x]$ and using (2.2), Eq. (2.1) may be rewritten as

$$u^{(1)}(x) = u'_{-1} + \int_{-1}^{x} b(s)ds + \sum_{j=0}^{1}\int_{-1}^{x} a_j(s)u^{(j)}(s)ds + \sum_{j=0}^{1}\int_{-1}^{x}\int_{-1}^{s}\tilde{K}_j(s,\tau)u^{(j)}(\tau)d\tau ds.$$

This equation can be split into Eqs. (2.3) and (2.5). Eq. (2.4) is clearly established in view of given initial condition (2.2).

Generally, the analysis of volterra integro-differential equations can be based on the integral equations that are equivalent to the original initial-value problem. This reformulation will not affect the regularity properties of solutions and the accuracy of numerical solutions (see, e.g., [1,11,13]). Consequently, it is reasonable that we consider the equivalent reformulation system (2.3)–(2.5). This transformation does not influence the stability of Legendre-collocation spectral approximation scheme (see, e.g., [5,23]).

Remark 2.2. Since $\sum_{p=0}^{N} u_p^{(1)} F_p(s)$ and $\sum_{p=0}^{N} v_p F_p(s)$ are polynomials of degree not exceeding N, we have

$$\begin{aligned}\int_{-1}^{x_i} \sum_{p=0}^{N} u_p^{(1)} F_p(s) ds &= \frac{1+x_i}{2} \int_{-1}^{1} \sum_{p=0}^{N} u_p^{(1)} F_p(s(x_i,\theta)) d\theta \\ &= \frac{1+x_i}{2} \sum_{p=0}^{N} u_p^{(1)} \sum_{k=0}^{N} F_p(s(x_i,\theta_k))\omega_k, \end{aligned} \tag{2.19}$$

$$\begin{aligned}\int_{-1}^{x_i} \sum_{p=0}^{N} v_p F_p(s) ds &= \frac{1+x_i}{2} \int_{-1}^{1} \sum_{p=0}^{N} v_p F_p(s(x_i,\theta)) d\theta \\ &= \frac{1+x_i}{2} \sum_{p=0}^{N} v_p \sum_{k=0}^{N} F_p(s(x_i,\theta_k))\omega_k. \end{aligned} \tag{2.20}$$

3. Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section.

Lemma 3.1. ([5]) *Assume that a $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight is used to integrate the product $u\phi$, where $u \in H^m(I)$ with $I := (-1,1)$ for some $m \geq 1$ and $\phi \in \mathscr{P}_N$. Then there exists a constant C independent of N such that*

$$\left| \int_{-1}^{1} u(x)\phi(x)dx - (u,\phi)_N \right| \leq CN^{-m} |u|_{H^{m;N}(I)} \|\phi\|_{L^2(I)}, \tag{3.1}$$

where

$$|u|_{H^{m;N}(I)} = \left(\sum_{k=\min(m,N+1)}^{m} \|u^{(k)}\|_{L^2(I)}^2 \right)^{\frac{1}{2}}, \tag{3.2}$$

$$(u,\phi)_N = \sum_{j=0}^{N} u(x_j)\phi(x_j)\omega_j. \tag{3.3}$$

Lemma 3.2. ([5]) *Assume that $u \in H^m(I)$ with $I := (-1,1)$ and denote $I_N u$ its interpolation polynomial associated with the $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{x_i\}_{i=0}^{N}$, namely,*

$$I_N u = \sum_{i=0}^{N} u(x_i) F_i(x). \tag{3.4}$$

Then the following estimates hold

$$\|u - I_N u\|_{L^2(I)} \le CN^{-m}|u|_{H^{m;N}(I)}, \tag{3.5}$$
$$\|u - I_N u\|_{H^l(I)} \le CN^{2l-1/2-m}|u|_{H^{m;N}(I)}, \qquad 1 \le l \le m. \tag{3.6}$$

Lemma 3.3. ([29]) *For every bounded function $v(x)$, there exists a constant C independent of v such that*

$$\sup_N \left\| \sum_{j=0}^{N} v(x_j)F_j(x) \right\|_{L^2(I)} \le C\|v\|_\infty. \tag{3.7}$$

Lemma 3.4. ([8]) *Suppose $C \ge 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \le t < T$, and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a(t) + C\int_0^t u(s)ds \tag{3.8}$$

on this interval; then

$$u(t) \le a(t) + C\int_0^t a(s)ds, \qquad 0 \le t < T. \tag{3.9}$$

Due to lemma 3.4, we have the following Gronwall inequality.

Lemma 3.5. *Suppose $C_1 \ge 0$, if a nonnegative integrable function $E(t)$ satisfies*

$$E(t) \le C_1\int_{-1}^t E(s)ds + G(t), \qquad -1 < t \le 1, \tag{3.10}$$

where G(t) is also a nonnegative integrable function, then

$$\|E\|_{L^p(I)} \le C\|G\|_{L^p(I)}, \qquad p \ge 1. \tag{3.11}$$

Lemma 3.6. ([6]) *Assume that $F_j(x)$ is the j-th Lagrange interpolation polynomial associated with the Legendre -Gauss, or Gauss-Radau, or Gauss-Lobatto points. Then*

$$\max_{x\in(-1,1)} \sum_{j=0}^{N} |F_j(x)| = 1 + \wedge_N^{leg}, \tag{3.12}$$

where

$$\wedge_N^{leg} = \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} N^{\frac{1}{2}} + B_0 + \mathcal{O}(N^{-\frac{1}{2}}),$$

with B_0 is a bounded constant.

