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Convergence Analysis of a Block-by-Block Method for Fractional Differential Equations

Jianfei Huang[1], Yifa Tang[1,*]and Luis Vázquez[2]

[1] *LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.*
[2] *Departamento de Matemática Aplicada, Facultad de Informática, Instituto de Matemática Interdisciplinar (IMI), Universidad Complutense de Madrid, 28040-Madrid, Spain.*

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Abstract. The block-by-block method, proposed by Linz for a kind of Volterra integral equations with nonsingular kernels, and extended by Kumar and Agrawal to a class of initial value problems of fractional differential equations (FDEs) with Caputo derivatives, is an efficient and stable scheme. We analytically prove and numerically verify that this method is convergent with order at least 3 for any fractional order index $\alpha > 0$.

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Key words: Fractional differential equation, Caputo derivative, block-by-block method, convergence analysis.

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1. Introduction

Fractional calculus [13, 14], almost as old as the familiar integer-order calculus, is now winning more and more scientific applications owing to its “memory" and “heredity" principle in a variety of areas, such as viscoelasticity [2], anomalous diffusion [3], control theory [15], finance [8, 16, 17] and hydrology [1, 18]. A recent panoramic view of the fractional calculus can be seen in [19].

Similarly to the integer-order differential equations, it is usually difficult to obtain the analytical solution for a fractional differential equation (FDE). So there has been a growing interest to develop numerical approaches in solving the FDEs. However, the theoretical studies of fractional numerical methods, including stability analysis and error estimation, are quite challenging due to the nonlocal property of fractional operators [5, 7, 12]. In this context, Diethelm *et al* [5, 7] took advantage of the fact that some kinds of FDEs can be formulated as Volterra integral equations of the second kind, then derived the fractional

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[*] Corresponding author. *Email addresses:* `jfhuang@lsec.cc.ac.cn` (J. Huang), `tyf@lsec.cc.ac.cn` (Y. Tang), `lvazquez@fdi.ucm.es` (L. Vázquez)

Adams-Bashforth-Moulton method from the classical case. Significantly, they gave convergence analysis, i.e., for any $\alpha > 0$ the described method is convergent with order at least one if the analytical solution $y(t)$ is twice continuously differentiable. In addition, Lin and Liu [12] developed a kind of linear multistep methods for fractional initial value problems based on Lubich's high-order approximations [10] to fractional derivatives and integrals. And they proved the consistence, convergence and stability of these methods. Nevertheless, the unavoidable shortcoming in these linear multistep methods is that one needs to spend much time in computing the starting weights.

In 2006, Kumar and Agrawal [9] also utilized the equivalent Volterra integral equation in [5] and extended the block-by-block method proposed by Linz [11] to some kinds of FDEs. Numerical examples have shown the efficiency and stability of this scheme, i.e., for a kind of FDEs the performance is better than that of Diethlm's Adams method [7]. However, it's a pity that the error estimate and convergence order analysis of this scheme was neglected. In the present paper, we will derive error estimate and precise convergence order of the block-by-block method under certain assumptions, and test the order via numerical experiments.

This paper is organized as follows. In Section 2, in order to facilitate the theoretical analysis, the block-by-block method is rewritten. We give in Section 3 some preparations and useful lemmas. The error estimate and convergence order analysis are given in Section 4. Numerical experiments are carried out in Section 5, which verify the theoretical results obtained in Section 4. Final section is the concluding remarks.

2. Block-by-block method

We consider the following nonlinear FDE

$$D_*^{\alpha} y(t) = f(t, y(t)), \quad 0 \le t \le T, \quad n-1 < \alpha \le n \tag{2.1}$$

subject to the initial conditions:

$$y^{(k)}(0) = c_k, \quad k = 0, 1, \cdots, n-1. \tag{2.2}$$

In (2.1), D_*^{α} denotes the Caputo derivative of order α, defined by

$$D_*^{\alpha} y(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n y(\tau)}{d\tau^n} d\tau.$$

Assume that $\Omega := [0, T] \times [c_0 - \lambda, c_0 + \lambda]$ with some $\lambda > 0$ and $f(t, y) \in C(\Omega)$. Furthermore, let f fulfill a Lipschitz condition with respect to the second variable on Ω, namely

$$|f(t, y) - f(t, z)| \le L|y - z|$$

for some constant $L > 0$. According to [6], there exists a unique solution $y(t)$ on $[0, T]$ for the initial value problem (IVP) (2.1-2.2).

As also mentioned in [6], if $f(t,y)$ is continuous, IVP (2.1-2.2) is equivalent to the following Volterra integral equation of the second kind

$$y(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \tag{2.3}$$

where

$$g(t) := \sum_{k=0}^{n-1} c_k \frac{t^k}{k!}.$$

Kumar and Agrawal [9] have extended the block-by-block method [11] for a kind of Volterra integral equations with nonsingular kernels to Eq.(2.3) in which the integral kernel is singular for $0 < \alpha < 1$. For convenience of analysis, we will rewrite this method in the sequel.

