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A Compact Difference Scheme for an Evolution Equation with a Weakly Singular Kernel

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Abstract. This paper is concerned with a compact difference scheme with the truncation error of order 3/2 for time and order 4 for space to an evolution equation with a weakly singular kernel. The integral term is treated by means of the second order convolution quadrature suggested by Lubich. The stability and convergence are proved by the energy method. A numerical experiment is reported to verify the theoretical predictions.

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Key words: Evolution equation, weakly singular kernel, compact difference scheme, stability, convergence, numerical experiment.

1. Introduction

We shall consider a compact difference scheme for the numerical solution of an evolution equation [2, 9, 12, 13, 15, 19]

$$u_t(x,t) - \int_0^t \beta(t-s)u_{xx}(x,s)ds = f(x,t), \qquad 0 < x < 1,\ 0 < t \leq T, \tag{1.1}$$

where the kernel $\beta(t) = (\pi t)^{-1/2}$ is singular at $t = 0$, with the boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 < t \leq T, \tag{1.2}$$

and the initial condition

$$u(x,0) = v(x), \qquad 0 \leq x \leq 1. \tag{1.3}$$

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Recall that, for $\gamma > 0$, the γ-th integral $I^{(\gamma)}f(t)$ is defined by the Riemann-Liouville operator (see [14]) as

$$I^{(\gamma)}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s)ds, \qquad t > 0.$$

Thus, the integral term can be viewed as the $1/2$-th integral of $u_{xx}(x,\cdot)$, equation (1.1) is intermediate between the diffusion and the wave equation [4, 5], and it can be termed a fractional partial differential equation of $3/2$-order in time. Equations similar to (1.1) can be found in the modelling of wave propagation involving viscoelastic forces, heat conduction in materials with memory and anomalous diffusion processes [1, 4, 5, 14]. They have recently attracted increasing interest in the physical, chemical and engineering literature (see the numerous papers citing [14]).

A number of people have studied the evolution equation. e.g., Chen, Thomee and Wahlbin [1] used backward Euler scheme in time, piecewise linear finite element method in space, the integral term by means of product integration, and gave the regularity and error boundness of the solution. Lopez-Marcos [9] studied a nonlinear partial integro-differential equation, used one order full discrete difference scheme. Mclean, Thomee [12] employed backward Euler, Crank-Nicolson and second order backward difference scheme, Galerkin finite element method for spatial variables and gave the regularity, stability and error estimate of (1.1)-(1.3). Sanz-Serna [15] studied this type of equations, used backward Euler scheme in time and one order convolution to the integral term, drove error boundedness for smooth and nonsmooth initial value. Xu [19] considered backward Euler and Crank-Nicolson scheme, with one and second order convolution quadrature to the integral term respectively, drove long time error boundness with weights. A practical difficulty of time discretization is that all $\mathbf{U}^n (1 \le n \le N)$ need be stored as they all enter the subsequent equations, which need much memory requirement. In order to conquer this problem, Huang [6] put forward an iterative scheme and reduced the memory requirement. Sloan, Thomee [16] proposed more economical schemes by using quadrature rules with higher order truncation errors.

It is well known that the Crank-Nicolson scheme has $O(k^2 + h^2)$ order accuracy and is unconditionally stable for any step-size ratio k/h^2 for the heat equation. However, it is not an optimum scheme. The six-point implicit difference scheme with minimum truncation error is $(\delta_t U_{j-1}^n + 10\delta_t U_j^n + \delta_t U_{j+1}^n)/12 = (\delta_x^2 U_j^n + \delta_x^2 U_j^{n-1})/2$, whose truncation error is $O(k^2 + h^4)$ [17]. In this paper, we will consider the scheme for the evolution equation with a weakly singular kernel.

Throughout the paper, for $0 \le t \le T$, $0 \le x \le 1$, we assume that there exists a positive constant C such that (see [9, (1.7)])

$$\begin{aligned} &|u_{tt}(x,t)| \le Ct^{-1/2}, \qquad &&|u_{ttt}(x,t)| \le Ct^{-3/2}, \\ &|u_{xxt}(x,0)| \le C, &&|u_{xxtt}(x,t)| \le Ct^{-1/2}. \end{aligned} \tag{1.4}$$

Remark 1.1. For sufficiently smooth $v(x)$ and $f(x,t)$, (1.1)-(1.3) exists a unique solution and satisfies the following regularities (see [1])

$$u \in C([0,T];H^2 \cap H_0^1),$$
$$u_t \in C([0,T];L_2) \cap L_1([0,T];H^2 \cap H_0^1),$$
$$u_{tt} \in L_1([0,T];L_2).$$