4. Convergence analysis

This section is devoted to provide a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed approximations. Firstly, we will carry our convergence analysis in L^2 space.

Theorem 4.1. *Let u be the exact solution of the second order VIDEs* (2.1) *with* (2.2). *Assume that*

$$U^{(1)}(x)=\sum_{i=0}^{N}u_i^{(1)}F_i(x),\qquad U(x)=\sum_{i=0}^{N}u_iF_i(x),\qquad V(x)=\sum_{i=0}^{N}v_iF_i(x),$$

where $\{u_i^{(1)}\}_{i=0}^N$, $\{u_i\}_{i=0}^N$ *and* $\{v_i\}_{i=0}^N$ *are given by* (2.16) − (2.18), $F_i(x)$ *is the i-th Lagrange basis function associated with the Legendre Gauss, or Legendre Gauss-Radau, or Legendre Gauss-Lobatto points* $\{x_i\}_{i=0}^N$. *If* $u\in H^{m+1}(I)$ *with* $I:=(-1,1)$ *for some* $m\geq 1$, *we have*

$$\begin{aligned}\|u^{(r)}-U^{(r)}\|_{L^2(I)}\leq& CN^{-m}\sum_{j=0}^{1}|a_j|_{H^{m;N}(I)}\|u^{(j)}\|_{L^2(I)}\\&+CN^{-m}\sum_{j=0}^{1}\max_{-1\leq x\leq 1}|\tilde{K}_j(x,\cdot)|_{H^{m;N}(I)}\|u^{(j)}\|_{L^2(I)}\\&+CN^{-m}\sum_{j=0}^{1}|u^{(j)}|_{H^{m;N}(I)}+CN^{-m}|b|_{H^{m+1;N}(I)}\end{aligned}\tag{4.1}$$

for $r\in\{0,1\}$ *provided that N is sufficiently large, where C is a constant independent of N.*

Proof. Following the notations of (3.3), we let

$$\big(R(s(x_i,\theta)),\phi(s(x_i,\theta))\big)_{N,x_i}=\sum_{k=0}^{N}R(s(x_i,\theta_k))\phi(s(x_i,\theta_k))\omega_k.$$

The numerical scheme (2.16)–(2.18) can be written as

$$u_i^{(1)}=u'_{-1}+\int_{-1}^{x_i}V(s)ds+\frac{1+x_i}{2}\sum_{j=0}^{1}\Big(a_j(s(x_i,\theta)),U^{(j)}(s(x_i,\theta))\Big)_{N,x_i},\tag{4.2}$$

$$u_i=u_{-1}+\int_{-1}^{x_i}U^{(1)}(s)ds,\tag{4.3}$$

$$v_i=b(x_i)+\frac{1+x_i}{2}\sum_{j=0}^{1}\Big(\tilde{K}_j(x_i,s(x_i,\theta)),U^{(j)}(s(x_i,\theta))\Big)_{N,x_i},\tag{4.4}$$

where Eqs. (4.2) and (4.3) are obtained in Remark 2.2. In order to use Lemma 3.1, we restate Eqs. (4.2) and (4.4) as

$$u_i^{(1)} = u'_{-1} + \int_{-1}^{x_i} V(s)ds + \frac{1+x_i}{2} \sum_{j=0}^{1} \int_{-1}^{1} a_j(s(x_i,\theta))U^{(j)}(s(x_i,\theta))d\theta - \sum_{j=0}^{1} I_j(x_i), \quad (4.5)$$

$$v_i = b(x_i) + \frac{1+x_i}{2} \sum_{j=0}^{1} \int_{-1}^{1} \tilde{K}_j(x_i,s(x_i,\theta))U^{(j)}(s(x_i,\theta))d\theta - \sum_{j=0}^{1} \hat{I}_j(x_i), \quad (4.6)$$

where

$$\begin{aligned} I_j(x) &= \frac{1+x}{2}\left(\int_{-1}^{1} a_j(s(x,\theta))U^{(j)}(s(x,\theta))d\theta - \Big(a_j(s(x,\theta)),U^{(j)}(s(x,\theta))\Big)_{N,x}\right), \\ \hat{I}_j(x) &= \frac{1+x}{2}\int_{-1}^{1} \tilde{K}_j(x,s(x,\theta))U^{(j)}(s(x,\theta))d\theta \\ &\quad - \frac{1+x}{2}\Big(\tilde{K}_j(x,s(x,\theta)),U^{(j)}(s(x,\theta))\Big)_{N,x}, \qquad j=0,1. \end{aligned}$$

It follows from (2.12) that

$$u_i^{(1)} = u'_{-1} + \int_{-1}^{x_i} V(s)ds + \sum_{j=0}^{1} \int_{-1}^{x_i} a_j(s)U^{(j)}(s)ds - \sum_{j=0}^{1} I_j(x_i), \quad (4.7)$$

$$v_i = b(x_i) + \sum_{j=0}^{1} \int_{-1}^{x_i} \tilde{K}_j(x_i,s)U^{(j)}(s)ds - \sum_{j=0}^{1} \hat{I}_j(x_i). \quad (4.8)$$