First divide the interval $[0,T]$ into $2N$ parts with stepsize $h = T/(2N)$, and set $t_j = jh (j = 0, 1, \cdots, 2N)$. The numerical solution of Eq.(2.3) at the point t_j is denoted by y_j. Let $g_j = g(t_j)$ and $f_j = f(t_j, y_j)$.

Now assume that $y_j (j = 0, 1, \cdots, 2m)$, the approximations of $y(t_j)(j = 0, 1, \cdots, 2m)$, are obtained. In order to get the numerical solutions y_{2m+1} and y_{2m+2}, the block-by-block method presented by Kumar and Agrawal can be described as follows, for $m = 0, 1, \cdots, N-1$,

$$\begin{aligned} y_{2m+2} = g_{2m+2} &+ \sum_{k=0}^{m-1} \left[W_{k,0}^{[2m+2]} f_{2k} + W_{k,1}^{[2m+2]} f_{2k+1} + W_{k,2}^{[2m+2]} f_{2k+2} \right] \\ &+ W_{m,0}^{[2m+2]} f_{2m} + W_{m,1}^{[2m+2]} f_{2m+1} + W_{m,2}^{[2m+2]} f_{2m+2}, \end{aligned} \tag{2.4a}$$

$$\begin{aligned} y_{2m+1} = g_{2m+1} &+ \sum_{k=0}^{m-1} \left[W_{k,0}^{[2m+1]} f_{2k} + W_{k,1}^{[2m+1]} f_{2k+1} + W_{k,2}^{[2m+1]} f_{2k+2} \right] \\ &+ W_{m,0}^{[2m+1]} f_{2m} + W_{m,1}^{[2m+1]} f_{2m+1} + W_{m,2}^{[2m+1]} f_{2m+2}, \end{aligned} \tag{2.4b}$$

where for $i = 0, 1, 2$,

$$W_{k,i}^{[2m+2]} := \frac{1}{\Gamma(\alpha)} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+2} - \tau)^{\alpha-1} \phi_{k,i}(\tau) d\tau, \quad k = 0, 1, \cdots m, \tag{2.5}$$

$$W_{k,i}^{[2m+1]} := \frac{1}{\Gamma(\alpha)} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+1} - \tau)^{\alpha-1} \phi_{k,i}(\tau) d\tau, \quad k = 0, 1, \cdots, m-1. \tag{2.6}$$

In Eq.(2.4b), if $k = m$, $W_{m,i}^{[2m+1]}$ $(i = 0, 1, 2)$ are defined as

$$W_{m,0}^{[2m+1]} := d_{2m} + \frac{3}{8} d_{2m+\frac{1}{2}}, \quad W_{m,1}^{[2m+1]} := \frac{3}{4} d_{2m+\frac{1}{2}} + d_{2m+1}, \quad W_{m,2}^{[2m+1]} := -\frac{1}{8} d_{2m+\frac{1}{2}} \tag{2.7}$$

with

$$\begin{aligned}
d_{2m} &:= \frac{1}{\Gamma(\alpha)} \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1} - \tau)^{\alpha-1} \psi_{m,0}(\tau) d\tau, \\
d_{2m+\frac{1}{2}} &:= \frac{1}{\Gamma(\alpha)} \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1} - \tau)^{\alpha-1} \psi_{m,1}(\tau) d\tau, \\
d_{2m+1} &:= \frac{1}{\Gamma(\alpha)} \int_{t_{2m}}^{t_{2m+1}} (t_{2m+1} - \tau)^{\alpha-1} \psi_{m,2}(\tau) d\tau.
\end{aligned}$$

Functions $\phi_{k,i}(t)$ $(i = 0, 1, 2)$ are quadratic Lagrange interpolating polynomials associated with points t_{2k}, t_{2k+1} and t_{2k+2}, precisely,

$$\begin{aligned}
&\phi_{k,0}(t) := \frac{(t - t_{2k+1})(t - t_{2k+2})}{2h^2}, \quad \phi_{k,1}(t) := \frac{(t - t_{2k})(t - t_{2k+2})}{-h^2}, \\
&\phi_{k,2}(t) := \frac{(t - t_{2k})(t - t_{2k+1})}{2h^2}.
\end{aligned}$$

Similarly, $\psi_{m,i}(t)$ $(i = 0, 1, 2)$ are Lagrange interpolating polynomials associated with points t_{2m}, $t_{2m+\frac{1}{2}}$ and t_{2m+1}. From (2.5)-(2.7), it is known that $W_{k,i}^{[2m+2]}$ and $W_{k,i}^{[2m+1]}$ can be explicitly calculated.