Remark 1.2. The regularity for the homogeneous equation of (1.1) was shown in [12, Theorem 5.5] and expressed in the terms of the semi-norm

$$|v|_r = \|A^{r/2}v\|, \quad r \in \mathscr{R}, \quad \text{where} \quad A = -\frac{\partial^2}{\partial x^2},$$
$$|u(\cdot,t)|_{r+2\theta} \le C(\alpha)t^{-(\alpha+1)\theta}|v|_r, \quad t \ge 0, \quad 0 \le \theta \le 1, \tag{1.5}$$

and similarly the time derivatives $D_t^m u(\cdot,t)(m \ge 1)$ satisfy

$$|D_t^m u(\cdot,t)|_{r+2\theta} \le C(m,\alpha)t^{-(\alpha+1)\theta-m}|v|_r, \qquad t \ge 0, -1 \le \theta \le 1. \tag{1.6}$$

If appropriate θ, r in (1.5) and (1.6) are chosen respectively, we have the following regularities ($\|\cdot\|_0$ is continuous L^2-norm) (see [19, (7.12)])

$$\|u_{tt}(x,t)\|_0 \le Ct^{-1/2}, \qquad \|u_{ttt}(x,t)\|_0 \le Ct^{-3/2},$$
$$\|u_{xxt}(x,0)\|_0 \le C, \qquad \|u_{xxtt}(x,t)\|_0 \le Ct^{-1/2},$$
$$\text{for } 0 \le t \le T,\ 0 \le x \le 1. \tag{1.7}$$

An overview of the paper follows. In Section 2, a compact finite difference scheme is introduced. Section 3 is devoted to the analysis of the stability and convergence of the scheme. In Section 4, a numerical example that is in total agreement with our analysis is reported. Concluding remark is in final section.

2. A compact finite difference scheme for the evolution equation

We introduce a grid $x_j = jh$, $j = 0,1,\cdots,J$, with $h = 1/J$ and J a positive integer. The step-length in time is denoted by k and a superscript n refers to the time level $t_n = nk$, $n = 0, 1,\cdots,N$ $(N = [T/k])$.

Let

$$V_h = \{\mathbf{U}|\mathbf{U} = (U_0,U_1,\cdots,U_{J-1},U_J), U_0 = U_J = 0\}$$

be the space of grid functions. Define the grid function

$$U_j^n = u(x_j,t_n), \qquad 0 \le j \le J,\ 0 \le n \le N.$$

For any grid functions $\mathbf{U}$, $\mathbf{W} \in V_h$, denote

$$\delta_t U_j^n = \frac{1}{k}(U_j^n - U_j^{n-1}), \qquad \delta_x U_j = \frac{1}{h}(U_{j+1} - U_j),$$
$$\delta_x^2 U_j = \frac{1}{h^2}(U_{j+1} - 2U_j + U_{j-1}), \qquad (UW)_j = U_j W_j,$$
$$\|U\|_\infty = \max_{1\le j\le J-1} |U_j|, \qquad \langle \mathbf{U}, \mathbf{W}\rangle = h\sum_{j=1}^{J-1} U_j W_j, \qquad \|\mathbf{U}\|^2 = \langle \mathbf{U}, \mathbf{U}\rangle. \tag{2.1}$$

We will introduce the following second order convolution quadrature formula employed by Lubich (see [10, 19])

$$q_n(\varphi) = k^{1/2}\sum_{p=0}^{n} \beta_p \varphi^{n-p} + w_{n0}\varphi^0, \tag{2.2}$$

where β_p are the coefficients of their generating power series

$$\tilde{\beta}\left(\frac{(1-z)(3-z)}{2}\right) = \left[\frac{(1-z)(3-z)}{2}\right]^{-\frac{1}{2}} = \sum_{p=0}^{\infty} \beta_p z^p. \tag{2.3}$$

Here $\tilde{\beta}(s) = \$[\beta(t)] = \int_0^\infty \beta(t)e^{-st}dt$ denotes the Laplace transform of $\beta(t)$. To approximate the integral formally to second order and we take the correction quadrature weights w_{n0} so that the quadrature formula becomes exact for polynomial $\varphi = 1$, namely

$$k^{1/2}\sum_{p=0}^{n} \beta_p + w_{n0} = \int_0^{t_n} \beta(t_n - s)ds = 2\left(\frac{t_n}{\pi}\right)^{\frac{1}{2}}.$$

We will give the quadrature error of $\varepsilon(\varphi)(t_n) = q_n(\varphi) - I^{(1/2)}\varphi(t_n)$, where $q_n(\varphi)$ is defined in (2.2).