With the aid of (3.1), we have

$$\begin{aligned} &|I_j(x)| \le CN^{-m}|a_j|_{H^{m;N}(I)}\|U^{(j)}\|_{L^2(I)}, \\ &|\hat{I}_j(x)| \le CN^{-m}|\tilde{K}_j(x,\cdot)|_{H^{m;N}(I)}\|U^{(j)}\|_{L^2(I)}, \qquad j=0,1. \end{aligned}$$

Multiplying $F_i(x)$ on both sides of Eqs. (4.7), (4.3) and (4.8) and summing up from $i=0$ to N yield

$$U^{(1)}(x) = u'_{-1} + I_N\int_{-1}^{x} V(s)ds + \sum_{j=0}^{1} I_N\int_{-1}^{x} a_j(s)U^{(j)}(s)ds - \sum_{j=0}^{1} J_j(x), \quad (4.9)$$

$$U(x) = u_{-1} + I_N\int_{-1}^{x} U^{(1)}(s)ds, \quad (4.10)$$

$$V(x) = I_N b(x) + \sum_{j=0}^{1} I_N\int_{-1}^{x} \tilde{K}_j(x,s)U^{(j)}(s)ds - \sum_{j=0}^{1} \hat{J}_j(x), \quad (4.11)$$

where

$$J_j(x)=\sum_{i=0}^{N} I_j(x_i)F_i(x),\quad \hat{J}_j(x)=\sum_{i=0}^{N}\hat{I}_j(x_i)F_i(x),\qquad j=0,1.$$

Similarly, multiplying $F_i(x)$ on both sides of Eqs. (2.6)-(2.8) and summing up from $i=0$ to N yield

$$I_N u^{(1)}(x)=u'_{-1}+I_N\int_{-1}^{x} v(s)ds+\sum_{j=0}^{1} I_N\int_{-1}^{x} a_j(s)u^{(j)}(s)ds, \tag{4.12}$$

$$I_N u(x)=u_{-1}+I_N\int_{-1}^{x} u^{(1)}(s)ds, \tag{4.13}$$

$$I_N v(x)=I_N b(x)+\sum_{j=0}^{1} I_N\int_{-1}^{x}\tilde{K}_j(x,s)u^{(j)}(s)ds. \tag{4.14}$$

It follows from (4.9)–(4.11) and (4.12)–(4.14) that

$$e_{u^{(1)}}(x)+I_N u^{(1)}(x)-u^{(1)}(x)$$
$$=I_N\int_{-1}^{x} e_v(s)ds+\sum_{j=0}^{1} J_j(x)+\sum_{j=0}^{1} I_N\int_{-1}^{x} a_j(s)e_{u^{(j)}}(s)ds, \tag{4.15}$$

$$e_u(x)+I_N u(x)-u(x)=I_N\int_{-1}^{x} e_{u^{(1)}}(s)ds, \tag{4.16}$$

$$e_v(x)+I_N v(x)-v(x)=\sum_{j=0}^{1} I_N\int_{-1}^{x}\tilde{K}_j(x,s)e_{u^{(j)}}(s)ds+\sum_{j=0}^{1}\hat{J}_j(x), \tag{4.17}$$

where

$$e_{u^{(1)}}(x)=u^{(1)}(x)-U^{(1)}(x),\quad e_u(x)=u(x)-U(x),\quad e_v(x)=v(x)-V(x).$$

Consequently,

$$e_{u^{(1)}}(x)=\int_{-1}^{x} e_v(s)ds+\sum_{j=0}^{1}\int_{-1}^{x} a_j(s)e_{u^{(j)}}(s)ds+\sum_{j=0}^{1} J_j(x)$$
$$+\tilde{J}_2(x)+\tilde{J}_3(x)+\sum_{j=0}^{1} H_j(x), \tag{4.18}$$

$$e_u(x)=\int_{-1}^{x} e_{u^{(1)}}(s)ds+\tilde{J}_4(x)+\tilde{J}_5(x), \tag{4.19}$$

$$e_v(x)=\sum_{j=0}^{1}\int_{-1}^{x}\tilde{K}_j(x,s)e_{u^{(j)}}(s)ds+\sum_{j=0}^{1}\hat{J}_j(x)+\tilde{J}_6(x)+\sum_{j=0}^{1}\hat{H}_j(x), \tag{4.20}$$

where

$$
\begin{aligned}
&\tilde{J}_2(x)=u^{(1)}(x)-I_N u^{(1)}(x),\ \ \tilde{J}_3(x)=I_N\int_{-1}^{x} e_v(s)ds-\int_{-1}^{x} e_v(s)ds,\\
&H_j(x)=I_N\int_{-1}^{x} a_j(s)e_{u^{(j)}}(s)ds-\int_{-1}^{x} a_j(s)e_{u^{(j)}}(s)ds, \qquad\qquad j=0,1,\\
&\tilde{J}_4(x)=u(x)-I_N u(x), \qquad \tilde{J}_5(x)=I_N\int_{-1}^{x} e_{u^{(1)}}(s)ds-\int_{-1}^{x} e_{u^{(1)}}(s)ds,\\
&\tilde{J}_6(x)=v(x)-I_N v(x),\\
&\hat{H}_j(x)=I_N\int_{-1}^{x}\tilde{K}_j(x,s)e_{u^{(j)}}(s)ds-\int_{-1}^{x}\tilde{K}_j(x,s)e_{u^{(j)}}(s)ds, \qquad j=0,1.
\end{aligned}
$$