For simplicity, we reduce Eq.(2.4) to

$$\begin{cases}
y_{2m+2} = g_{2m+2} + h^\alpha \sum\limits_{j=0}^{2m} \omega_{2m+2-j} f_j + h^\alpha \omega_1 f_{2m+1} + h^\alpha \omega_0 f_{2m+2}, \\
y_{2m+1} = g_{2m+1} + h^\alpha \sum\limits_{j=0}^{2m} \varpi_{2m+2-j} f_j + h^\alpha \varpi_1 f_{2m+1} + h^\alpha \varpi_0 f_{2m+2},
\end{cases} \tag{2.8}$$

where

$$\begin{aligned}
&\omega_0 := W_{m,2}^{[2m+2]}/h^\alpha, \quad \omega_{2k+1} := W_{m-k,1}^{[2m+2]}/h^\alpha, && k = 0, 1, \cdots, m; \\
&\omega_{2k} := W_{m-k,2}^{[2m+2]}/h^\alpha + W_{m-k+1,0}^{[2m+2]}/h^\alpha, && k = 1, 2, \cdots, m; \\
&\omega_{2m+2} := W_{0,0}^{[2m+2]}/h^\alpha,
\end{aligned}$$

and ϖ_j is defined similarly just by replacing the $W_{k,i}^{[2m+2]}$ in the definition of ω_j with $W_{k,i}^{[2m+1]}$ $(j = 0, 1, \cdots, 2m + 2)$.

In the remainder of this paper, we will be devoted to convergence analysis of the block-by-block method under the assumptions $D_*^\alpha y(t) \in C^3[0, T]$ and $f_y(x, y) \in C(\Omega)$. Therefore, it is necessary to relate the smoothness properties of a given function to smoothness properties of its Caputo derivatives.

Lemma 2.1. ([7]) *For any* $\alpha > 0$, $y(t) \in C^{3+\lceil\alpha\rceil}[0, T]$, *we have*

$$D_*^\alpha y(t) = \sum_{k=0}^{2} \frac{y^{(k+\lceil\alpha\rceil)}(0)}{\Gamma(\lceil\alpha\rceil - \alpha + k + 1)} t^{\lceil\alpha\rceil - \alpha + k} + \phi(t)$$

with some function $\phi(t) \in C^3[0,T]$.

Remark. By above lemma, for $y(t) \in C^{3+\lceil\alpha\rceil}[0,T]$, $D_*^\alpha y(t) \in C^3[0,T]$ if and only if $y^{(\lceil\alpha\rceil)}(0) = y^{(1+\lceil\alpha\rceil)}(0) = y^{(2+\lceil\alpha\rceil)}(0) = 0$. These conditions seem to be quite stringent and limit the application of block-by-block method. Notice that there exist such kinds of functions as the most simple one $y(t) = t^\alpha$, which satisfy $D_*^\alpha y(t) \in C^3[0,T]$ but $y(t) \notin C^{3+\lceil\alpha\rceil}[0,T]$.

3. Preliminary lemmas

In our subsequent analysis in Section 4, the following lemmas are needed.

Lemma 3.1. *Let*

$$\begin{aligned} a_k = &(k+1)^{\alpha+1} + k^{\alpha+1} + \left(k+\frac{1}{2}\right)^{\alpha+1} + 6\frac{k^{\alpha+2}-(k+1)^{\alpha+2}}{\alpha+2} \\ &+ 12\frac{(k+1)^{\alpha+3}+k^{\alpha+3}-2\left(k+\frac{1}{2}\right)^{\alpha+3}}{(\alpha+2)(\alpha+3)}. \end{aligned} \tag{3.1}$$

Then for $\alpha \geq 0$, *we have*

$$\sum_{k=0}^{m} a_k = \mathcal{O}(m^\alpha), \quad m \geq 2. \tag{3.2}$$

Proof. For $k \geq 2$, we have

$$\begin{aligned} a_k =& k^{\alpha+1}\left[\left(1+\frac{1}{k}\right)^{\alpha+1} + 1 + \left(1+\frac{1}{2k}\right)^{\alpha+1}\right] + \frac{6k^{\alpha+2}}{\alpha+2}\left[1-\left(1+\frac{1}{k}\right)^{\alpha+2}\right] \\ &+ \frac{12k^{\alpha+3}}{(\alpha+2)(\alpha+3)}\left[\left(1+\frac{1}{k}\right)^{\alpha+3} + 1 - 2\left(1+\frac{1}{2k}\right)^{\alpha+3}\right] \\ =& k^{\alpha+1}\left[3+\sum_{j=1}^{+\infty}\frac{(2^j+1)(\alpha+1)\cdots(\alpha+1-j+1)}{j!2^jk^j}\right] - \frac{6k^{\alpha+2}}{\alpha+2}\sum_{j=1}^{+\infty}\frac{(\alpha+2)\cdots(\alpha+2-j+1)}{j!k^j} \\ &+ \frac{12k^{\alpha+3}}{(\alpha+2)(\alpha+3)}\sum_{j=1}^{+\infty}\frac{(2^j-2)(\alpha+3)\cdots(\alpha+3-j+1)}{j!2^jk^j} \\ =& k^{\alpha+1}\sum_{j=2}^{+\infty}\frac{(j-1)(j-2)2^j+(j-1)(j+4)}{(j+1)(j+2)2^j}\frac{(\alpha+1)\cdots(\alpha+2-j)}{j!k^j} \\ \doteq& \frac{(\alpha+1)\alpha}{k^{1-\alpha}}\sum_{j=0}^{+\infty}\frac{b_j}{k^j}. \end{aligned} \tag{3.3}$$