Lemma 2.1 ([19]). *If $\beta(t) = (\pi t)^{-1/2}$, then for any $n \ge 1$*

$$|\varepsilon(\varphi)(t_n)|$$
$$\le Ck^2 t_n^{-1/2}|\varphi_t(0)| + Ck^{3/2}\int_{t_{n-1}}^{t_n} |\varphi_{tt}(s)|ds + Ck^2\int_0^{t_{n-1}} (t_n - s)^{-1/2}|\varphi_{tt}(s)|ds.$$

The boundness of $\varepsilon(u_{xx})(t_n) = q_n(u_{xx}) - I^{(1/2)}u_{xx}(\cdot, t_n)ds$ will be given.

Lemma 2.2 ([2]). *Let u_{xx} be real, continuously differentiable function in $0 \le t \le T$, and u_{xxtt} continuous and integrable in $0 < t < T$. There exists a positive constant C that depends only on T, such that*

$$|\,\varepsilon(u_{xx})(t_n)\,| = |q_n(u_{xx}) - I^{(\frac{1}{2})}u_{xx}(\cdot, t_n)ds| \le Ck(k/n)^{1/2},\ 1 \le n \le N.$$

In this article, we will present the compact difference scheme for (1.1)-(1.3) as follows

$$\begin{aligned}&\frac{1}{12}\left(\delta_t U_{j-1}^n+10\delta_t U_j^n+\delta_t U_{j+1}^n\right)-q_{n-1/2}\left(\delta_x^2 U_j\right)\\=&\frac{1}{12}\left(f_{j-1}^{n-1/2}+10f_j^{n-1/2}+f_{j+1}^{n-1/2}\right),\end{aligned}$$

where

$$\begin{aligned}&q_{n-1/2}(\delta_x^2 U_j)=\frac{1}{2}\left(q_n(\delta_x^2 U_j)+q_{n-1}(\delta_x^2 U_j)\right),\\&f_j^{n-1/2}=\frac{1}{2}\left(f(x_j,t_n)+f(x_j,t_{n-1})\right),\qquad j=1,\cdots,J-1,\ n=1,\cdots,N.\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{1}{12}\left(\delta_t U_{j-1}^n+10\delta_t U_j^n+\delta_t U_{j+1}^n\right)\\&-\frac{1}{2}\left(k^{1/2}\sum_{p=0}^{n}\beta_p\delta_x^2 U_j^{n-p}+\omega_{n0}\delta_x^2 U_j^0+k^{1/2}\sum_{p=0}^{n-1}\beta_p\delta_x^2 U_j^{n-1-p}+\omega_{n-1,0}\delta_x^2 U_j^0\right)\\=&\frac{1}{12}\left(f_{j-1}^{n-1/2}+10f_j^{n-1/2}+f_{j+1}^{n-1/2}\right),\qquad j=1,\cdots,J-1,\ n=1,\cdots,N.\end{aligned}\tag{2.4}$$

Furthermore

$$U_0^n=U_J^n=0,\qquad n=0,1,\cdots,N.\tag{2.5}$$

$$U_j^0=v(x_j),\qquad j=1,\cdots,J-1.\tag{2.6}$$

Remark 2.1. In this paper, we use a two-level six-point difference scheme for the time derivative, second order difference scheme in space, second order convolution quadrature formula for the integral term. And we will show that the proposed method is stable and convergent with the convergence order $O(k^{3/2}+h^4)$ in the L^2-norm. However, Chen and Xu [2] used second order backward difference scheme for the time derivative, second order difference scheme in space, and the second order convolution quadrature formula for the integral term, they only obtained $O(k^{3/2}+h^2)$ order convergence.

3. Analysis of the compact difference scheme

3.1. Stability

We first introduce the following lemmas which will be used in the stability analysis.

Lemma 3.1 ([2]). *1) Let* $\mathbf{U},\mathbf{W}\in V_h$. *We have*

$$\langle\delta_x^2\mathbf{U},\mathbf{W}\rangle=-h\sum_{j=0}^{J-1}(\delta_x U_j)(\delta_x W_j).$$

2) Let $\mathbf{U}^m, \mathbf{U}^n \in V_h$. *We have*

$$|\langle \delta_x^2 \mathbf{U}^m, \mathbf{U}^n \rangle| \leq \frac{4}{h^2} \|U^m\| \cdot \|U^n\|.$$

We will give a general result on the nonnegative character of certain real quadratic form with convolution structure. In order to treat more general choices, we say that q_n is β_0-positive [11, 12] if

$$Q_N(\Phi) = k \sum_{n=0}^{N} q_n(\varphi)\varphi^n \geq -\beta_0(\varphi^0)^2, \qquad \forall\ N \geq 1, \qquad \Phi = (\varphi^0, \cdots, \varphi^N)^T.$$