Due to Eqs. (4.18)–(4.20) and using the $Dirichlet's$ formula which sates

$$
\int_{-1}^{x}\int_{-1}^{s}\Phi(s,\tau)d\tau ds=\int_{-1}^{x}\int_{\tau}^{x}\Phi(s,\tau)dsd\tau
$$

provided the integral exists, we obtain

$$
e_{u^{(1)}}(x)=\int_{-1}^{x} H(s,x)e_{u^{(1)}}(s)ds+J^*(x), \tag{4.21}
$$

where

$$
\begin{aligned}
H(s,x)=&\int_{s}^{x}\left(\int_{s}^{\tau}\tilde{K}_0(\tau,z)dz\right)d\tau+\int_{s}^{x}\tilde{K}_1(\tau,s)d\tau+\int_{s}^{x} a_0(\tau)d\tau+a_1(s),\\
J^*(x)=&\int_{-1}^{x}\left(\int_{-1}^{s}\tilde{K}_0(s,\tau)(\tilde{J}_4+\tilde{J}_5)(\tau)d\tau\right)ds+\int_{-1}^{x} a_0(s)(\tilde{J}_4+\tilde{J}_5)(s)ds\\
&+\int_{-1}^{x}\left(\sum_{j=0}^{1}(\tilde{J}_j(s)+\tilde{H}_j(s))+\tilde{J}_6(s)\right)ds+\sum_{j=0}^{1}(J_j(x)+H_j(x))+\tilde{J}_2(x)+\tilde{J}_3(x).
\end{aligned}
$$

By (4.21), we have

$$
|e_{u^{(1)}}(x)|\le\max_{(s,x)\in\hat{I}\times\hat{I}}|H(s,x)|\int_{-1}^{x}|e_{u^{(1)}}(s)|ds+|J^*(x)|.
$$

Using the Gronwall inequality in Lemma 3.5, we deduce that

$$
\begin{aligned}
\|e_{u^{(1)}}(x)\|_{L^2(I)}\le& C\|J^*(x)\|_{L^2(I)}\\
\le& C\sum_{j=0}^{1}\Big(\|J_j(x)\|_{L^2(I)}+\|\hat{J}_j(x)\|_{L^2(I)}+\|H_j(x)\|_{L^2(I)}\\
&+\|\hat{H}_j(x)\|_{L^2(I)}\Big)+C\sum_{k=2}^{6}\|\tilde{J}_k(x)\|_{L^2(I)}.
\end{aligned} \tag{4.22}
$$

Eq. (4.19) leads to

$$\|e_u(x)\|_{L^2(I)} \le \|e_{u^{(1)}}(x)\|_{L^2(I)} + \|\tilde{J}_4(x)\|_{L^2(I)} + \|\tilde{J}_5(x)\|_{L^2(I)}. \tag{4.23}$$

We now apply Lemma 3.3 to obtain that

$$\begin{aligned}
\|J_j(x)\|_{L^2(I)} &\le C\|I_j(x)\|_\infty \le CN^{-m}|a_j|_{H^{m;N}(I)}\Big(\|u^{(j)}(x)\|_{L^2(I)} + \|e_{u^{(j)}}(x)\|_{L^2(I)}\Big),\\
\|\hat{J}_j(x)\|_{L^2(I)} &\le C\|\hat{I}_j(x)\|_\infty \le CN^{-m}\max_{-1\le x\le 1}|\tilde{K}_j(x,\cdot)|_{H^{m;N}(I)}\Big(\|u^{(j)}(x)\|_{L^2(I)}\\
&\qquad + \|e_{u^{(j)}}(x)\|_{L^2(I)}\Big), \qquad j=0,1.
\end{aligned}$$

Using the L^2 error bounds for the interpolation polynomials (see Lemma 3.2) gives

$$\begin{aligned}
&\|\tilde{J}_2(x)\|_{L^2(I)} \le CN^{-m}|u^{(1)}|_{H^{m;N}(I)}, \quad \|\tilde{J}_4(x)\|_{L^2(I)} \le CN^{-m}|u|_{H^{m;N}(I)},\\
&\|\tilde{J}_6(x)\|_{L^2(I)} \le CN^{-(m+1)}|v|_{H^{m+1;N}(I)}\\
&\qquad\qquad \le CN^{-m}\Big(|b|_{H^{m+1;N}(I)} + |u|_{H^{m;N}(I)} + |u^{(1)}|_{H^{m;N}(I)}\Big).
\end{aligned}$$

By virtue of (3.5) with $m=1$,

$$\begin{aligned}
&\|\tilde{J}_3(x)\|_{L^2(I)} \le CN^{-1}\|e_v(x)\|_{L^2(I)}, \quad \|\tilde{J}_5(x)\|_{L^2(I)} \le CN^{-1}\|e_{u^{(1)}}(x)\|_{L^2(I)},\\
&\|H_j(x)\|_{L^2(I)} \le CN^{-1}\|a_j(x)e_{u^{(j)}}(x)\|_{L^2(I)}\\
&\qquad\qquad \le CN^{-1}\|e_{u^{(j)}}(x)\|_{L^2(I)}, \qquad j=0,1,\\
&\|\hat{H}_j(x)\|_{L^2(I)} \le CN^{-1}\left\|\tilde{K}_j(x,x)e_{u^{(j)}}(x) + \int_{-1}^{x}\partial_x\tilde{K}_j(x,s)e_{u^{(j)}}(s)ds\right\|_{L^2(I)}\\
&\qquad\qquad \le CN^{-1}\|e_{u^{(j)}}(x)\|_{L^2(I)}, \qquad j=0,1.
\end{aligned}$$