When $0 < \alpha < 1$, it is easy to check that for any $k \geq 2$, $\left\{ b_j \middle| j = 0,1,\cdots \right\}$ is an alternate series with

$$b_0 = \frac{1}{16} > 0, \ \ b_1 = \frac{1}{32}(\alpha-1) < 0, \ \ b_2 = \frac{1}{96}(\alpha-1)(\alpha-2) > 0$$

and $\left\{|b_j|\,\middle|\, j=0,1,\cdots\right\}$ monotonically approaches 0. Hence it deduces that the infinity series of the last term of (3.3) converges and

$$\frac{(\alpha+1)\alpha}{k^{1-\alpha}}\left(b_0+\frac{b_1}{k}\right) < a_k < \frac{(\alpha+1)\alpha b_0}{k^{1-\alpha}}, \tag{3.4a}$$

$$\sum_{k=2}^{m}\frac{(\alpha+1)\alpha b_0}{k^{1-\alpha}}+\sum_{k=2}^{m}\frac{(\alpha+1)\alpha b_1}{k^{2-\alpha}} < \sum_{k=2}^{m} a_k < \sum_{k=2}^{m}\frac{(\alpha+1)\alpha b_0}{k^{1-\alpha}}. \tag{3.4b}$$

Consequently,

$$\begin{aligned}\sum_{k=0}^{m} a_k &= \sum_{k=2}^{m} a_k+\mathscr{O}(1)=\sum_{k=2}^{m}\frac{(\alpha+1)\alpha b_0}{k^{1-\alpha}}\left[1+o(1)\right]+\mathscr{O}(1)\\ &=(\alpha+1)b_0 m^{\alpha}\left[1+o(1)\right]=\frac{(\alpha+1)}{16}m^{\alpha}\left[1+o(1)\right].\end{aligned} \tag{3.5}$$

Note that for $\alpha=0$,

$$a_k=3k+\frac{3}{2}+3\left[k^2-(k+1)^2\right]+2\left[(k+1)^3+k^3-2\left(k+\frac{1}{2}\right)^3\right]=0;$$

and for $\alpha=1$,

$$a_k=(k+1)^2+k^2+\left(k+\frac{1}{2}\right)^2+2\left[k^3-(k+1)^3\right]+(k+1)^4+k^4-2\left(k+\frac{1}{2}\right)^4=\frac{1}{8}.$$

When $n-1<\alpha\le n$ for some integer $n\ge 2$, it can be checked that for any $k\ge 2$, b_0, $\cdots$, b_{n-1} are positive, and $\left\{b_j\,\middle|\, j=n-1,n,\cdots\right\}$ is an alternate series and $\left\{|b_j|\,\middle|\, j=n-1,n,\cdots\right\}$ monotonically approaches 0. According to the similar analysis dealing with the case $0<\alpha<1$, we can also obtain for $\alpha>1$, the result (3.2) holds. This completes the proof. ☐

Lemma 3.2. *For $\alpha>0$ and $j\ge 2$, the following statements hold*

$$\omega_j=\mathscr{O}\left(j^{\alpha-1}\right),\quad \varpi_j=\mathscr{O}\left(j^{\alpha-1}\right), \tag{3.6}$$

where ω_j and ϖ_j are defined in Eq.(2.8).

Proof. Here we prove $\omega_j=\mathscr{O}\left(j^{\alpha-1}\right)$; the proof of $\varpi_j=\mathscr{O}\left(j^{\alpha-1}\right)$ is similar. Without loss of generality, we only need to prove

$$\omega_{2m+1-2k}=\frac{W_{k,1}^{[2m+2]}}{h^{\alpha}}=\mathscr{O}\left((2m+1-2k)^{\alpha-1}\right); \tag{3.7}$$

the other results can be obtained by using the same method. According to the definition of $W_{k,1}^{[2m+2]}$, it results in

$$\frac{W_{k,1}^{[2m+2]}}{h^{\alpha}}=\frac{1}{\Gamma(\alpha)}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\frac{(\tau-t_{2k})(\tau-t_{2k+2})}{-h^{2+\alpha}}d\tau \tag{3.8}$$