Lemma 3.2 ([9, 19]). *If* $\{a_0, a_1, \cdots, a_n, \cdots\}$ *is a real-valued sequence such that* $\hat{a}(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic in* $D = \{z \in \mathscr{C} : |z| \leq 1\}$, *then for any positive integer* N *and for any* $(U^0, U^1, \cdots, U^N) \in \mathbb{R}^{N+1}$,

$$\sum_{n=0}^{N} \left(\sum_{p=0}^{n} a_p U^{n-p} \right) U^n \geq 0,$$

if and only if

$$\Re \hat{a}(z) \geq 0, \qquad for \qquad z \in D.$$

We know easily $\beta(t)$ is positive type, viz. $\Re\tilde{\beta}(s) \geq 0$, when $\Re(s) \geq 0$. If $|z| < 1$, we have $\Re(3/2 - 2z + 1/2z^2) > 0$ (see [11, Lemma 4.1]). So for $|z| < 1$, $\Re\tilde{\beta}(((1-z)(3-z))/2) = \Re(3/2 - 2z + 1/2z^2) > 0$, the generating function (2.3) satisfies the conditions of Lemma 3.2.

We can now establish the stability of the scheme by means of the energy method.

Lemma 3.3. *Let* $\mathbf{U}^n = (U_1^n, U_2^n, \cdots, U_{J-1}^n)$ *be the solution of*

$$\begin{aligned} &\frac{1}{12}(\delta_t U_{j-1}^n + 10\delta_t U_j^n + \delta_t U_{j+1}^n) - \frac{1}{2}\Bigg(k^{1/2} \sum_{p=0}^{n} \beta_p \delta_x^2 U_j^{n-p} + \omega_{n0} \delta_x^2 U_j^0 \\ &\quad + k^{1/2} \sum_{p=0}^{n-1} \beta_p \delta_x^2 U_j^{n-1-p} + \omega_{n-1,0} \delta_x^2 U_j^0 \Bigg) = g_j^n, \\ &\qquad\qquad j = 1, \cdots, J-1, \quad n = 1, \cdots, N, \end{aligned} \tag{3.1}$$

where

$$U_0^n = U_J^n = 0, \qquad n = 0, 1, \cdots, N. \tag{3.2}$$

$$\mathbf{U}^0 = \big(v(x_1), v(x_2), \cdots, v(x_{J-1})\big) \qquad given. \tag{3.3}$$

Then for $N \geq 1$, *we have*

$$\|\mathbf{U}^N\| \leq C(T)\|\mathbf{U}^0\| + 3k \sum_{n=1}^{N} \|\mathbf{g}^n\|. \tag{3.4}$$

Proof. We denote

$$\overline{U}_j^n = \frac{1}{2}(U_j^n + U_j^{n-1}), \qquad \Gamma_j^n = \delta_t U_{j-1}^n + 10\delta_t U_j^n + \delta_t U_{j+1}^n,$$
$$\omega_n = \omega_{n0} + \omega_{n-1,0}, \qquad j = 1,\cdots,J-1, \qquad n = 1,\cdots,N. \tag{3.5}$$

Then (3.1) is identity to the following expression

$$\frac{1}{12}\left(\delta_t U_{j-1}^n + 10\delta_t U_j^n + \delta_t U_{j+1}^n\right) - \left(k^{1/2}\sum_{p=0}^{n}\beta_p \delta_x^2 \overline{U}_j^{n-p} + \omega_n \delta_x^2 \overline{U}_j^0\right) = g_j^n,$$
$$j = 1,\cdots,J-1, \qquad n = 1,\cdots,N. \tag{3.6}$$

Multiplication by $h\overline{U}_j^n$ each side in (3.6) and summation in j $(1 \le j \le J-1)$ yield

$$\left\langle \frac{1}{12}\Gamma^n, \overline{U}^n \right\rangle - k^{1/2}\sum_{p=0}^{n}\beta_p \left\langle \delta_x^2 \overline{U}^{n-p}, \overline{U}^n \right\rangle - \omega_n \left\langle \delta_x^2 \overline{U}^0, \overline{U}^n \right\rangle = \left\langle g^n, \overline{U}^n \right\rangle. \tag{3.7}$$

When $N \ge 1$, we have

$$\begin{aligned} & k\sum_{n=1}^{N}\left\langle \frac{1}{12}\Gamma^n, \overline{U}^n \right\rangle \\ =& k^{3/2}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p \left\langle \delta_x^2 \overline{U}^{n-p}, \overline{U}^n \right\rangle + k\sum_{n=1}^{N}\omega_n \left\langle \delta_x^2 \overline{U}^0, \overline{U}^n \right\rangle + k\sum_{n=1}^{N}\left\langle g^n, \overline{U}^n \right\rangle. \end{aligned} \tag{3.8}$$