The above estimates, together with (4.22) and (4.23), yield

$$\begin{aligned}
\|e_{u^{(r)}}(x)\|_{L^2(I)} \le\ & CN^{-m}\sum_{j=0}^{1}|a_j|_{H^{m;N}(I)}\|u^{(j)}\|_{L^2(I)}\\
&+ CN^{-m}\sum_{j=0}^{1}\max_{-1\le x\le 1}|\tilde{K}_j(x,\cdot)|_{H^{m;N}(I)}\|u^{(j)}\|_{L^2(I)}\\
&+ CN^{-m}\sum_{j=0}^{1}|u^{(j)}|_{H^{m;N}(I)} + CN^{-m}|b|_{H^{m+1;N}(I)}
\end{aligned} \tag{4.24}$$

for $r \in \{0,1\}$ which leads to (4.1). This completes the proof of the theorem. □

Next, we will extend the L^2 error estimate to the L^∞ space. The key technique is to use an extrapolation between L^2 and H^1.

Theorem 4.2. *Let u be the exact solution of the second order VIDEs* (2.1) *with* (2.2). $U^{(1)}(x), U(x)$ *and $V(x)$ are defined in Theorem* 4.1. *If $u \in H^{m+1}(I)$ with $I := (-1,1)$ for some $m \geq 1$, we have*

$$\begin{aligned}\|u^{(r)} - U^{(r)}\|_{L^\infty(I)} \leq & CN^{\frac{1}{2}-m}\sum_{j=0}^{1}|a_j|_{H^{m;N}(I)}\|u^{(j)}\|_{L^2(I)} \\ & +CN^{\frac{1}{2}-m}\sum_{j=0}^{1}\max_{-1\leq x\leq 1}|\tilde{K}_j(x,\cdot)|_{H^{m;N}(I)}\|u^{(j)}\|_{L^2(I)} \\ & +CN^{\frac{3}{4}-m}\sum_{j=0}^{1}|u^{(j)}|_{H^{m;N}(I)} + CN^{\frac{3}{4}-m}|b|_{H^{m+1;N}(I)} \end{aligned} \tag{4.25}$$

for $r \in \{0,1\}$ provided that N is sufficiently large, where C is a constant independent of N.

Proof. Applying the inequality in the Sobolev space (see [5])

$$\|w\|_{L^\infty(a,b)} \leq \left(\frac{1}{b-a}+2\right)^{\frac{1}{2}}\|w\|^{\frac{1}{2}}_{L^2(a,b)}\|w\|^{\frac{1}{2}}_{H^1(a,b)}, \quad \forall w \in H^1(a,b), \tag{4.26}$$

and Lemma 3.2, we get

$$\begin{aligned}\|u - I_N u\|_{L^\infty(I)} &\leq C\|u - I_N u\|^{\frac{1}{2}}_{L^2(I)}\|u - I_N u\|^{\frac{1}{2}}_{H^1(I)} \\ &\leq C\left(N^{-m}|u|_{H^{m;N}(I)}\right)^{\frac{1}{2}}\left(N^{\frac{3}{2}-m}|u|_{H^{m;N}(I)}\right)^{\frac{1}{2}} \\ &\leq CN^{\frac{3}{4}-m}|u|_{H^{m;N}(I)}. \end{aligned} \tag{4.27}$$

Following the same procedure as in the proof of Theorem 4.1, we have the L^∞ error estimate with the help of (4.27) and Lemma 3.6. □

5. Numerical experiments

Writing $U_N^{(1)} = (u_0^{(1)}, u_1^{(1)}, \cdots, u_N^{(1)})^T, U_N = (u_0, u_1, \cdots, u_N)^T$ and $V_N = (v_0, v_1, \cdots, v_N)^T$, we obtain the following equations of the matrix form from (2.16)-(2.18):

$$\begin{aligned} &U_N^{(1)} = U'_{-1} + AU_N + BU_N^{(1)} + CV_N, \\ &U_N = U_{-1} + CU_N^{(1)}, \\ &V_N = b_N + LU_N + MU_N^{(1)}, \end{aligned}$$

where

$$U'_{-1} = u'_{-1} \times \overbrace{(1, \cdots, 1)}^{N+1}{}^T, \quad U_{-1} = u_{-1} \times \overbrace{(1, \cdots, 1)}^{N+1}{}^T, \quad b_N = \big(b(x_0), \cdots, b(x_N)\big)^T.$$

The entries of the matrices are given by

$$A_{ij} = \frac{1+x_i}{2}\sum_{k=0}^{N} a_0(s(x_i,\theta_k))F_j(s(x_i,\theta_k))\omega_k,$$
$$B_{ij} = \frac{1+x_i}{2}\sum_{k=0}^{N} a_1(s(x_i,\theta_k))F_j(s(x_i,\theta_k))\omega_k,$$
$$C_{ij} = \frac{1+x_i}{2}\sum_{k=0}^{N} F_j(s(x_i,\theta_k))\omega_k,$$
$$L_{ij} = \frac{1+x_i}{2}\sum_{k=0}^{N} K_0(x_i,s(x_i,\theta_k))F_j(s(x_i,\theta_k))\omega_k,$$
$$M_{ij} = \frac{1+x_i}{2}\sum_{k=0}^{N} K_1(x_i,s(x_i,\theta_k))F_j(s(x_i,\theta_k))\omega_k.$$