After some operations, the right side of (3.8) becomes

$$\begin{aligned}
&\frac{-1}{\Gamma(\alpha)h^{2+\alpha}}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}[\tau^2-(t_{2k}+t_{2k+2})\tau+t_{2k}t_{2k+2}]d\tau\\
=&\frac{-1}{\Gamma(\alpha+2)}\left[-2(2m+2-2k)^{1+\alpha}-2(2m-2k)^{1+\alpha}+\frac{2(2m+2-2k)^{2+\alpha}}{2+\alpha}-\frac{2(2m-2k)^{2+\alpha}}{2+\alpha}\right]\\
=&\frac{-2n^{1+\alpha}}{\Gamma(\alpha+2)}\left[-(1+\frac{1}{n})^{1+\alpha}-(1-\frac{1}{n})^{1+\alpha}+n\left(\frac{(1+\frac{1}{n})^{2+\alpha}}{2+\alpha}-\frac{(1-\frac{1}{n})^{2+\alpha}}{2+\alpha}\right)\right]\\
=&\frac{-2n^{1+\alpha}}{\Gamma(\alpha+2)}\left[(1+\alpha)\alpha\left(\frac{2}{3!}-\frac{2}{2!}\right)\frac{1}{n^2}+(1+\alpha)\alpha(\alpha-1)(\alpha-2)\left(\frac{2}{5!}-\frac{2}{4!}\right)\frac{1}{n^4}\right.\\
&\left.+(1+\alpha)\alpha\cdots(\alpha-4)\left(\frac{2}{7!}-\frac{2}{6!}\right)\frac{1}{n^6}+(1+\alpha)\alpha\cdots(\alpha-6)\left(\frac{2}{9!}-\frac{2}{8!}\right)\frac{1}{n^8}+\cdots\right]\\
=&\frac{4n^{\alpha-1}}{\Gamma(\alpha)}\left[\left(\frac{1}{2!}-\frac{1}{3!}\right)+(\alpha-1)(\alpha-2)\left(\frac{1}{4!}-\frac{1}{5!}\right)\frac{1}{n^2}\right.\\
&\left.+(\alpha-1)\cdots(\alpha-4)\left(\frac{1}{6!}-\frac{1}{7!}\right)\frac{1}{n^4}+(\alpha-1)\cdots(\alpha-6)\left(\frac{1}{8!}-\frac{1}{9!}\right)\frac{1}{n^6}+\cdots\right]\\
=&\ \mathscr{O}(n^{\alpha-1}),
\end{aligned}\tag{3.9}$$

where $n=2m+1-2k$. This completes the proof. □

Lemma 3.3. *(Gronwall Inequality) Let* $C_1>0$ *independent* $h>0$, $C_2\geq 0$, *and* $\{z_n\}$ *satisfy the inequality*

$$|z_n|\leq h^\alpha C_1\sum_{j=0}^{n-1}(n-j)^{\alpha-1}|z_j|+C_2,\quad j=0,1,\cdots,n-1,\quad nh\leq T,\tag{3.10}$$

with $0<\alpha\leq 1$. *Then*

$$|z_n|\leq C_2E_\alpha\left(C_1\Gamma(\alpha)T^\alpha\right),\quad nh\leq T,\tag{3.11}$$

where E_α *denotes the Mittag-Leffler function defined as*

$$E_\alpha(x):=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)},\quad \alpha>0.$$

In particular, when $\alpha=1$, *the inequality* (3.11) *results in*

$$|z_n|\leq C_2e^{C_1T},\quad nh\leq T.\tag{3.12}$$

The proof of this lemma can be found in [4].

4. Convergence analysis

The objective of this section is to analyze the block-by-block method (2.4) or (2.8). First, we derive the error estimate.

Theorem 4.1. *For* $\alpha>0$, *the truncation error order of the block-by-block method* (2.4) *is at least 3.*

Proof. It is known that the error of quadratic Lagrange interpolating polynomial is

$$\frac{f^{(3)}(\xi_k, y(\xi_k))}{6}(t - t_{2k})(t - t_{2k+1})(t - t_{2k+2}), \ \ \xi_k \in (t_{2k}, t_{2k+2}).$$

Notice that $f(t, y(t))$ is three times continuously differentiable, so there exists a constant C such that

$$\frac{|f^{(3)}(t, y(t))|}{6} \le C, \quad t \in [0, T].$$

Then the truncation error of the first formula of Eq.(2.4) is

$$\begin{aligned}&\left|\sum_{k=0}^{m} \frac{1}{\Gamma(\alpha)} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+2} - \tau)^{\alpha-1} \frac{f^{(3)}(\xi_k, y(\xi_k))}{6} (\tau - t_{2k})(\tau - t_{2k+1})(\tau - t_{2k+2}) d\tau\right| \\ &\le \frac{C}{\Gamma(\alpha)} \sum_{k=0}^{m} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+2} - \tau)^{\alpha-1} \left|(\tau - t_{2k})(\tau - t_{2k+1})(\tau - t_{2k+2})\right| d\tau. \qquad (4.1)\end{aligned}$$

Let estimate the integrals on the right-hand side of (4.1):