Now each term will be estimated. First, we have

$$\begin{aligned} \frac{1}{12}k\langle \Gamma^n, \overline{U}^n \rangle =& \frac{k}{12}h\sum_{j=1}^{J-1}\left(\delta_t U_{j-1}^n + 10\delta_t U_j^n + \delta_t U_{j+1}^n\right)\overline{U}_j^n \\ =& kh\sum_{j=1}^{J-1}\left(\delta_t U_j^n + \frac{h^2}{12}\delta_x^2 \delta_t U_j^n\right)\overline{U}_j^n \\ =& k\left(\delta_t U^n, \overline{U}^n\right) + \frac{kh^2}{12}\left(\delta_x^2 \delta_t U^n, \overline{U}^n\right) \\ =& \frac{1}{2}\left(\|U^n\|^2 - \|U^{n-1}\|^2\right) - \frac{kh^2}{12}\left(\|\delta_x U^n\|^2 - \|\delta_x U^{n-1}\|^2\right) \\ =& \frac{1}{2}\left[\left(\|U^n\|^2 - \frac{h^2}{12}\|\delta_x U^n\|^2\right) - \left(\|U^{n-1}\|^2 - \frac{h^2}{12}\|\delta_x U^{n-1}\|^2\right)\right], \end{aligned} \tag{3.9}$$

where

$$\|\delta_x U^n\|^2 = h\sum_{j=0}^{J-1}|\delta_x U_j^n|^2.$$

It follows from

$$\|\delta_x U^n\|^2 \le \frac{4}{h^2}\|U^n\|^2$$

that

$$\frac{2}{3}\|U^n\|^2 \le \|U^n\|^2 - \frac{h^2}{12}\|\delta_x U^n\|^2 \le \|U^n\|^2.$$

Consequently,

$$\begin{aligned} k\sum_{n=1}^{N}\left\langle \frac{1}{12}\Gamma^n, \overline{U}^n \right\rangle =& \frac{1}{2}\left[\left(\|U^N\|^2 - \frac{h^2}{12}\|\delta_x U^N\|^2\right) - \left(\|U^0\|^2 - \frac{h^2}{12}\|\delta_x U^0\|^2\right)\right] \\ \ge& \frac{2}{3}\|U^N\|^2 - \|U^0\|^2. \end{aligned} \tag{3.10}$$

Secondly, the first term of the right equality is β_0-positive (see [11,12]). This follows from Lemma 3.1, on permuting the summation indices and using, for each fixed j, and Lemma 3.2:

$$\begin{aligned} \sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\left\langle \delta_x^2\overline{U}^{n-p}, \overline{U}^n\right\rangle =& -\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\sum_{j=0}^{J-1}h\left(\delta_x\overline{U}_j^{n-p}\right)\left(\delta_x\overline{U}_j^{n}\right) \\ =& -h\sum_{j=0}^{J-1}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\left(\delta_x\overline{U}_j^{n-p}\right)\left(\delta_x\overline{U}_j^{n}\right) \\ =& -h\sum_{j=0}^{J-1}\left[\sum_{n=0}^{N}\sum_{p=0}^{n}\beta_p\left(\delta_x\overline{U}_j^{n-p}\right)\left(\delta_x\overline{U}_j^{n}\right) - \beta_0\left(\delta_x\overline{U}_j^{0}\right)^2\right] \\ \le& \beta_0\sum_{j=0}^{J-1}h\left(\delta_x\overline{U}_j^{0}\right)^2 \le \frac{2}{h^2}\beta_0\|\overline{U}^0\|^2. \end{aligned} \tag{3.11}$$

Then using (3.10), (3.11), Lemma 3.1 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\frac{1}{3}\|U^N\|^2 - \frac{1}{2}\|U^0\|^2 \le 2\frac{k^{3/2}}{h^2}\beta_0\|\overline{U}^0\|^2 + k\sum_{n=1}^{N}\omega_n\left\langle \delta_x^2\overline{U}^0, \overline{U}^n\right\rangle + \sum_{n=1}^{N}k\left\langle g^n, \overline{U}^n\right\rangle \\ \le& 2\frac{k^{3/2}}{h^2}\beta_0\|\overline{U}^0\|^2 + \frac{4k}{h^2}\sum_{n=1}^{N}|\omega_n|\|\overline{U}^0\|\|\overline{U}^n\| + \sum_{n=1}^{N}k\|g^n\|\|\overline{U}^n\|, \end{aligned} \tag{3.12}$$

so we have

$$\|U^N\|^2 \le \frac{3}{2}\|U^0\|^2 + 6\frac{k^{3/2}}{h^2}\beta_0\|\overline{U}^0\|^2 + 12\frac{k}{h^2}\sum_{n=1}^{N}|\omega_n|\|\overline{U}^0\|\|\overline{U}^n\| + 3\sum_{n=1}^{N}k\|g^n\|\|\overline{U}^n\|.$$