Without lose of generality, we will only use the Legendre-Gauss points (i.e., the zeros of $L_{N+1}(x)$) as the collocation points. Our numerical evidences show that the other two kinds of Legendre points produce results with similar accuracy. For the Legendre-Gauss points, the corresponding weights are

$$\omega_j = \frac{2}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, \qquad 0 \le j \le N.$$

Here, we simply introduce the computation of Gauss-Legendre quadrature rule nodes and weights (see the detailed algorithm and download related codes in [26]). The Gauss-Legendre quadrature formula is used to numerically calculate the integral

$$\int_{-1}^{1} f(x)dx, \qquad x \in [-1,1],$$

by using the formula

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=0}^{N} f(x_i)\omega_i.$$

With the help of a change in the variables (which changes both weights ω_i and nodes x_i), we can get onto the arbitrary interval [a,b].

The Gauss-Legendre quadrature formula for a given order N is completely defined by the set of nodes x_i and weights ω_i (see more details in [3,4]).

Example 5.1. Consider the second order VIDE

$$u^{(2)}(x) = b(x) + (\cos x + e^{x^2})u(x) + \sin x u^{(1)}(x) + \frac{1}{6}\int_{-1}^{x} se^{xs}u^{(1)}(s)ds,$$
$$u(-1) = e^{-6}, \quad u^{(1)}(-1) = 6e^{-6}, \qquad x \in [-1,1],$$

Table 1: Example 5.1: The L^∞ and L^2 errors for $u(x)=e^{6x}$.

N	10	12	14	16	18
L^∞-error	5.5989e-002	3.0014e-003	1.1655e-004	3.1288e-006	3.9801e-008
L^2-error	8.0971e-003	3.5724e-004	1.2658e-005	4.3927e-007	1.9046e-008
N	20	22	24	28	30
L^∞-error	1.3835e-009	1.2855e-010	3.5982e-011	2.9274e-011	3.2060e-011
L^2-error	8.8486e-010	3.8866e-011	1.0436e-011	8.0367e-012	9.0488e-012

Table 2: Example 5.1: The L^∞ and L^2 errors for $u^{(1)}(x)=6e^{6x}$.

N	10	12	14	16	18
L^∞-error	4.3394e-001	2.8993e-002	1.6610e-003	8.8405e-005	4.5336e-006
L^2-error	2.0403e-002	2.3457e-004	1.0517e-004	1.0010e-005	6.6451e-007
N	20	22	24	28	30
L^∞-error	2.2215e-007	1.0269e-008	5.5343e-010	1.0186e-010	1.1232e-010
L^2-error	3.5992e-008	1.6845e-009	7.8804e-011	2.8307e-011	3.1698e-011

with $b(x)$ chosen so that $u(x)=e^{6x}$. By calculation

$$b(x)=(36-6\sin x-\cos x)e^{6x}-\frac{1}{(x+6)^2}\left(e^{-(x+6)}-e^{x(x+6)}\right)\\ -\frac{1}{x+6}\left(e^{-(x+6)}+(2x+6)e^{x(x+6)}\right).$$

Table 1 shows the errors for $u(x)=e^{6x}$ and Table 2 shows the errors for $u^{(1)}(x)=6e^{6x}$ obtained by using the spectral methods described above. It is observed that the desired exponential rate of convergence is obtained. Fig. 1 presents the numerical and exact solutions for $u(x)$ and $u^{(1)}(x)$, which are found in excellent agreement. In Fig. 2, numerical errors of $u(x)$ and $u^{(1)}(x)$ are plotted for $2\le N\le 30$ in both L^∞ and L^2 norms.

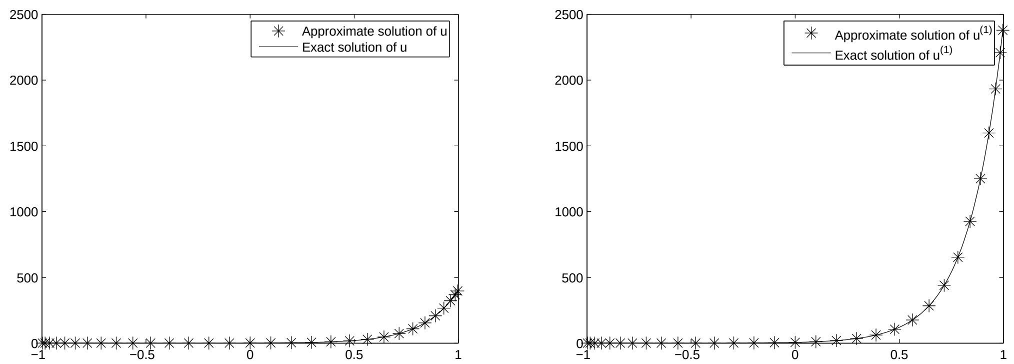

Figure 1: Example 5.1: Comparison between numerical and exact solution of $u(x)$ (left) and $u^{(1)}(x)$ (right).