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)} \sum_{k=0}^{m} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+2} - \tau)^{\alpha-1} \left|(\tau - t_{2k})(\tau - t_{2k+1})(\tau - t_{2k+2})\right| d\tau \\ =&\frac{2^{\alpha+2} h^{\alpha+3}}{\Gamma(\alpha+2)} \sum_{k=0}^{m} \Bigg[(k+1)^{\alpha+1} + k^{\alpha+1} + \left(k + \frac{1}{2}\right)^{\alpha+1} + 6\frac{k^{\alpha+2} - (k+1)^{\alpha+2}}{\alpha+2} \\ &+12 \frac{(k+1)^{\alpha+3} + k^{\alpha+3} - 2\left(k + \frac{1}{2}\right)^{\alpha+3}}{(\alpha+2)(\alpha+3)} \Bigg] \\ =&\frac{2^{\alpha+2} h^{\alpha+3}}{\Gamma(\alpha+2)} \sum_{k=0}^{m} a_k. \qquad (4.2)\end{aligned}$$

According to Lemma 3.1, we have

$$h^{3+\alpha} \sum_{k=0}^{m} a_k = h^3 \mathcal{O}([hm]^\alpha) = h^3 \mathcal{O}(T^\alpha) = \mathcal{O}(h^3). \qquad (4.3)$$

Consequently the order of error is at least 3. The error analysis for Eq.(2.4b) has the same conclusion at the expense of more additional work. ☐

Remark. In fact, instead of using complicated Lemma 3.1, one can obtain Theorem 4.1 from formula (4.1) directly. The reason why we established Lemma 3.1 is that based on the method of proving this lemma, one can explicitly explain why the numerical convergence order often demonstrate as $3 + \alpha$ with $0 < \alpha \le 1$ and 4 with $\alpha > 1$ in many numerical experiments. Actually, when the right hand side function $f(t, y(t))$ of Eq.(2.1) is three-time continuously differentiable, then the values of $f^{(3)}(t, y(t))$ do not occur fierce changes on a small interval. Thus the truncation error of formula (2.4) may be approximately represented as

$$\begin{aligned}&\frac{C_1}{6\Gamma(\alpha)} \sum_{k=0}^{m} \int_{t_{2k}}^{t_{2k+2}} (t_{2m+2} - \tau)^{\alpha-1} (\tau - t_{2k})(\tau - t_{2k+1})(\tau - t_{2k+2}) d\tau \\ =&\frac{C_1 2^{\alpha+2} h^{\alpha+3}}{6\Gamma(\alpha+2)} \sum_{k=0}^{m} a'_k, \qquad (4.4)\end{aligned}$$

where C_1 is a value of $f^{(3)}(t,y(t))$ at a certain point and

$$a'_k = 12\frac{(k+1)^{3+\alpha}-k^{3+\alpha}}{(2+\alpha)(3+\alpha)} - 6\frac{(k+1)^{2+\alpha}+k^{2+\alpha}}{2+\alpha} + (k+1)^{1+\alpha} - k^{1+\alpha}.$$

Using similar method for proving Lemma 3.1, one can know the series $\sum_{k=0}^{\infty} a'_k$ is convergent for $0<\alpha\le 1$, and

$$\sum_{k=0}^{m} a'_k = \mathcal{O}(m^{\alpha-1}) \ \text{ for } \ \alpha > 1.$$

Thus we know the reason that some numerical convergence order approach $3+\alpha$ with $0<\alpha\le 1$ and 4 with $\alpha>1$.

Finally, we state the main result of this paper, i.e., the convergence order of the block-by-block method is at least 3. Thus this method can provide enough accuracy in practical computation.

Theorem 4.2. *The block-by-block scheme* (2.8) *for Eq.* (2.1) *is convergent. Moreover,*

$$|e_n| \equiv |y(t_n)-y_n| = \mathcal{O}(h^3) \ \textit{ for } \ n=1,2,\cdots. \tag{4.5}$$

Proof. According to the mean value theorem, there exists L_j holding that

$$f(t_j,y(t_j)) - f(t_j,y_j) = L_j(y(t_j)-y_j) = L_j e_j, \quad j=0,1,\cdots,2N,$$

where $e_j = y(t_j)-y_j$. In terms of $f(t,y)$ satisfying Lipschitz condition and $f_y(x,y)\in C(\Omega)$, then $|L_j|\le L$. Thanks to Lemma 3.2, there exists a constant C satisfying

$$\max\{|\omega_0|,|\omega_1|,|\varpi_0|,|\varpi_1|\}\le C, \quad \max\{|\omega_j|,|\varpi_j|\}\le Cj^{\alpha-1}, \quad j=2,\cdots,2m+2.$$