Now, choosing M so that $\|U^M\| = \max_{0\le n\le N}\|U^n\|$, gives

$$\|U^M\|^2 \le \frac{3}{2}\|U^0\|^2 + 6\frac{k^{3/2}}{h^2}\beta_0\|\overline{U}^0\|^2 + 12\frac{k}{h^2}\sum_{n=1}^{M}|\omega_n|\|\overline{U}^0\|\|\overline{U}^n\| + 3\sum_{n=1}^{M}k\|g^n\|\|\overline{U}^n\|,$$

so

$$\|U^M\|^2 \le \frac{3}{2}\|U^0\|\|U^M\| + 6\frac{k^{3/2}}{h^2}\beta_0\|\overline{U}^0\|\|U^M\| + 12\frac{k}{h^2}\sum_{n=1}^{M}|\omega_n|\|\overline{U}^0\|\|U^M\| + 3\sum_{n=1}^{M}k\|g^n\|\|U^M\|.$$

Therefore, for $N \ge 1$, we obtain

$$\|U^N\| \le \|U^M\| \le \frac{3}{2}\|U^0\| + 6\frac{k^{3/2}}{h^2}\beta_0\|\overline{U}^0\| + 12\frac{k}{h^2}\sum_{n=1}^{N}|\omega_n|\|\overline{U}^0\| + 3\sum_{n=1}^{N}k\|g^n\|. \tag{3.13}$$

Because of $w_{n0} = O(k^{1/2}n^{-1/2})$ (see [11, Theorem 2.4.(1)]) $(1 \le n \le N)$, we have

$$\sum_{n=1}^{N}|w_{n0}| \le C\sum_{n=1}^{N}(k^{1/2}n^{-1/2}) \le C(Nk)^{1/2} \le C(T). \tag{3.14}$$

Using (3.13) and (3.14), we finish the proof. □

3.2. Convergence

The following lemmas will be used in the derivation of the convergence of compact difference scheme.

Lemma 3.4 ([20]). *Let $y \in C^3[t_{n-1}, t_n]$. It holds that*

$$\begin{aligned}
&\frac{1}{2}\Big[y'(t_n) + y'(t_{n-1})\Big] - \frac{1}{k}\Big[y(t_n) - y(t_{n-1})\Big] \\
=&\frac{k^2}{16}\int_0^1 \left[y^{(3)}\left(t_{n-\frac{1}{2}} + \frac{sk}{2}\right) + y^{(3)}\left(t_{n-\frac{1}{2}} - \frac{sk}{2}\right)\right]\left(1 - s^2\right)ds \\
=&\frac{k^2}{12}y^{(3)}\left(t_n + \eta_n k\right), \qquad \eta_n \in (-1,1),
\end{aligned}$$

where $t_{n-1/2} = (n - 1/2)k$.

Lemma 3.5 ([18]). *Suppose $p(x) \in C^6[x_{j-1}, x_{j+1}]$, and $\zeta(s) = 5(1-s)^3 - 3(1-s)^5$, we have*

$$\begin{aligned}
&\frac{1}{12}\Big[p''(x_{j-1}) + 10p''(x_j) + p''(x_{j+1})] - \frac{1}{h^2}[p(x_{j-1}) - 2p(x_j) + p(x_{j+1})\Big] \\
=&\frac{h^4}{360}\int_0^1 \Big[p^{(6)}(x_j - sh) + p^{(6)}(x_j + sh)\Big]\zeta(s)ds \\
=&\frac{h^4}{240}p^{(6)}\left(x_j + \theta_j h\right), \qquad \theta_j \in (-1,1).
\end{aligned}$$

We can derive the convergence of the numerical method (2.4)-(2.6) as a direct consequence of Lemma 3.3.

Theorem 3.1. *Assume that the solution u of* (1.1)-(1.3) *satisfies the smoothness requirements listed in the hypothesis* (1.4), *and that* $(U^0,\cdots,U^N)$ $(N=[T/k])$ *are solutions of* (2.4)-(2.6). *Then for sufficiently smooth data $v(x)$ and $f(x,t)$, we have*

$$\max_{0\le n\le N}\|U^n-u^n\|=C(T)(k^{3/2}+h^4). \tag{3.15}$$

Proof. Let $e_j^n=U_j^n-u_j^n$, where $u_j^n=u(x_j,t_n)$. Then

$$\begin{aligned}&\frac{1}{12}\left(\delta_t e_{j-1}^n+10\delta_t e_j^n+\delta_t e_{j+1}^n\right)-\left(k^{1/2}\sum_{p=0}^{n}\beta_p\delta_x^2\overline{e}_j^{n-p}+w_n\delta_x^2\overline{e}_j^0\right)\\ =&\frac{1}{12}\left(\overline{f}_{j-1}^n+10\overline{f}_j^n+\overline{f}_{j+1}^n\right)-\frac{1}{12}\left(\delta_t u_{j-1}^n+10\delta_t u_j^n+\delta_t u_{j+1}^n\right)\\ &-\left(k^{1/2}\sum_{p=0}^{n}\beta_p\delta_x^2\overline{u}_j^{n-p}+w_n\delta_x^2\overline{u}_j^0\right),\qquad j=1,\cdots,J-1,\qquad n=1,\cdots,N,\end{aligned}$$