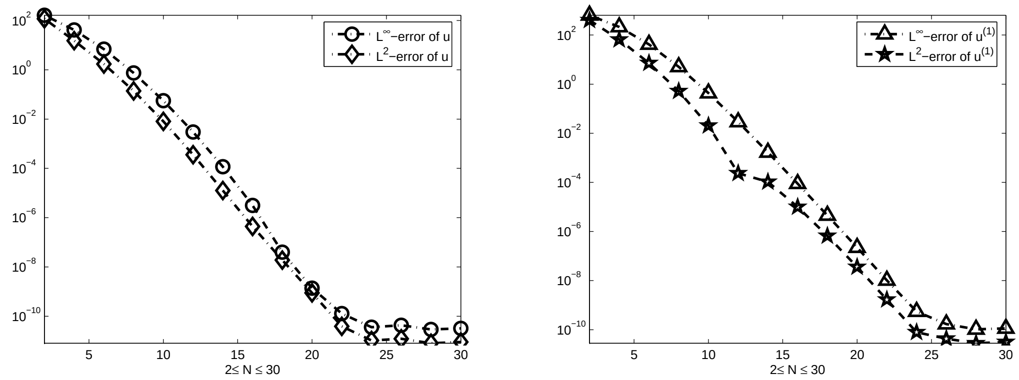

Figure 2: Example 5.1: The errors for $u(x)$(left) and $u^{(1)}(x)$(right) versus the number of collocation points.

Example 5.2. Consider the second order VIDE

$$\begin{aligned} w^{(2)}(x) &= 2(1-x)\sin 1+\cos 1-(4x+1)w(x)+(x^2-2)w^{(1)}(x) \\ &\quad +\int_{-1}^{x} s^2 w(s)ds + 2x\int_{-1}^{x} w^{(1)}(s)ds, \\ w(-1) &= -\sin 1, \;\; w^{(1)}(-1)=\cos 1, \;\; x\in[-1,1]. \end{aligned}$$

The corresponding exact solution is given by $w(x)=\sin x$.

The numerical and exact solutions for $w(x)$ and $w^{(1)}(x)$ are displayed in Fig. 3. Fig. 4 plots the errors of $w(x)$ and $w^{(1)}(x)$ for $2\le N\le 16$ in both L^∞ and L^2 norms. Moreover, the corresponding errors with several values of N are displayed in Table 3 for $w(x)$ and in Table 4 for $w^{(1)}(x)$. As expected that the errors decay exponentially which confirmed our theoretical predictions.

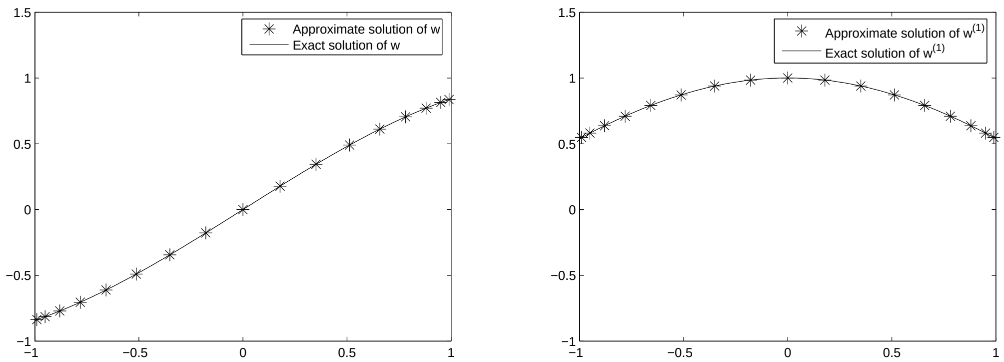

Figure 3: Example 5.2: Comparison between numerical and exact solution of $w(x)$ (left) and $w^{(1)}(x)$ (right).

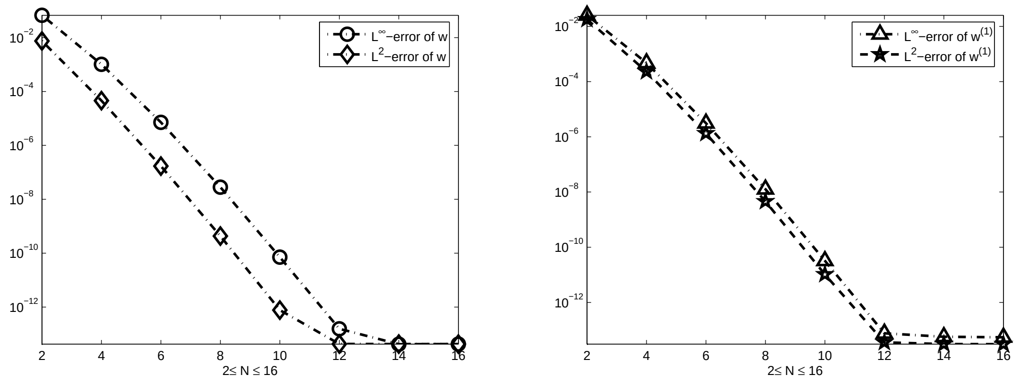

Figure 4: Example 5.2: The errors for $w(x)$ (left) and $w^{(1)}(x)$ (right) versus the number of collocation points.

Table 3: Example 5.2: The L^∞ and L^2 errors for $w(x)=\sin x$.

N	2	4	6	8
L^∞-error	6.7546e-002	1.0186e-003	7.1545e-006	2.8260e-008
L^2-error	7.5119e-003	4.5823e-005	1.7043e-007	4.3103e-010
N	10	12	14	16
L^∞-error	7.1157e-011	1.5699e-013	4.2411e-014	4.1966e-014
L^2-error	7.7423e-013	4.2567e-014	4.2436e-014	4.2349e-014

Table 4: Example 5.2: The L^∞ and L^2 errors for $w^{(1)}(x)=\cos x$.