Note that

$$\begin{aligned} y(t_{2m+2}) =& g(t_{2m+2}) + h^\alpha \sum_{j=0}^{2m} \omega_{2m+2-j} f(t_j,y(t_j)) + h^\alpha \omega_1 f(t_{2m+1},y(t_{2m+1})) \\ &+ h^\alpha \omega_0 f(t_{2m+2},y(t_{2m+2})) + \mathcal{O}(h^3), \\ y(t_{2m+1}) =& g(t_{2m+1}) + h^\alpha \sum_{j=0}^{2m} \varpi_{2m+2-j} f(t_j,y(t_j)) + h^\alpha \varpi_1 f(t_{2m+1},y(t_{2m+1})) \\ &+ h^\alpha \varpi_0 f(t_{2m+2},y(t_{2m+2})) + \mathcal{O}(h^3), \end{aligned}$$

and

$$\begin{aligned} y_{2m+2} =& g_{2m+2} + h^\alpha \sum_{j=0}^{2m} \omega_{2m+2-j} f(t_j,y_j) + h^\alpha \omega_1 f(t_{2m+1},y_{2m+1}) \\ &+ h^\alpha \omega_0 f(t_{2m+2},y_{2m+2}), \\ y_{2m+1} =& g_{2m+1} + h^\alpha \sum_{j=0}^{2m} \varpi_{2m+2-j} f(t_j,y_j) + h^\alpha \varpi_1 f(t_{2m+1},y_{2m+1}) \\ &+ h^\alpha \varpi_0 f(t_{2m+2},y_{2m+2}). \end{aligned}$$

We then obtain

$$\begin{cases} e_{2m+2} = h^{\alpha} \sum\limits_{j=0}^{2m} \omega_{2m+2-j} L_j e_j + h^{\alpha}\omega_1 L_{2m+1} e_{2m+1} + h^{\alpha}\omega_0 L_{2m+2} e_{2m+2} + \mathscr{O}(h^3), \\ e_{2m+1} = h^{\alpha} \sum\limits_{j=0}^{2m} \varpi_{2m+2-j} L_j e_j + h^{\alpha}\varpi_1 L_{2m+1} e_{2m+1} + h^{\alpha}\varpi_0 L_{2m+2} e_{2m+2} + \mathscr{O}(h^3). \end{cases} \tag{4.6}$$

Consequently,

$$\begin{cases} |e_{2m+2}| \le LCh^{\alpha} \sum\limits_{j=0}^{2m} (2m+2-j)^{\alpha-1} |e_j| + LCh^{\alpha}|e_{2m+1}| + LCh^{\alpha}|e_{2m+2}| + \mathscr{O}(h^3), \\ |e_{2m+1}| \le LCh^{\alpha} \sum\limits_{j=0}^{2m} (2m+2-j)^{\alpha-1} |e_j| + LCh^{\alpha}|e_{2m+1}| + LCh^{\alpha}|e_{2m+2}| + \mathscr{O}(h^3). \end{cases} \tag{4.7}$$

Set $|\epsilon_{2i+1}| = |\epsilon_{2i+2}| = \max\{|e_{2i+1}|, |e_{2i+2}|\}$ for $i = 0, 1, \cdots, m$, and note that $|\epsilon_0| = |e_0| = 0$. For the sufficient small h and any $\alpha > 0$, there exists a constant C_3 such that

$$1 < (1 - 2LCh^{\alpha})^{-1} \le C_3.$$

For $0 < \alpha \le 1$, the inequalities in (4.7) lead to

$$|\epsilon_{2m+1}| \le LCh^{\alpha} \sum_{j=0}^{2m} (2m+1-j)^{\alpha-1} |\epsilon_j| + 2LCh^{\alpha}|\epsilon_{2m+1}| + \mathscr{O}(h^3). \tag{4.8}$$

After further transformation and by Lemma 3.3, we obtain

$$|\epsilon_{2m+1}| \le \mathscr{O}(h^3) E_{\alpha}(C_3 LC\Gamma(\alpha) T^{\alpha}) = \mathscr{O}(h^3). \tag{4.9}$$

For $\alpha > 1$, from (4.7) we get

$$|\epsilon_{2m+1}| \le LCT^{\alpha-1} h \sum_{j=0}^{2m} |\epsilon_j| + 2LCh^{\alpha}|\epsilon_{2m+1}| + \mathscr{O}(h^3).$$

With a similar treatment for $0 < \alpha \le 1$, it deduces that

$$|\epsilon_{2m+1}| \le \mathscr{O}(h^3) e^{C_3 LCT^{\alpha}} = \mathscr{O}(h^3). \tag{4.10}$$

Therefore the block-by-block method (2.8) is convergent with order 3. ☐

5. Numerical experiments

In this section, we verify the convergence order by numerical experiments.