where

$$\begin{aligned}&u_t(x_j,t_n)-\int_0^{t_n}\beta(t_n-s)u_{xx}(x_j,s)ds=f(x_j,t_n),\\ &u_t(x_j,t_{n-1})-\int_0^{t_{n-1}}\beta(t_{n-1}-s)u_{xx}(x_j,s)ds=f(x_j,t_{n-1}).\end{aligned}$$

Therefore the error equation can be obtained as follows

$$\frac{1}{12}\left(\delta_t e_{j-1}^n+10\delta_t e_j^n+\delta_t e_{j+1}^n\right)-\left(k^{1/2}\sum_{p=0}^{n}\beta_p\delta_x^2\overline{e}_j^{n-p}+w_n\delta_x^2\overline{e}_j^0\right)=\tau_{1j}^n-\tau_{2j}^n,$$

where

$$\begin{aligned}\tau_{1j}^n=&\frac{1}{12}\Bigg[\left[\frac{1}{2}\left(u_t(x_{j-1},t_n)+u_t(x_{j-1},t_{n-1})\right)-\delta_t u_{j-1}^n\right]\\ &+10\left[\frac{1}{2}\left(u_t(x_j,t_n)+u_t(x_j,t_{n-1})\right)-\delta_t u_j^n\right]\\ &+\left[\frac{1}{2}\left(u_t(x_{j+1},t_n)+u_t(x_{j+1},t_{n-1})\right)-\delta_t u_{j+1}^n\right]\Bigg]\end{aligned}$$

and

$$\begin{aligned}\tau_{2j}^n=&\frac{1}{24}\Bigg(\left[\int_0^{t_n}\beta(t_n-s)\left(u_{xx}(x_{j-1},s)+10u_{xx}(x_j,s)+u_{xx}(x_{j+1},s)\right)ds-12q_n(\delta_x^2u_j)\right]\\ &+\left[\int_0^{t_{n-1}}\beta(t_{n-1}-s)\left(u_{xx}(x_{j-1},s)+10u_{xx}(x_j,s)+u_{xx}(x_{j+1},s)\right)ds-12q_{n-1}(\delta_x^2u_j)\right]\Bigg).\end{aligned}$$

Using Lemma 3.3, we have

$$\|e^n\| \le C(T)\|e^0\| + 3k\sum_{n=1}^{N}\|\tau_1^n - \tau_2^n\| \le 3k\sum_{n=1}^{N}(\|\tau_1^n\| + \|\tau_2^n\|). \tag{3.16}$$

First, using Lemma 3.4, we obtain

$$\left|\frac{1}{2}(u_t(x_j,t_n) + u_t(x_j,t_{n-1})) - \delta_t u_j^n\right| = \frac{k^2}{12}\left|u_{ttt}(x_j,t_n+\eta_n k)\right|, \quad \eta_n \in (-1,1).$$

Using hypothesis (1.4), we have

$$\sum_{n=1}^{N} k\|\tau_1^n\| \le \sum_{n=1}^{N} k\left(\sum_{j=1}^{J-1} h|\tau_{1j}^n|^2\right)^{\frac{1}{2}} \le Ck^{3/2}. \tag{3.17}$$

Next, using Lemmas 2.2 and 3.5 yields

$$\begin{aligned}
&\left|\int_0^{t_n}\beta(t_n-s)\frac{1}{12}\Big(u_{xx}(x_{j-1},s)+10u_{xx}(x_j,s)+u_{xx}(x_{j+1},s)\Big)ds - q_n(\delta_x^2 u_j)\right| \\
\le&\left|\int_0^{t_n}\beta(t_n-s)\left(\delta_x^2 u(x_j,s)+\frac{h^4}{240}u_{x^6}(x_j+\theta_j h,s)\right)ds - q_n(\delta_x^2 u_j)\right| \\
\le&\left|\int_0^{t_n}\beta(t_n-s)\left(\frac{h^4}{240}u_{x^6}(x_j+\theta_j h,s)\right)ds + \int_0^{t_n}\beta(t_n-s)\delta_x^2 u(x_j,s)ds - q_n(\delta_x^2 u_j)\right| \\
\le&\left|\int_0^{t_n}\beta(t_n-s)\left(\frac{h^4}{240}u_{x^6}(x_j+\theta_j h,s)\right)ds\right| + Ck(k/n)^{1/2} \\
\le& C(T)\left(h^4 + k\left(\frac{k}{n}\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Consequently, we have

$$\sum_{n=1}^{N} k\|\tau_2^n\| \le \sum_{n=1}^{N} k\left(\sum_{j=1}^{J-1} h|\tau_{2j}^n|^2\right)^{\frac{1}{2}} \le C(T)(h^4 + k^{3/2}). \tag{3.18}$$