N	2	4	6	8
L^∞-error	2.5387e-002	4.7764e-004	3.1264e-006	1.2636e-008
L^2-error	1.7908e-002	2.3136e-004	1.3243e-006	4.5661e-009
N	10	12	14	16
L^∞-error	3.2349e-011	7.6050e-014	5.6177e-014	5.4623e-014
L^2-error	1.0378e-011	3.5851e-014	3.1254e-014	3.0840e-014

6. Comparison between polynomial spline collocation methods and the spectral methods

Second order VIDEs of the form (1.1) have been solved numerically using polynomial spline spaces [9]. In order to describe these approximating polynomial spline spaces, let $\prod_{\tilde{N}}: 0=t_0<t_1<\cdots<t_{\tilde{N}}=T$ be the mesh for the interval $\tilde{I}$, and set

$$\begin{aligned}&\sigma_n:=[t_n,t_{n+1}],\quad h_n:=t_{n+1}-t_n,\qquad n=0,1,\cdots,\tilde{N}-1,\\&h=\max\{h_n:0\le n\le\tilde{N}-1\}\qquad\text{(mesh diameter)},\\&Z_{\tilde{N}}:=\{t_n:n=1,2,\cdots,\tilde{N}-1\}.\end{aligned}\tag{6.1}$$

Let Π_{m+d} be the set of (real) polynomials of degree not exceeding $m+d$, where $m\ge1$ and $d\ge-1$ are given integers. The solution y to the initial-value problem (1.1)-(1.2) will be approximated by an element $\tilde{y}$ in the polynomial spline space,

$$S_{m+d}^{(d)}(Z_{\tilde{N}}) := \Big\{ \tilde{y} := \tilde{y}(t)\,|_{t\in\sigma_n} := \tilde{y}_n(t) \in \Pi_{m+d}, \quad n = 0,1,\cdots,\tilde{N}-1, \\ \tilde{y}_{n-1}^{(j)}(t_n) = \tilde{y}_n^{(j)}(t_n) \text{ for } j = 0,1,\cdots,d, \;\; t_n \in Z_{\tilde{N}} \Big\}, \tag{6.2}$$

that is, by a polynomial spline function of degree $m+d$ which possesses the knots $Z_{\tilde{N}}$ and is d times continuously differential on $\tilde{I}$. If $d=-1$, then the elements of $S_{m-1}^{(-1)}(Z_{\tilde{N}})$ may have jump discontinuities at the knots $Z_{\tilde{N}}$. We denote the collocation parameters by $\{c_j\}_{j=1}^m$, where $0 < c_1 < c_2 < \cdots < c_m \leq 1$.

To provide a comparison between polynomial spline collocation methods and the spectral methods, we consider the following second order VIDE

$$y^{(2)}(t) = q(t) + y(t) + \int_0^t tsy(s)ds, \quad y(0)=1, \quad y^{(1)}(0)=1, \qquad t\in[0,1], \tag{6.3}$$

with $q(t) = 2 - t - t^2 - \frac{t^5}{4} + te^t(1-t)$, so that it has the exact solution $y(t) = t^2 + e^t$.

Table 5: Polynomial spline collocation methods: Approximate error when $m=3$ and ($\{c_1 = 1/3,\ c_2 = 2/3,\ c_3 = 1\}$).

d	0	1	2	3
$\tilde{e_1}$	1.5165e-002	3.0370e-004	3.0370e-004	3.0369e-004
$\tilde{e_2}$	1.7211e-001	1.2941e-002	1.2941e-002	1.2941e-002

Table 6: Spectral methods: Approximate error versus the number of collocation points.

N	4	6	8	10
e_1	1.7120e-006	1.8697e-009	1.2388e-012	4.5297e-014
e_2	5.5285e-005	9.6240e-008	9.4301e-011	2.2649e-013

Following Tables 5 and 6 illustrate the error approximations as

$$\tilde{e_1} := |y(0.5) - \tilde{y}(0.5)|, \quad e_1 := |y(0.5) - Y(0.5)|, \\ \tilde{e_2} := |y(1) - \tilde{y}(1)|, \qquad e_2 := |y(1) - Y(1)|,$$

(here we select only two points $t = 0.5$ and $t = 1$ as the comparative object, the other points are similar) where $\tilde{y} \in S_{m+d}^{(d)}(Z_{\tilde{N}})$ is the approximate solution obtained by using the polynomial spline collocation methods if $t_i = ih \in Z_{\tilde{N}}$ with $h = 0.01$ ($i.e.$ $\tilde{N} = 100$) and Y is the numerical solution obtained by using the Legendre-collocation methods described in Section 2. As we can see from the above tables, the spectral methods yield better approximations than the polynomial spline collocation methods.

7. Conclusion and future work

This paper proposes a numerical method for second order Volterra integro-differential equations based on a Legendre-spectral approach. To facilitate the use of the method, we first restate the original second order Volterra integro-differential equation as three simple integral equations of the second kind. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of approximations decay exponentially in L^2-norm and L^∞-norm, which is a desired feature for a spectral method.

In our future work, the spectral collocation methods will be studied for Volterra integro-differential equations with a weakly singular kernel.

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