Example 1. Consider the following equations where $y(t) \in C^{3+\lceil\alpha\rceil}[0,1]$ and $D_*^{\alpha} y(t) \in C^3[0,1]$

$$D_*^{\alpha} y(t) = \frac{\Gamma(5+\alpha)}{24} t^4 + t^{8+2\alpha} - y^2(t) \tag{5.1}$$

with initial condition $y(0) = 0$ for the case $0 < \alpha \le 1$ and $y(0) = y\prime(0) = 0$ for $1 < \alpha \le 2$. The exact solution of this equation is given as

$$y(t) = t^{4+\alpha}.$$

Notice that the function $y(t)$ satisfies the Lemma 2.1 and its remark. The comparisons of numerical solution and exact solution for $\alpha = 0.5$ and $\alpha = 1.5$ are shown in Fig.1. For this case, we take stepsize $h = 0.05$. It can be seen that our numerical results are in excellent agreement with the exact solution. From Tables 1 and 2, we find that as the stepsize h decreasing, the error is reduced. For $\alpha = 0.5$, the numerical convergence order is about 3.5; while $\alpha = 1.5$, the convergence order almost approaches 4. Thus the numerical results are consistent with the theoretical analysis in Section 4.

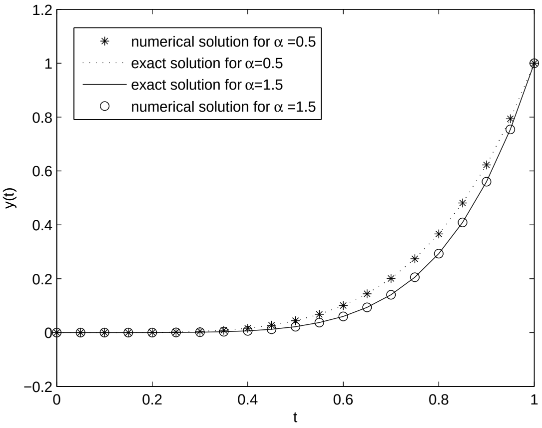

Figure 1: The comparison of numerical solution and exact solution for different α in Example 1.

Table 1: The errors for different stepsize h and $\alpha = 0.5$ in Example 1.

stepsize h	max $\lvert y(t_i) - y_i \rvert$	convergence order
1/10	6.790480279114108e-004	
1/20	6.115764018488346e-005	3.47290897206878
1/40	5.326621723589220e-006	3.52124000532623
1/80	4.578516905606733e-007	3.54026857517293
1/160	3.923653379978020e-008	3.54461084656701
1/320	3.360732248047782e-009	3.54535008607147

Example 2. The following equations where $y(t) \notin C^{3+\lceil\alpha\rceil}[0,1]$ and $D_*^\alpha y(t) \in C^3[0,1]$

$$D_*^\alpha y(t) = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)} t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1) + \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^3 - [y(t)]^{3/2} \tag{5.2}$$

subject to the initial conditions $y^{(k)}(0) = 0$, $k = 0, \cdots, \lceil\alpha\rceil - 1$ with $0 < \alpha < 2$. The exact solution is

$$y(t) = t^8 - 3t^{4+\alpha/2} + \frac{9}{4}t^\alpha.$$

From Tables 3 and 4, we know that these numerical results are in good agreement with the theoretical analysis.

Table 2: The errors for different stepsize h and $\alpha = 1.5$ in Example 1.

stepsize h	max $\left\|y\left(t_i\right) - y_i\right\|$	convergence order
1/10	3.925338639648723e-004	
1/20	2.535081665921979e-005	3.95271299240205
1/40	1.598769163946301e-006	3.98699866595346
1/80	9.970496439581922e-008	4.00315250268271
1/160	6.191967605317927e-009	4.00919551060856
1/320	3.851163832280236e-010	4.00703152050329

Table 3: The errors for different stepsize h and $\alpha = 0.4$ in Example 2.

stepsize h	max $\left\|y\left(x_i\right) - y_i\right\|$	convergence order
1/10	0.00338693729077	
1/20	4.105770707192313e-004	3.04425631016298
1/40	4.457995720869024e-005	3.20318592336832
1/80	4.585147059199546e-006	3.28135532062425
1/160	4.576406376077813e-007	3.32468093459055
1/320	4.486848470541816e-008	3.35044079896980

Table 4: The errors for different stepsize h and $\alpha = 1.6$ in Example 2.

stepsize h	max $\left\|y\left(x_i\right) - y_i\right\|$	convergence order
1/10	0.00186448622110	
1/20	1.309261948120866e-004	3.83195245959574
1/40	8.614536457729471e-006	3.92583679231980
1/80	5.493969965075785e-007	3.97085223961853
1/160	3.454578240136286e-008	3.99126753153549
1/320	2.160079393132008e-009	3.99935334239282

6. Concluding remarks

A block-by-block method, proposed by Kumar and Agrawal for a class of initial value problems of fractional differential equations with Caputo derivatives, has been rewritten. On this basis, the error estimate and the proof of convergence are given under the assumptions $D_*^\alpha y(t) \in C^3[0,T]$ and $f_y(x,y) \in C(\Omega)$. And the convergence order of this scheme is shown to be at least 3. The numerical examples have verified the theoretical results. It is demonstrated that this block-by-block method is an effective and convergent numerical scheme in solving a variety of FDEs.

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