Using (3.16), (3.17) and (3.18) completes the proof. □

Remark 3.1. Although we employ second order method in time and spatial, because of the lack of smoothness of the exact solution $u(x,t)$ at $t=0$, second order accuracy could not be obtained in time if we use a uniform time step k. However, an improvement over the first-order method (see [9]) is manifest.

To obtain second-order accurate in time, we can employ a family of non-uniform meshes that concentrate the time levels near $t=0$ to compensate for the singular behavior of the exact solution at $t=0$ (see [13]).

4. Numerical example

In this section, the coefficients β_p in the second order convolution quadrature formula (2.2) will be confirmed, and the results of some computations using discretization (2.4)-(2.6) will be described, together with second order convolution quadrature formula (2.2).

The main difficulty is to confirm the coefficients β_p in the process of computations. One order convolution quadrature formula have been gave and its coefficients c_p be confirmed by Sanz-Serna [15]

$$(1-z)^{-1/2}=\sum_{p=0}^{\infty}c_p z^p, \tag{4.1}$$

where $c_p=(-1)^p\binom{-1/2}{p}=\frac{(2p-1)!!}{(2p)!!}$ (see [12, p.49]), furthermore

$$\sum_{p=0}^{n-1}c_p=2nc_n=2n^{1/2}\pi^{-1/2}+O(n^{-1/2}). \tag{4.2}$$

Following we will confirm β_p, which are the coefficients of the generating power series

$$\begin{aligned}\sum_{p=0}^{\infty}\beta_p z^p=&\tilde{\beta}\left(\frac{(1-z)(3-z)}{2}\right)=\left(\frac{(1-z)(3-z)}{2}\right)^{-1/2}\\ =&2^{1/2}(1-z)^{-1/2}(3-z)^{-1/2}=\frac{6^{1/2}}{3}(1-z)^{-1/2}(1-\frac{z}{3})^{-1/2}\\ =&\frac{6^{1/2}}{3}\left(\sum_{p=0}^{\infty}c_p z^p\right)\left(\sum_{p=0}^{\infty}c_p\left(\frac{z}{3}\right)^p\right)=\frac{6^{1/2}}{3}\sum_{p=0}^{\infty}\left(\sum_{n=0}^{p}c_n c_{p-n}/3^{p-n}\right)z^p.\end{aligned}$$

By comparing there coefficients, we can obtain

$$\beta_p=\frac{\sqrt{6}}{3}\sum_{n=0}^{p}c_n\frac{c_{p-n}}{3^{p-n}},\qquad p=0,1,\cdots,N. \tag{4.3}$$

In our example, the exact solution is given by ([13])

$$u(x,t)=\sin\pi x-\frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}\sin 2\pi x, \tag{4.4}$$

so the initial data is $v(x)=\sin\pi x$ and the inhomogeneous term is

$$f(x,t)=\frac{2t^{\frac{1}{2}}}{\sqrt{\pi}}(\pi^2\sin\pi x-\sin 2\pi x)-2\pi^2t^2\sin 2\pi x. \tag{4.5}$$

In the calculation we set $T=1$. In Table, we list the errors ($\max_{1\le n\le N}\|e^n\|$) and computed rates of convergence in time. The numerical results reflect a convergence rate $\approx 3/2$ in time. There are in good agreement with the theoretical prediction of Theorem 3.1. Numerical result is computed by Matlab 7.0.

Table 1: Errors and convergence rates in time, $J = 50$.

N	Error	Rate
10	4.58D-2	
20	1.74D-2	1.3963
40	6.62D-3	1.3942
80	2.46D-3	1.4282

5. Concluding remark

The numerical scheme can be slightly readjusted to cater to the weakly singular equation obtained from (1.1) by replacing the integral term $I^{(\frac{1}{2})}u_{xx}$ by

$$I^{(\alpha)}u_{xx}(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u_{xx}(x,s)ds, \qquad 0<\alpha<1. \tag{5.1}$$

An analysis for stability and convergence can be obtained for (5.1) in a similar way.

Furthermore, the scheme can be extended [9, 15] to cover general integral terms with convolution structure

$$\int_0^t a(t-s)u_{xx}(x,s)ds.$$

The stability and convergence of the numerical method will be preserved provided that the convolution quadrature employed satisfies the requirement that its associated quadratic form is nonnegative or β_0-positive.

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